

## Research Article

# Global Optimization for Sum of Linear Ratios Problem Using New Pruning Technique

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A global optimization algorithm is proposed for solving sum of general linear ratios problem (P) using new pruning technique. Firstly, an equivalent problem (P1) of the (P) is derived by exploiting the characteristics of linear constraints. Then, by utilizing linearization method the relaxation linear programming (RLP) of the (P1) can be constructed and the proposed algorithm is convergent to the global minimum of the (P) through the successive refinement of the linear relaxation of feasible region and solutions of a series of (RLP). Then, a new pruning technique is proposed, this technique offers a possibility to cut away a large part of the current investigated feasible region by the optimization algorithm, which can be utilized as an accelerating device for global optimization of problem (P). Finally, the numerical experiments are given to illustrate the feasibility of the proposed algorithm.

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## 1. Introduction

Consider the following sum of general linear ratios problem:

$$(P) : \begin{cases} \min & h(x) = \sum_{j=1}^p \frac{c_{j0} + c_j^T x}{d_{j0} + d_j^T x}, \\ \text{s.t.} & Ax \leq b, \quad x \in R^N, \end{cases} \quad (1.1)$$

where  $A \in R^{M \times N}$ ,  $b \in R^M$ ,  $p$  is a natural number,  $c_{j0} + c_j^T x$  and  $d_{j0} + d_j^T x$  are all finite affine functions on  $R^N$  such that  $d_{j0} + d_j^T x \neq 0$  for all  $x \in X = \{x \mid Ax \leq b\}$ ,  $j = 1, \dots, p$ .

Problem (P) has attracted the interest of researchers for many years. This is because problem (P) has a number of important applications, including multistage shipping problems, cluster analysis, and multiobjective bond portfolio [1, 2]. However, some computational difficulties can be encountered, since multiple local optima of problem (P) that are not globally optimal exist.

In the last decades, many solution algorithms have been proposed for globally solving special cases of the (P), which are intended only for the sum of positive linear ratios problem with assumption that  $c_{j0} + c_j^T x \geq 0$  and  $d_{j0} + d_j^T x > 0$  for all  $x \in X = \{x \mid Ax \leq b\}$  [2, 3]. In [4], Kuno proposed a new method for solving the maximization of sum of linear ratios, which used a concave function to overestimate the optimal value of the original problem. A global optimization method was considered by Jiao et al. [5] by introducing parameters, then the global optimal solution can be derived using linear relaxation and branch and bound algorithm. Recently, in [6], Ji et al. presented a deterministic global optimization algorithm for the linear sum-of-ratios problem, and Jiao and Chen [7] give a short extensive application for the algorithm proposed in [6]. Though optimization methods for special forms of the (P) are ubiquitous, to our knowledge, little work has been done in the literature for globally solving the sum of general linear ratios problem (P) which the numerators and denominators of the ratios may be arbitrary value except that the denominators of the ratios are nonzero over the feasible region considered in this paper.

The purpose of this paper is to develop a deterministic algorithm for solving sum of general linear ratios problem (P) which the numerators and denominators of the ratios may be arbitrary value except that the denominators of the ratios are nonzero over the feasible region. The main feature of the algorithm is described as follows. (1) An equivalent optimization problem (P1) of the (P) is derived by exploiting the characteristics of this linear constraints. (2) A new linearization method is proposed to linearize the objective function of the (P1), and the linear relaxation of the (P1) is easier to be obtained and need not introduce new variables and constraints compared with the method in [5, 8]. (3) A new pruning technique is given, and this technique offers the possibility to cut away a large part of the current investigated feasible region. Using the new technique as an accelerating device and applying it to the proposed algorithm, we can largely reduce current investigated feasible region to improve the convergence of the algorithm. (4) The proposed algorithm is convergent to the global minimum through the successive refinement of the linear relaxation of feasible region of the objective function and solutions of a series of (RLP). Finally, the numerical results show the feasibility and effectiveness of the proposed algorithm.

The organization of this article is as follows. In Section 2, we show how to convert the (P) into equivalent problem (P1), and generate the relaxed linear programming (RLP) of the (P1). In Section 3, the proposed branch-and-bound algorithm in which the relaxed subproblems are embedded is described, and its convergence is shown. Some numerical results are reported in Section 4 and Section 5 provides some concluding remarks.

## 2. Linear relaxation programming

In this section, first we convert the (P) into an equivalent nonconvex programming problem (P1). In order to globally solve the (P), the branch and bound algorithm to be presented can be applied to the (P1).

