Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2008, Article ID 627983, 9 pages
doi:10.1155/2008/627983

Research Article

Application of He's Variational Iteration Method to Solve Semidifferential Equations of *n*th Order

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Received 3 October 2007; Accepted 4 February 2008

Recommended by Oleg Gendelman

He's variational iteration method is applied to solve *n*th order semidifferential equations. Comparison is made between collocation spline method based on Lagrange interpolation and the present method. In this method, the solution is calculated in the form of a convergent series with an easily computable component. This approach does not need linearization, weak nonlinearity assumptions, or perturbation theory. Some examples are given to illustrate the effectiveness of the method; the results show that He's method provides a straightforward and powerful mathematical tool for solving various semidifferential equations of the *n*th order.

1. Introduction

He's variational iteration method (HVIM) [1–14] proposed by He has been recently intensively studied by scientists and engineers and favorably applied to various kinds of nonlinear problems [15–25]. The method has been shown to solve effectively, easily, and accurately a large class of nonlinear problems, generally one or two iterations lead to high accurate solutions. This method is, in fact, a modifying of the general Lagrange multiplier method into an iteration method, which is called correction functional. Generally speaking, the solution procedure of He's method is very effective, straightforward, and convenient. For a relatively comprehensive survey on the method, new interpretation, and new development, the reader is referred to the review articles [14, 26].

A fractional differential equation of the form [27]:

$$\left[D^{n/2} + c_1 D^{(n-1)/2} + \cdots + c_n D^0\right] y(t) = f(t), \tag{1.1}$$

is called a semidifferential equation of nth order, where $c_1, \ldots, c_n \in \mathbf{R}$ and $f(t)$ is a given function from $\mathbf{I}$ into $\mathbf{R}$, $\mathbf{I}$ is the interval $[0, T]$.

The analytic results on existence and uniqueness of solutions to fractional differential equations have been investigated by many authors (see, e.g., [28–30]).

In (1.1)D^q denotes the fractional differential operator of order $q \notin \mathbf{N}$ in the sense of Caputo, and is given by [29–31]

$$D^q y(t) = J^{m-q} D^m y(t) = \frac{1}{\Gamma(m-q)} \int_0^t (t-s)^{m-q-1} D^m y(s) ds, \tag{1.2}$$

where $m \in \mathbf{N} D^m$ is the usual integer differential operator of order m, and J^q is the Riemann-Liouville fractional integral operator of order $q > 0$, defined by

$$J^q y(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s) ds, \quad t > 0. \tag{1.3}$$

For $q, p > 0$, $m - 1 < q \leq m$, and $\gamma > -1$, we have the following properties:

$$\begin{gathered} \left(J^q J^p y\right)(t) = \left(J^p J^q y\right)(t) = \left(J^{q+p} y\right)(t), \\ D^q t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+q)} t^{\gamma+q}, \\ \left(J^q D^q y\right)(t) = y(t) - \sum_{k=0}^{m-1} y^{(k)}\left(0^+\right) \frac{t^k}{k!}. \end{gathered} \tag{1.4}$$

The Caputo fractional derivative is considered here because it allows traditional, initial, and boundary conditions to be included in the formulation of the problem. For more details on the geometric and physical interpretation for fractional derivatives of both the Riemann-Liouville and Caputo types see [29–31].

The theory for the derivatives of fractional order was developed in the 19th century. In earlier work, the main application of the fractional calculus has been as a technique for solving integral equations, for more details (see, [31]). Recently, fractional derivatives have proved to be tools in the modeling of many physical phenomena (see, [32–36]). We mention the important example: the Bagley-Torvik equation,

$$\left[D^2 + c_1 D^{3/2} + c_4 D^0\right] y(t) = f(t), \qquad y(0) = a_1, \qquad y'(0) = a_2, \tag{1.5}$$

which arises, for example, in the modeling of the motion of a rigid plate immersed in a Newtonian fluid. Studying the numerical solution of (1.1) has been increased in the last two decades. A survey of some numerical methods is given by Podlubny [37]. Blank [38] proposed the collocation spline method and also Rawashdeh [27] applied the collocation spline method to solve semidifferential equations. It is well-known that the main disadvantage of the presented methods in [27, 37, 38] is the complex and difficult procedure. In order to overcome the demerit, in this paper we will apply He's variational iteration method for solving semidifferential equations of nth order. This will make the solution procedure easier, more effective, and more straightforward.