Firstly, we solve the following  $2N + p$  linear programming problems:

$$\begin{aligned} \underline{x}_i &= \begin{cases} \min & x_i, \\ \text{s.t.} & Ax \leq b, \end{cases} \\ \bar{x}_i &= \begin{cases} \min & x_i, \\ \text{s.t.} & Ax \leq b, \end{cases} \\ r_j &= \begin{cases} \min & c_{j0} + c_j^T x, \\ \text{s.t.} & Ax \leq b, \end{cases} \end{aligned} \quad (2.1)$$

where  $i = 1, 2, \dots, N$ ,  $j = 1, \dots, p$ . Then, we can get initial partition rectangle  $X^0 = \{x \mid \underline{x}_i \leq x_i \leq \bar{x}_i, i = 1, 2, \dots, N\}$ . Without loss of generality, assume  $r_j \geq 0$  ( $j = 1, \dots, q$ ),  $r_j < 0$  ( $j = q + 1, \dots, p$ ), the problem (P) can be rewritten in the following form:

$$(P1) : \begin{cases} \min & h(x) = \sum_{j=1}^q \frac{c_{j0} + c_j^T x}{d_{j0} + d_j^T x} + \sum_{j=q+1}^p \frac{c_{j0} + c_j^T x - r_j}{d_{j0} + d_j^T x} - \sum_{j=q+1}^p \frac{-r_j}{d_{j0} + d_j^T x}, \\ \text{s.t.} & Ax \leq b, \quad x \in X^0. \end{cases} \quad (2.2)$$

Obviously, we have  $c_{j0} + c_j^T x \geq 0$  for each  $j \in \{1, \dots, q\}$ , and  $c_{j0} + c_j^T x - r_j \geq 0$  and  $-r_j > 0$  for each  $j \in \{q + 1, \dots, p\}$ .

**Theorem 2.1.** *Problems (P) and the (P1) have the same global optimal solution.*

*Proof.* Obviously, if  $x$  is feasible to the (P), then  $x \in X^0$ . Conversely, if  $x$  is feasible to the (P1), then  $Ax \leq b$ . So they have the same feasible region, then conclusion is followed.

The linear relaxation of the (P1) can be realized by underestimating function  $h(x)$  with a linear function  $h^l(x)$ . All the details of this linearization technique for generating relaxations will be given in the following theorems.

Given any  $X = [\underline{x}, \bar{x}] \subseteq X^0$  and for all  $x = (x_i)_{n \times 1} \in X$ , the following notations are introduced:

$$h^l(x) = \sum_{j=1}^q \frac{c_{j0} + c_j^T x}{d_j^u} + \sum_{j=q+1}^p \frac{c_{j0} + c_j^T x - r_j}{d_j^u} - \sum_{j=q+1}^p \frac{-r_j}{d_j^l}, \quad (2.3)$$

$$h^u(x) = \sum_{j=1}^q \frac{c_{j0} + c_j^T x}{d_j^l} + \sum_{j=q+1}^p \frac{c_{j0} + c_j^T x - r_j}{d_j^l} - \sum_{j=q+1}^p \frac{-r_j}{d_j^u}, \quad (2.4)$$

$$d_j^l = \min_{x \in X} (d_{j0} + d_j^T x), \quad d_j^u = \max_{x \in X} (d_{j0} + d_j^T x), \quad j = 1, \dots, p. \quad (2.5)$$

□

**Theorem 2.2.** *Let  $d_j(x) \neq 0$  for any  $x \in X$  and let  $d_j^l$  and  $d_j^u$  defined in (2.5). Then,  $1/d_j^u \leq 1/d_j(x) \leq 1/d_j^l$  ( $j = 1, \dots, p$ ), for all  $x \in X$ .*

*Proof.* Obviously, by the continuity of the function  $d_j(x)$ , there must exist some  $x^1 \in X$  and  $x^2 \in X$ , such that  $d_j(x^1) = d_j^l$  and  $d_j(x^2) = d_j^u$ . If  $d_j^u d_j^l < 0$ , then  $d_j^l < 0$  and  $d_j^u > 0$ , that is,  $d_j(x^1) < 0$  and  $d_j(x^2) > 0$ . Then, there must exist one positive number  $\sigma \in (0, 1)$  such that  $d_j(x^1 + \sigma(x^2 - x^1)) = 0$ , where  $x^1 + \sigma(x^2 - x^1) \in X$ , which contradicts  $d_j(x) \neq 0$  for any  $x \in X$ . Therefore, we have  $d_j^u d_j^l > 0$ , that is,  $d_j^u \geq d_j^l > 0$  or  $d_j^l \leq d_j^u < 0$ . Obviously, for  $\forall x \in X$ , the following two conclusions hold.

- (i) If  $d_j^u \geq d_j^l > 0$ , then  $d_j^u \geq d_j(x) \geq d_j^l > 0$ , we have  $0 < 1/d_j^u \leq 1/d_j(x) \leq 1/d_j^l$ .
- (ii) If  $d_j^l \leq d_j^u < 0$ , then  $d_j^l \leq d_j(x) \leq d_j^u < 0$ , we have  $1/d_j^u \leq 1/d_j(x) \leq 1/d_j^l < 0$ .

By (i) and (ii), for for all  $x \in X$ , we have  $1/d_j^u \leq 1/d_j(x) \leq 1/d_j^l$ . □

**Theorem 2.3.** For any  $x \in X$ , consider the functions  $h(x)$ ,  $h^l(x)$ , and  $h^u(x)$  defined in (2.3) and (2.4). Then, the following two statements hold.