2. He's Variational Iteration Method (HVIM)

To illustrate its basic idea of the method, He [14, 26] considered the following general nonlinear equation:

$$Lu(t) + Nu(t) = f(t), \tag{2.1}$$

where L is a linear operator, N is a nonlinear operator, and $f(t)$ is a given continuous function. The basic character of the method is to construct a correction functional for the system, which reads

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s)\{Lu_n(s) + N\tilde{u}_n - f(s)\}ds, \tag{2.2}$$

where λ is a Lagrange multiplier which can be identified optimally via variational theory [7], u_n is the nth approximate solution, and $\tilde{u}_n$ denotes a restricted variation, that is, $\delta\tilde{u}_n = 0$. In most cases, the integration in (2.2) is not easily evaluated or needs a huge computational work. Therefore, in the following, we introduce an application of the HVIM to reduce the size of calculations and to overcome the difficulty arising in calculating complicated integrals.

An application of He's variational iteration method can be proposed based on this assumption that the function $f(s)$ can be divided into two parts, namely, $f_0(s)$ and $f_1(s)$,

$$f(s) = f_0(s) + f_1(s). \tag{2.3}$$

According to this assumption, $f(s) = f_0(s) + f_1(s)$, we construct the following iteration formula:

$$\begin{aligned} u_1(t) &= u_0(t) + \int_0^t \lambda(s)\{Lu_0(s) + Nu_0 - f_0(s)\}ds, \\ u_{n+1}(t) &= u_n(t) + \int_0^t \lambda(s)\{Lu_n(s) + Nu_n - (f_0(s) + f_1(s))\}ds, \quad n \geq 1. \end{aligned} \tag{2.4}$$

It is shown that this method is very effective and easy for linear problem, its exact solution can be obtained by only one iteration because λ can be exactly identified. But for nonlinear problems, there are secular terms, which should be considered [1]. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximation $u_n(t)$, $n \geq 0$ of the solution $u(t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. The zeroth approximation u_0 may be selected any function that just satisfies at least the initial and boundary conditions. With λ determined, then several approximations $u_n(t)$, $n \geq 0$ follow immediately. Consequently, the exact solution may be obtained by using

$$u(t) = \lim_{n\to\infty} u_n(t). \tag{2.5}$$

The HVIM has been shown to solve effectively, easily, and accurately a large class of nonlinear problems with approximations converges rapidly to accurate solutions.

3. Implementation of the Method

In this section, an application of He's variational iteration method is discussed, for solving the Bagley-Torvik equation and the initial value problem, which are semidifferential equations of order 4, [27].

Example 3.1. As an example that arises in application, we solve the Bagley-Torvik equation [27]

$$[D^2 + D^{3/2} + D^0]y(t) = 2 + 4\sqrt{t/\pi} + t^2, \qquad y(0) = y'(0) = 0. \tag{3.1}$$

According to He's variational iteration method, we consider the correction functional in the following form (see, [14, 26]):

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda\{y_n^{(2)}(s) + \tilde{y}_n^{(3/2)}(s) + \tilde{y}_n(s) - 2 - 4\sqrt{s/\pi} - s^2\}ds, \tag{3.2}$$

where λ is the general Lagrange multiplier, y_0 is an initial approximation which must be chosen suitably, and $\tilde{y}_n$ is the restricted variation, that is, $\delta\tilde{y}_n = 0$. Under these conditions, its stationary conditions of the above correction functional can be expressed as follows:

$$\begin{aligned} &\frac{\partial^2\lambda(t,s)}{\partial s^2} = 0, \\ &1 - \left.\frac{\partial\lambda(t,s)}{\partial s}\right|_{t=s} = 0, \\ &\partial\lambda(t,s)|_{t=s} = 0. \end{aligned} \tag{3.3}$$

The Lagrange multiplier, therefore, can be easily identified as

$$\lambda = s - t, \tag{3.4}$$

leading to the following iteration formula:

$$y_{n+1}(t) = y_n(t) + \int_0^t (s-t)\{y_n^{(2)}(s) + y_n^{(3/2)}(s) + y_n(s) - 2 - 4\sqrt{s/\pi} - s^2\}ds. \tag{3.5}$$

We start with the initial approximation $y_0(t) = 0$. In Table 1, we have listed the approximation solution $y_3(t)$ of (3.5), the exact solution, and the absolute error.