- (i) The functions  $h(x)$ ,  $h^l(x)$ , and  $h^u(x)$  satisfy  $h^l(x) \leq h(x) \leq h^u(x)$ .
- (ii) The maximal errors of bounding  $h(x)$  using  $h^l(x)$  and  $h^u(x)$  satisfy

$$\lim_{\|\bar{x}-x\| \rightarrow 0} \text{Error}_{\max}^L = \lim_{\|\bar{x}-x\| \rightarrow 0} \text{Error}_{\max}^U = 0, \quad (2.6)$$

where

$$\text{Error}_{\max}^L = \max_{x \in X} (h(x) - h^l(x)), \quad \text{Error}_{\max}^U = \max_{x \in X} (h^u(x) - h(x)). \quad (2.7)$$

*Proof.* The proof the theorem can be found in [7]. □

For convenience in exposition, in the following we assume that  $X^k = [\underline{x}^k, \bar{x}^k]$  represents either the initial bounds on the variables of the problem (P1), or modified bounds as defined for some partitioned subproblem in a branch-and-bound scheme. By means of Theorem 2.3, we can give the linear relaxation of the (P1). Let  $X^k = [\underline{x}^k, \bar{x}^k] \subseteq X^0$ , consequently we construct the corresponding approximation relaxation linear programming (RLP) of (P1) in  $X^k$  as follows:

$$\text{RLP}(X^k) : \begin{cases} \min & h^l(x) = \sum_{j=1}^q \frac{c_{j0} + c_j^T x}{d_j^u} + \sum_{j=q+1}^p \frac{c_{j0} + c_j^T x - r_j}{d_j^u} - \sum_{j=q+1}^p \frac{-r_j}{d_j^l}, \\ \text{s.t.} & Ax \leq b, \quad x \in X^k. \end{cases} \quad (2.8)$$

Based on the above linear underestimators, every feasible point of (P1) in subdomain  $X^k$  is feasible in (RLP); and the value of the objective function for (RLP) is less than or equal to that of (P1) for all points in  $X^k$ . Thus, (RLP) provides a valid lower bound for the solution of (P1) over the partition set  $X^k$ . It should be noted that problem (RLP) contains only the necessary constraints to guarantee convergence of the algorithm.

### 3. New pruning technique

In this section, we pay more attention to how to form the new pruning technique to delete or reduce a large part of regions in which there exists no global optimal solution so that we can accelerate the convergence of the proposed algorithm. Let  $X = (X_i)_{N \times 1}$  with  $X_i = [\underline{x}_i, \bar{x}_i]$  be a subrectangle of  $X^0$ , that is,  $X \subseteq X^0$ . Moreover, assume that  $\bar{h}$  are currently known upper bound of optimal objective value  $h^*$  of problem (P1). For convenience of the following discussions, we introduce some notations as follows:

$$\begin{aligned}
E &= \sum_{j=1}^p \frac{c_{j0}}{d_j^u} + \sum_{j=q+1}^p \left( \frac{r_j}{d_j^l} - \frac{r_j}{d_j^u} \right), \\
\beta_i &= \sum_{j=1}^p \frac{c_{ji}}{d_j^u}, \quad i = 1, 2, \dots, N, \\
S &= \sum_{i=1}^N \min \{ \beta_i \underline{x}_i, \beta_i \bar{x}_i \}, \\
L_t &= \sum_{i=1}^N \min \{ a_{ti} \underline{x}_i, a_{ti} \bar{x}_i \}, \quad t = 1, 2, \dots, M, \\
s_m &= \min_{x \in X} \sum_{i=1, i \neq m}^N \beta_i x_i, \\
z_m &= \frac{\bar{h} - s_m - E}{\beta_m}, \\
z_{tm}^1 &= \frac{b_t - L_t + a_{tm} \underline{x}_m}{a_{tm}}, \\
z_{tm}^2 &= \frac{b_t - L_t + a_{tm} \bar{x}_m}{a_{tm}}.
\end{aligned} \tag{3.1}$$

**Theorem 3.1.** Consider the subrectangle  $X = (X_i)_{N \times 1} \subseteq X^0$ , we have the following conclusions.

- (i) If  $S + E > UB$ , then  $\min_{x \in X} h(x) > h^*$ .
- (ii) If  $S + E \leq UB$ , then if there exists some index  $m$  ( $m \in \{1, 2, \dots, N\}$ ) such that  $\beta_m > 0$  and  $z_m < \bar{x}_m$ , then  $\min_{x \in \hat{X}^1} h(x) > h^*$ ; if there exists some index  $m$  ( $m \in \{1, 2, \dots, N\}$ ) such that  $\beta_m < 0$  and  $z_m > \underline{x}_m$ , then  $\min_{x \in \hat{X}^2} h(x) > h^*$ , where

$$\begin{aligned}
\hat{X}^1 &= (\hat{X}_i^1)_{N \times 1} \subseteq X, & \hat{X}^2 &= (\hat{X}_i^2)_{N \times 1} \subseteq X, \\
\hat{X}_i^1 &= \begin{cases} X_i, & \text{if } i \neq m, \quad i = 1, 2, \dots, N, \\ (z_m, \bar{x}_m] \cap X_m, & \text{if } i = m, \end{cases} \\
\hat{X}_i^2 &= \begin{cases} X_i, & \text{if } i \neq m, \quad i = 1, 2, \dots, N, \\ [\underline{x}_m, z_m) \cap X_m, & \text{if } i = m. \end{cases}
\end{aligned} \tag{3.2}$$

*Proof.* (i) By assumption and definitions of  $\beta_i, S, E$  in (3.1), if  $S + E > \text{UB}$ , then  $h^l(x) \geq S + E > \text{UB} \geq h^*$  for any  $x \in X$ , that is,  $\min_{x \in X} h(x) > h^*$  for any  $x \in X$ .

(ii) By assumption and definitions of  $\beta_i, S, E$  in (3.1), if  $S + E \leq \text{UB}$ , then we have the following two conclusions:

- (a) If there exists some index  $m$  ( $m \in \{1, 2, \dots, N\}$ ) such that  $\beta_m > 0$  and  $z_m < \bar{x}_m$ , then for any  $x = (x_i)_{N \times 1} \in \hat{X}^1$ , let  $\beta_m > 0$  and  $z_m < \bar{x}_m$  for some  $m$ , we first show that  $h(x) \geq \bar{h}$ . By assumption and definitions of  $\beta_m, s_m$ , and  $z_m$  in (3.1), we have  $x_m > z_m$ , that is,  $x_m > (\bar{h} - s_m - E)/\beta_m$ . Furthermore, from the definitions of  $h^l(x)$ , we get

$$\begin{aligned}
 \bar{h} &< \beta_m x_m + s_m + E \\
 &= \beta_m x_m + \min_{x \in \hat{X}^1} \sum_{i=1, i \neq m}^N \beta_i x_i + E \\
 &\leq \sum_{i=1}^N \beta_i x_i + E \\
 &= \sum_{i=1}^N \sum_{j=1}^p \frac{c_{ji}}{d_j^u} x_i + \sum_{j=1}^p \frac{c_{j0}}{d_j^u} + \sum_{j=q+1}^p \left( \frac{r_j}{d_j^l} - \frac{r_j}{d_j^u} \right) \\
 &= \sum_{j=1}^p \sum_{i=1}^N \frac{c_{ji}}{d_j^u} x_i + \sum_{j=1}^p \frac{c_{j0}}{d_j^u} + \sum_{j=q+1}^p \left( \frac{r_j}{d_j^l} - \frac{r_j}{d_j^u} \right) \\
 &= h^l(x).
 \end{aligned} \tag{3.3}$$

By the above discussion and assumption, it follows that  $h^* \leq \bar{h} < h^l(x) \leq h(x)$  for any  $x \in \hat{X}^1$ . Therefore,  $\min_{x \in \hat{X}^1} h(x) > \bar{h} \geq h^*$ , that is, there exists no global optimal solution in  $\hat{X}^1$ . This proof of part (a) is completed.

- (b) If there exists some index  $m$  ( $m \in \{1, 2, \dots, N\}$ ) such that  $\beta_m < 0$  and  $z_m > \underline{x}_m$ , then  $\min_{x \in \hat{X}^2} h(x) > h^*$ , since the proof of the part (b) is similar to the part (a) of the theorem, it is omitted here.  $\square$

**Theorem 3.2.** Consider the subrectangle  $X = (X_i)_{N \times 1} \subseteq X^0$ , we have the following conclusions.

- (i) If there exists some  $t$  ( $t \in \{1, \dots, M\}$ ) such that  $L_t > b_t$  on the subrectangle  $X$ , then there exists no global optimization solution in  $X$ .
- (ii) If  $L_t \leq b_t$  ( $t \in \{1, \dots, M\}$ ) on the subrectangle  $X$ , then if there exists some index  $m$  ( $m \in \{1, 2, \dots, N\}$ ) such that  $a_{tm} > 0$  and  $z_{tm}^1 < \bar{x}_m$ , then there exists no global optimization solution in  $\hat{X}^3$ ; if there exists some index  $m$  ( $m \in \{1, 2, \dots, N\}$ ) such that  $a_{tm} < 0$  and  $z_{tm}^2 > \underline{x}_m$ , then there exists no global optimization solution in  $\hat{X}^4$ , where

$$\begin{aligned}
\widehat{X}^3 &= (\widehat{X}_i^3)_{N \times 1} \subseteq X, & \widehat{X}^4 &= (\widehat{X}_i^4)_{N \times 1} \subseteq X, \\
\widehat{X}_i^3 &= \begin{cases} X_i, & \text{if } i \neq m, i = 1, 2, \dots, N, \\ (z_{im}^1, \bar{x}_m] \cap X_m, & \text{if } i = m, \end{cases} \\
\widehat{X}_i^4 &= \begin{cases} X_i, & \text{if } i \neq m, i = 1, 2, \dots, N, \\ [\underline{x}_m, z_{im}^2) \cap X_m, & \text{if } i = m. \end{cases}
\end{aligned} \tag{3.4}$$

*Proof.* Since the proof of Theorem 3.2 is similar to that of Theorem 3.1, it is omitted here.  $\square$

Based on Theorems 3.1 and 3.2, we can give new pruning technique to cut away or reduce a large part of region in which there exists no optimal solution.

Next, we will show how this new pruning technique is formed, that is, we provide a process that show how a subrectangle  $X \subseteq X^0$  can be deleted or reduced, where  $X = (X_i)_{N \times 1}$  with  $X_i = [\underline{x}_i, \bar{x}_i]$ . Let  $Y_m$  denote the discarded interval in  $X_m$ . First, we calculate  $E, \beta_m, S, s_m, z_m, L_t, z_{im}^1, z_{im}^2$  according to (3.1). Then, the eliminated interval  $Y_m$  can be determined according to the following rules, which is called new pruning technique.