Here, an application of the HVIM can be introduced based on the assumption that the function $f(t)$ can be divided into two parts, namely, $f_0(t)$ and $f_1(t)$,

$$f(t) = f_0(t) + f_1(t). \tag{3.6}$$

Table 1: Numerical solutions and absolute errors of Example 3.1 using the original HVIM.

t	$y(t) = t^2$	$y_3(t)$	$\lvert t^2 - y_3(t)\rvert$
0.1	0.01	0.1005487432e−1	0.5487432e−4
0.2	0.04	0.4063125562e−1	0.63125562e−3
0.3	0.09	0.9266557142e−1	0.266557142e−2
0.4	0.16	0.1674801219	0.74801219e−2
0.5	0.25	0.2667959208	0.167959208e−1
0.6	0.36	0.3927730722	0.327730722e−1
0.7	0.49	0.5480653520	0.580653520e−1
0.8	0.64	0.7358850823	0.958850823e−1
0.9	0.81	0.9600768448	0.1500768448
1.0	1.00	1.2251995000	0.2251995000

According to this assumption, $f(t) = f_0(t) + f_1(t)$, we construct the following iteration formula:

$$
\begin{aligned}
y_1(t) &= y_0(t) + \int_0^t (s-t)\left\{y_0^{(2)}(s) + y_0^{(3/2)}(s) + y_0(s) - f_0(s)\right\} ds, \\
y_2(t) &= y_1(t) + \int_0^t (s-t)\left\{y_1^{(2)}(s) + y_1^{(3/2)}(s) + y_1(s) - \left(f_0(s) + f_1(s)\right)\right\} ds, \\
&\ \vdots \\
y_{n+1}(t) &= y_n(t) + \int_0^t (s-t)\left\{y_n^{(2)}(s) + y_n^{(3/2)}(s) + y_n(s) - \left(f_0(s) + f_1(s)\right)\right\} ds, \quad n \geq 1.
\end{aligned}
\tag{3.7}
$$

Consequently, begin with the initial approximation $y_0(t) = 0$, in view of the iteration formulas (3.7) and taking $f_0(t) = 2$ and $f_1(t) = 4\sqrt{t/\pi} + t^2$, we can obtain the following approximations:

$$
\begin{aligned}
y_0(t) &= 0, \\
y_1(t) &= t^2, \\
y_n(t) &= t^2, \quad n \geq 1.
\end{aligned}
\tag{3.8}
$$

We, therefore, obtain

$$
y(t) = t^2. \tag{3.9}
$$

This is the exact solution. We note that the success of obtaining the exact solution is a result of the proper selection of $f_0(t)$ and $f_1(t)$.

Example 3.2. Consider the following initial value problem [27]

$$\left[D^2 - 2D + D^{1/2} + D^0\right] y(t) = 6t - 6t^2 + \frac{16}{5\sqrt{\pi}} t^{5/2} + t^3, \qquad y(0) = y'(0) = 0. \tag{3.10}$$

In view of the correction functional (2.2), the Lagrange multiplier can be identified as

$$\lambda = s - t. \tag{3.11}$$

As a result, we obtain the following iteration formula:

$$y_{n+1}(t) = y_n(t) + \int_0^t (s-t)\left\{ y_n^{(2)}(s) - y_n^{(1)} + y_n^{(1/2)}(s) + y_n(s) - 6s + 6s^2 - \frac{16}{5\sqrt{\pi}} s^{5/2} - s^3 \right\} ds. \tag{3.12}$$

We start with the initial approximation $y_0(t) = 0$. The comparison between the exact solution and the approximation solution$y_3(x)$ of (3.12) can be seen in Table 2.