*Rule 1*

If  $S + E > UB$ , then

$$Y_m = [\underline{x}_m, \bar{x}_m], \quad m = 1, \dots, N. \tag{3.5}$$

*Rule 2*

If  $S + E \leq UB$ , then

(a) If  $\beta_m > 0$  and  $z_m < \bar{x}_m$ , then

$$Y_m = \begin{cases} (z_m, \bar{x}_m], & \text{if } \underline{x}_m \leq z_m, \\ [\underline{x}_m, \bar{x}_m], & \text{if } z_m \leq \underline{x}_m, \end{cases} \tag{3.6}$$

(b) If  $\beta_m < 0$  and  $z_m > \underline{x}_m$ , then

$$Y_m = \begin{cases} [\underline{x}_m, z_m), & \text{if } z_m \leq \bar{x}_m, \\ [\underline{x}_m, \bar{x}_m], & \text{if } z_m \geq \bar{x}_m. \end{cases} \tag{3.7}$$

*Rule 3*

If  $L_t > b_t$  for some  $t$  ( $t \in \{1, \dots, M\}$ ), then

$$Y_m = [\underline{x}_m, \bar{x}_m], \quad m = 1, \dots, N. \tag{3.8}$$

Rule 4

If  $L_t \leq b_t$  for some  $t$  ( $t \in \{1, \dots, M\}$ ), then

(c) If  $a_{tm} > 0$  and  $z_{tm}^1 < \bar{x}_m$ , then

$$Y_m = \begin{cases} [\underline{x}_m, \bar{x}_m], & \text{if } z_{tm}^1 < \underline{x}_m, \\ (z_{tm}^1, \bar{x}_m], & \text{if } z_{tm}^1 \geq \underline{x}_m, \end{cases} \quad (3.9)$$

(d) If  $a_{tm} < 0$  and  $z_{tm}^2 > \bar{x}_m$ , then

$$Y_m = \begin{cases} [\underline{x}_m, \bar{x}_m], & \text{if } z_{tm}^2 > \bar{x}_m, \\ [\underline{x}_m, z_{tm}^2), & \text{if } z_{tm}^2 \leq \bar{x}_m. \end{cases} \quad (3.10)$$

Consequently, under some assumption the discarded part of  $X_m$  is  $X_m \cap Y_m$  for some  $m$  ( $m \in \{1, 2, \dots, N\}$ ), the new rectangle reducing tactics provides the possibility to cut away or reduce a large part of the rectangle that is currently investigated by the procedure. The rest part of  $X$  denoted by  $\tilde{X} = (\tilde{X}_i)_{N \times 1}$  is left for further considering, where  $\tilde{X}_i = X_i$  if  $i \neq m$  ( $i = 1, 2, \dots, N$ );  $\tilde{X}_m = X_m \setminus Y_m$  if  $i = m$ .

#### 4. Algorithm and its convergence

In this section, by connecting the former branch-and-bound algorithm with new pruning technique a global optimization algorithm is proposed for solving problem (P1). This algorithm needs to solve a sequence of relaxation linear programming over partitioned subsets of  $X^0$  in order to find a global optimum solution.

The branch and bound approach is based on partitioning the set  $X^0$  into sub-hyperrectangles, each concerned with a node of the branch and bound tree, and each node is associated with a relaxation linear subproblem in each sub-hyperrectangle. Hence, at any stage  $k$  of the algorithm, suppose that we have a collection of active nodes denoted by  $Q_k$ , say, each associated with a hyperrectangle  $X \subseteq X^0$ , for all  $X \in Q_k$ . For each such node  $X$ , we will have computed a lower bound of the optimal value of (P1) via the solution  $LB(X)$  of the RLP, so that the lower bound of optimal value of (P1) on the whole initial box region  $X^0$  at stage  $k$  is given by  $LB_k = \min\{LB(X), \text{for all } X \in Q_k\}$ . Whenever the solution of the relaxation linear programming (RLP) turns out to be feasible to the problem (P1), we update the upper bound of incumbent solution  $UB$  if necessary. Then, the active nodes collection  $Q_k$  will satisfy  $LB(X) < UB$ , for all  $X \in Q_k$ , for each stage  $k$ . We now select an active node to partition its associated hyperrectangle into two sub-hyperrectangles as described below, computing the lower bounds for each new node as before. Upon fathoming any nonimproving nodes,



we obtain a collection of active nodes for the next stage, and this process is repeated until convergence is obtained.

The critical element in guaranteeing convergence to a global minimum is the choice of a suitable partitioning strategy. In our paper, we choose a simple and standard bisection rule. This method is sufficient to ensure convergence since it drives all the intervals to zero for all variables. This branching rule is given as follows.

Assume that the sub-hyperrectangle  $X' = [\underline{x}', \bar{x}'] \subseteq X^0$  is going to be divided. Then, we select the branching variable  $x'_\gamma$ , satisfying  $\gamma = \arg \max\{\bar{x}'_i - \underline{x}'_i : i = 1, \dots, N\}$  and partition  $X'$  by bisection the interval  $[\underline{x}'_\gamma, \bar{x}'_\gamma]$  into the subintervals  $[\underline{x}'_\gamma, (\underline{x}'_\gamma + \bar{x}'_\gamma)/2]$  and  $[(\underline{x}'_\gamma + \bar{x}'_\gamma)/2, \bar{x}'_\gamma]$ .

The basic steps of the proposed algorithm are summarized as follows. Let  $LB(X^k)$  refer to the optimal objective function value of P1 for the sub-hyperrectangles  $X^k$  and  $x^k = x(X^k)$  refer to an element of corresponding argmin.

#### Algorithm statement

*Step 1* (initialization). Initialize the iteration counter  $k := 0$ , the set of all active node  $Q_0 = \{X^0\}$ , the upper bound  $UB = \infty$ , and the set of feasible points  $F := \emptyset$ . Solve the problem (RLP) for  $X = X^0$ , obtaining  $LB_0 := LB(X)$  and  $x^0 := x(X)$ . If  $x^0$  is feasible to (P1) update  $F$  and  $UB$ , if necessary. If  $UB \leq LB_0 + \epsilon$ , where  $\epsilon > 0$  is some accuracy tolerance, then stop with  $x^0$  as the prescribed solution to problem (P1). Otherwise, proceed to Step 2.

*Step 2* (midpoint check). Select the midpoint  $x^m$  of  $X^k$ , if  $x^m$  is feasible to the (P1), then  $F := F \cup \{x^m\}$ . Define the upper bound  $UB := \min_{x \in F} h(x)$ . If  $F \neq \emptyset$ , the best known feasible point is denoted  $b := \operatorname{argmin}_{x \in F} h(x)$ .

*Step 3* (branching). Choose a branching variable  $x_\gamma$  to partition  $X^k$  to get two new sub-hyperrectangles according to the above selected branching rule. Call the set of new partition rectangles as  $\bar{X}^k$ .

*Step 4* (pruning). (1) If  $\bar{X}^k \neq \emptyset$ , then calculate  $E, \beta_i, S, L_t, s_m, z_m, z_{tm}^1$ , and  $z_{tm}^2$  for each  $X \in \bar{X}^k$ . If one of assumption conditions in rules 1–4 is satisfied, then the proposed new pruning technique can be applied into each  $X \in \bar{X}^k$ , the rest parts of  $X$  and  $\bar{X}^k$  are denoted by  $\tilde{X}$  and  $\tilde{X}^k$ , respectively.

(2) If  $\tilde{X}^k \neq \emptyset$ , solve RLP to obtain  $LB(\tilde{X})$  and  $x(\tilde{X})$ , for each  $\tilde{X} \in \tilde{X}^k$ . If  $LB(\tilde{X}) > UB$ , set  $\tilde{X}^k := \tilde{X}^k \setminus \tilde{X}$ , otherwise, update the best available solution  $UB, F$  and  $b$  if possible, as in Step 2.

*Step 5* (updating lower bound). The partition set remaining is now  $Q_k := (Q_k \setminus X^k) \cup \tilde{X}^k$  giving a new lower bound  $LB_k := \inf_{X \in Q_k} LB(X)$ .

*Step 6* (convergence check). Fathom any nonimproving nodes by setting  $Q_{k+1} = Q_k \setminus \{X : UB - LB(X) \leq \epsilon, X \in Q_k\}$ . If  $Q_{k+1} = \emptyset$ , then stop with  $UB$  is the solution of (P1), and  $b$  is an optimal solution. Otherwise,  $k := k+1$ , and select an active node  $X^k$  such that  $X^k = \operatorname{argmin}_{X \in Q_k} LB(X)$ ,  $x^k := x(X^k)$ , and return to Step 2.

### Convergence of the algorithm

Let  $X^a$  be the set of accumulation points of  $\{x^k\}$ , and let  $X^*$  be  $\arg \min_{x \in G} h(x)$ , where  $G \neq \emptyset$  is the feasible space of the (P1).

**Theorem 4.1.** *The above algorithm either terminates finitely with the incumbent solution being optimal to the (P1) or generates an infinite sequence of iteration such that along any infinite branch of the branch and bound tree, any accumulation point of the sequence  $\{LB_k\}$  will be global minimum of the (P1).*

*Proof.* If the above proposed algorithm terminates finitely at some iteration  $k$ , obviously, UB is global optimal value and  $x(X^k)$  is optimal solution for the (P1). If the algorithm is infinite, it generates at least one infinitely sequence  $\{X^k\}$  such that  $X^{k+1} \subset X^k$  for any  $k$ . In the case, since partition sets used by the proposed algorithm are all rectangular and compact, by Tuy [9], it follows that this rectangular subdivision is exhaustive. Hence, for every iteration,  $k = 0, 1, 2, \dots$ , by design of the algorithm, we have

$$LB_k \leq \min_{x \in G} h(x), \quad X^k \in \arg \min_{X \in Q_k} LB(X), \quad x^k = x(X^k) \in X^k \subseteq X^0. \quad (4.1)$$