Now, we choose

$$\begin{aligned} f_0(s) &= 6s, \\ f_1(s) &= -6s^2 + \frac{16}{5\sqrt{\pi}} S^{5/2} - s^3. \end{aligned} \tag{3.13}$$

Under the above assumption, we construct the following iteration formulas:

$$\begin{aligned} y_1(t) &= y_0(t) + \int_0^t (s-t)\left\{ y_0^{(2)}(s) - y_0^{(1)} + y_0^{(1/2)}(s) + y_0(s) - 6s \right\} ds, \\ y_2(t) &= y_1(t) + \int_0^t (s-t)\left\{ y_1^{(2)}(s) - y_1^{(1)} + y_1^{(1/2)}(s) + y_1(s) + 6s^2 - \frac{16}{5\sqrt{\pi}} s^{5/2} - s^3 \right\} ds, \\ &\vdots \\ y_{n+1}(t) &= y_n(t) + \int_0^t (s-t)\left\{ y_n^{(2)}(s) - y_n^{(1)} + y_n^{(1/2)}(s) + y_n(s) + 6s^2 - \frac{16}{5\sqrt{\pi}} s^{5/2} - s^3 \right\} ds. \end{aligned} \tag{3.14}$$

With starting the initial approximation $u_0(t) = 0$ and in view of the iteration formulas (3.14), therefore, obtain the following approximations:

$$\begin{aligned} y_0(t) &= 0, \\ y_1(t) &= t^3, \\ y_n(t) &= t^3, \quad n \geq 1. \end{aligned} \tag{3.15}$$

Table 2: Numerical solutions and absolute errors of Example 3.2 using the original HVIM.

t	$y(t)=t^3$	$y_3(t)$	$\lvert t^3 - y_3(t)\rvert$
0.1	0.001	0.9745626669e−3	0.254373331e−4
0.2	0.008	0.7586906052e−2	0.413093948e−3
0.3	0.027	0.2488146380e−1	0.211853620e−2
0.4	0.064	0.5722919465e−1	0.677080535e−2
0.5	0.125	0.1083125979	0.166874021e−1
0.6	0.216	0.1811260017	0.348739983e−1
0.7	0.343	0.2779910635	0.650089365e−1
0.8	0.512	0.4005870763	0.1114129237
0.9	0.729	0.5499953629	0.1790046371
1.0	1.000	0.7267567745	0.2732432255

Table 3: Numerical solution of $y(t)$ in Examples 3.1 and 3.2.

t_n ($T=5$)	N ($h=T/N$)	Absolute error of Example 3.1	Absolute error of Example 3.2
0.1	50	0.3×10^{-10}	0.3×10^{-11}
	100	0.2×10^{-10}	0.18×10^{-10}
1	50	0.37×10^{-7}	0
	100	0.44×10^{-7}	0.534×10^{-9}
2.5	50	0.66×10^{-6}	0.44×10^{-6}
	100	0.25×10^{-5}	0.25×10^{-6}
4	50	0.5×10^{-7}	0.54×10^{-5}
	100	0.15×10^{-6}	0.88×10^{-5}
5	50	0.18×10^{-5}	0.15×10^{-4}
	100	0.22×10^{-4}	0.39×10^{-4}

Thus we obtain

$$y(t)=t^3, \tag{3.16}$$

which is the exact solution.

4. Comparison with Collocation Spline Method

Rawashdeh in [27] applied the collocation spline method based on Lagrange interpolation for solving semidifferential equations of order 4. It is clear that the main disadvantage of the collocation spline method is its complex and difficult procedure. In order to make solution procedure easier and more effective, in this paper, we applied He's VIM to overcome the demerit. Also Rawashdeh in [27] reported the computed absolute error (error between exact and approximate value) with $N = 50, 100$ (N is the division number of the given interval) for Examples 3.1 and 3.2, see Table 3. In the studies by Rawashdeh, much time was spent and boring operations were done by collocation spline method based on Lagrange interpolation to get approximate solutions. In our study, however, the exact solutions are computed easily using this technique. Generally speaking, He's VIM is reliable and more efficient as compared with collocation spline method based on Lagrange interpolation.

5. Conclusion

In this paper, we have applied He's variational iteration method to various semidifferential equations of nth order. The obtained solution shows that He's method is a very convenient and effective for various semidifferential equations of nth order, only one iteration leads to exact solutions.

Acknowledgment

The authors would like to thank the referees for their valuable suggestions and recommendations, which greatly improved the paper.

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