Horst [10] gives that  $\{LB_k\}$  is a nondecreasing sequence bounded above by  $\min_{x \in G} h(x)$ , which guarantees the existence of the limit  $LB := \lim_{k \rightarrow \infty} LB_k \leq \min_{x \in G} h(x)$ .  $\{x^k\}$  is a sequence on a compact set, therefore, it has a convergent subsequence. For any  $\hat{x} \in X^a$ , suppose that there exists a subsequence  $\{x^r\}$  of  $\{x^k\}$  with  $\lim_{r \rightarrow \infty} x^r = \hat{x}$ . By the proposed algorithm and [9], it follows that the subdivision of partition sets in Step 3 is exhaustive on  $X^0$ , and the selection of elements to be partitioned in Step 3 is bound improving. Thus, there exists a decreasing subsequence  $\{X^q\} \subset X^r$ , where  $X^r \in Q_r$  with  $x^q \in X^q$ ,  $LB_q = LB(X^q) = h^l(x^q)$ ,  $\lim_{q \rightarrow \infty} x^q = \hat{x}$ . From the construction method of linear lower bound relaxation functions for objective function of the (P1), we know that the linear subfunctions  $h^l(x)$  used in (RLP) are strongly consistent on  $X^0$ . Thus, it follows that  $\lim_{q \rightarrow \infty} LB_q = LB = h(\hat{x})$ .  $\square$

## 5. Numerical experiments

To verify performance of the proposed algorithm, some common used test problems are implemented on Pentium IV (433 MHz) microcomputer. The algorithm is coded in C++ language and each linear programming is solved by simplex method, and the convergence tolerance  $\epsilon_c$  set to  $10^{-8}$  in our experiment. Below, we describe some of these sample problems and solution results are summarized in Table 1. In Table 1, the notations have been used for column headers. Iter: number of algorithm iteration; maxnode: the maximal number of active nodes necessary; time: execution time in seconds;  $\epsilon_f$ : feasibility tolerance.

*Example 5.1* (see [3]). We have

$$\begin{aligned} \min \quad & \frac{-x_1 + 2x_2 + 2}{3x_1 - 4x_2 + 5} + \frac{4x_1 - 3x_2 + 4}{-2x_1 + x_2 + 3} \\ \text{s.t.} \quad & x_1 + x_2 \leq 1.5, \end{aligned}$$

$$\begin{aligned} x_1 - x_2 &\leq 0, \\ 0 &\leq x_1, x_2 \leq 1. \end{aligned} \quad (5.1)$$

*Example 5.2* (see [2]). We have

$$\begin{aligned} \min \quad & -\frac{4x_1 + 3x_2 + 3x_3 + 50}{3x_2 + 3x_3 + 50} - \frac{3x_1 + 4x_3 + 50}{4x_1 + 4x_2 + 5x_3 + 50} \\ & - \frac{x_1 + 2x_2 + 5x_3 + 50}{x_1 + 5x_2 + 5x_3 + 50} - \frac{x_1 + 2x_2 + 4x_3 + 50}{5x_2 + 4x_3 + 50}, \\ \text{s.t.} \quad & 2x_1 + x_2 + 5x_3 \leq 10, \\ & x_1 + 6x_2 + 3x_3 \leq 10, \\ & 5x_1 + 9x_2 + 2x_3 \leq 10, \\ & 9x_1 + 7x_2 + 3x_3 \leq 10, \\ & x_1, x_2, x_3 \geq 0. \end{aligned} \quad (5.2)$$

If there exists sum of general linear ratios in constraint functions of problem (P1), by using the same linear relaxation method proposed in Section 2, we can construct the corresponding linear relaxation programming of problem (P1). Therefore, the above proposed algorithm can be extensively applied to solve the following sum of linear ratios problem with sum of linear ratios constraints.

*Example 5.3.* We have

$$\begin{aligned} \min \quad & \frac{2x_1 + x_2 + x_3 + 1}{x_1 + 2x_2 + x_3 + 2} + \frac{x_1 + 2x_2 + x_3 + 2}{2x_1 + 2x_2 + x_3 + 3} - \frac{x_1 + x_2 + 3x_3 + 5}{x_1 + 2x_2 + 3x_3 + 4} - \frac{1.5x_1 + x_2 + x_3 + 6}{x_1 + x_2 + 1.5x_3 + 5}, \\ \text{s.t.} \quad & \frac{2x_1 + x_2 + x_3 + 2}{2x_1 + x_2 + x_3 + 3} - \frac{2x_1 + 2x_2 + x_3 + 5}{x_1 + 2x_2 + x_3 + 4} - \frac{2x_1 + 3x_2 + x_3 + 6}{x_1 + 2x_2 + 2x_3 + 5} - \frac{1.5x_1 + x_2 + 2x_3 + 7}{1.5x_1 + 2x_2 + x_3 + 6} \\ & \leq -2.4, \\ & \frac{1.5x_1 + x_2 + x_3 + 3}{x_1 + 1.5x_2 + x_3 + 4} + \frac{2x_1 + x_2 + x_3 + 4}{x_1 + x_2 + 2x_3 + 5} + \frac{x_1 + 2x_2 + x_3 + 5}{x_1 + 2x_2 + x_3 + 6} + \frac{x_1 + x_2 + 3x_3 + 6}{x_1 + 3x_2 + x_3 + 7} \leq 3.8, \\ & \frac{x_1 + x_2 + x_3 + 4}{x_1 + x_2 + x_3 + 5} + \frac{x_1 + x_2 + x_3 + 5}{x_1 + x_2 + x_3 + 6} + \frac{x_1 + x_2 + 3x_3 + 6}{x_1 + x_2 + x_3 + 7} + \frac{x_1 + x_2 + x_3 + 7}{x_1 + x_2 + x_3 + 8} \leq 3.9, \\ & \frac{x_1 + x_2 + x_3 + 5}{x_1 + x_2 + x_3 + 6} + \frac{x_1 + x_2 + x_3 + 6}{x_1 + x_2 + x_3 + 7} - \frac{x_1 + x_2 + x_3 + 9}{x_1 + x_2 + x_3 + 8} - \frac{x_1 + x_2 + x_3 + 10}{x_1 + x_2 + x_3 + 9} \leq 0.1, \\ & 1.0 \leq x_1 \leq 3.0, \quad 1.0 \leq x_2 \leq 3.0, \quad 1.0 \leq x_3 \leq 3.0. \end{aligned} \quad (5.3)$$

**Table 1:** Computational results of test examples.

Example	optimal solution			optimal value	Iter	Maxnode	Time	$\epsilon_c$
	$x_1$	$x_2$	$x_3$					
1. [3]	0.0	0.282165		1.623188	36	16	<1	$10^{-6}$
This paper	0.0	0.283243180		1.623183349	11	5	0	$10^{-8}$
2. [2]	1	0	0	-4.081481	38	3	2	$10^{-6}$
This paper	0.999999991	0.000000056	0.000000237	-4.081481483	22	11	<1	$10^{-8}$
3.	1.0	1.0	1.0	-0.534313724	35538	2655	6	$10^{-8}$
4.	1.0	1.0	1.0	-0.63452382	9056	992	<3	$10^{-8}$

*Example 5.4.* We have

$$\begin{aligned}
\min \quad & \frac{x_1 + x_2 + x_3 + 1}{x_1 + x_2 + x_3 + 2} + \frac{x_1 + x_2 + x_3 + 2}{x_1 + x_2 + x_3 + 3} - \frac{x_1 + x_2 + x_3 + 5}{x_1 + 2x_2 + 2x_3 + 4} - \frac{x_1 + x_2 + x_3 + 6}{x_1 + x_2 + x_3 + 5}' \\
\text{s.t.} \quad & \frac{x_1 + x_2 + x_3 + 2}{x_1 + x_2 + x_3 + 3} - \frac{x_1 + x_2 + x_3 + 5}{x_1 + x_2 + x_3 + 4} - \frac{x_1 + x_2 + x_3 + 6}{x_1 + x_2 + x_3 + 5} - \frac{x_1 + x_2 + x_3 + 7}{x_1 + x_2 + x_3 + 6} \leq -2.0, \\
& \frac{x_1 + x_2 + x_3 + 3}{x_1 + x_2 + x_3 + 4} + \frac{x_1 + x_2 + x_3 + 4}{x_1 + x_2 + x_3 + 5} + \frac{x_1 + x_2 + x_3 + 5}{x_1 + x_2 + x_3 + 6} + \frac{x_1 + x_2 + x_3 + 6}{x_1 + x_2 + x_3 + 7} \leq 3.59, \\
& \frac{x_1 + x_2 + x_3 + 4}{x_1 + x_2 + x_3 + 5} + \frac{x_1 + x_2 + x_3 + 5}{x_1 + x_2 + x_3 + 6} + \frac{x_1 + x_2 + 3x_3 + 6}{x_1 + x_2 + x_3 + 7} + \frac{x_1 + x_2 + x_3 + 7}{x_1 + x_2 + x_3 + 8} \leq 3.6, \\
& \frac{x_1 + x_2 + x_3 + 5}{x_1 + x_2 + x_3 + 6} + \frac{x_1 + x_2 + x_3 + 6}{x_1 + x_2 + x_3 + 7} + \frac{x_1 + x_2 + x_3 + 7}{x_1 + x_2 + x_3 + 8} + \frac{x_1 + x_2 + x_3 + 8}{x_1 + x_2 + x_3 + 9} \leq 3.7, \\
& 1.0 \leq x_1 \leq 3.0, \quad 1.0 \leq x_2 \leq 3.0, \quad 1.0 \leq x_3 \leq 3.0.
\end{aligned} \tag{5.4}$$

From Table 1, numerical results show that our algorithm can globally solve sum of general linear ratios problem (P) on a microcomputer.

## 6. Concluding remarks

A global optimization algorithm is proposed for solving sum of general linear ratios problem (P). To globally solve the (P), we first convert (P) into an equivalent problem (P1), then one new linearization method is proposed to construct the linear relaxation programming of the (P1). Then, a new pruning technique is proposed, this technique offers a possibility to cut away a large part of the current investigated feasible region by the algorithm, which can be utilized as an accelerating device for global optimization of problem (P). The proposed algorithm is convergent to the global minimum of (P1) through the successive refinement of linear relaxation of the feasible region and the subsequent solutions of a series of (RLP). Finally, the numerical experiments are given to illustrate the feasibility and effectiveness of the proposed algorithm.

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