Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2008, Article ID 754262, 18 pages
doi:10.1155/2008/754262

Research Article

Complete Solutions to Extended Stokes' Problems

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Received 20 January 2008; Revised 30 May 2008; Accepted 4 September 2008

Recommended by Francesco Pellicano

The main object of the present study is to theoretically solve the viscous flow of either a finite or infinite depth, which is driven by moving plane(s). Such a viscous flow is usually named as Stokes' first or second problems, which indicates the fluid motion driven by the impulsive or oscillating motion of the boundary, respectively. Traditional Stokes' problems are firstly revisited, and three extended problems are subsequently examined. Using some mathematical techniques and integral transforms, complete solutions which can exactly capture the flow characteristics at any time are derived. The corresponding steady-state and transient solutions are readily determined on the basis of complete solutions. Current results have wide applications in academic researches and are of significance for future studies taking more boundary conditions and non-Newtonian fluids into account.

1. Introduction

Stokes' problems are of great significance in numerous fields, which include the industry manufacturing, chemical engineering, geophysical flows, and heat conduction problems. Early in 1851, Stokes [1] proposed a well-known paper on pendulums in which the problems of the impulsive and oscillatory motions of a plane were studied. For a viscous flow motivated by a moving plane, Stokes' problems are one-dimensional, initial and boundary value problems. The momentum equation which is simplified from Navier-Stokes equations is a partial differential equation with differentiations with respect to only two variables, time and the coordinate normal to the plane. This simplification enables the viscous flow to be mathematically solvable. The associated solutions either for the transient state or the steady state have been well discussed [2–5].

In addition to previous conditions considered in Stokes' problems, much more boundary conditions are considered to simulate the flows in practice. For example, Zeng and Weinbaum [6] theoretically studied Stokes' problems for moving half-planes. Their results can be applied to many practical problems, for example, the flow induced by either

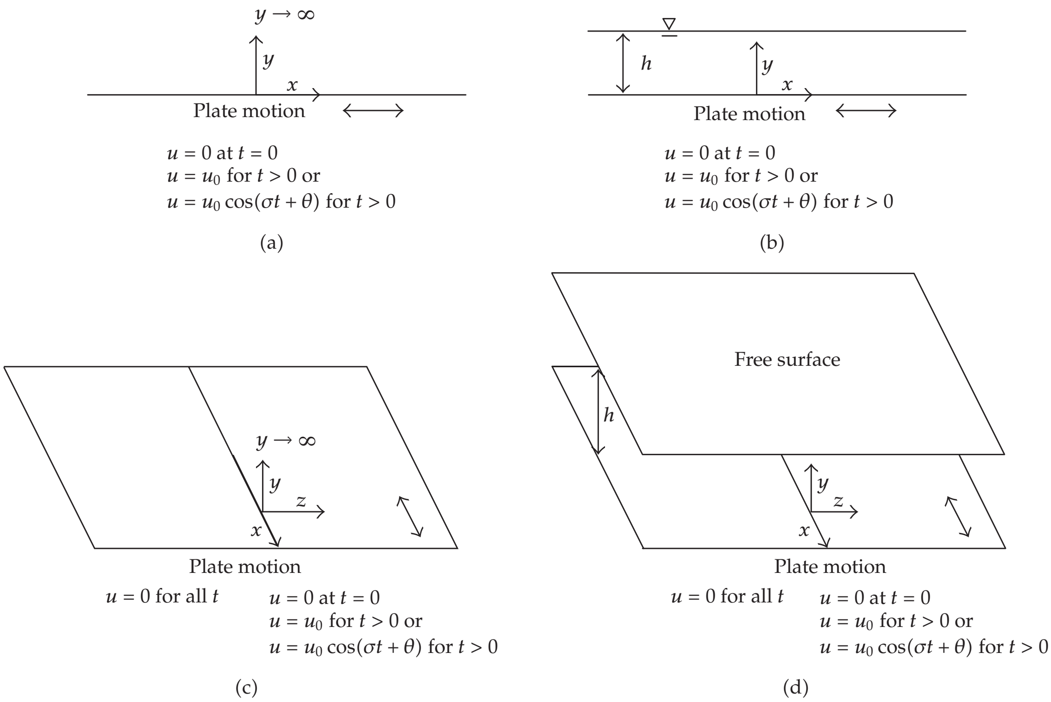

Figure 1: Traditional Stokes' problems and three extended Stokes' problems.

earthquakes or fracture of ice sheets. However, their steady-state solutions cannot capture the transient-state phenomenon while most of the earthquakes and other specific problems usually occur in a very short period. Up to now, the transient solutions are still lack and unclear. Moreover, a flow of a finite depth instead of an infinite depth should be further investigated as well. These new conditions constitute the extended Stokes' problems.

Due to above reasons, complete solutions to extended Stokes' problems are theoretically derived in the present paper. The organization of the present paper is as follows. Traditional Stokes' problems (see Figure 1(a)) are revisited in Section 2. Three extended Stokes' problems (see Figure 1(b) to Figure 1(d)) are examined in Sections 3 to 5. According to the derived solutions, results and conclusions are addressed in Section 6.

2. Review of traditional Stokes' problems

Traditionally, Stokes' problems are classified into two types according to the motions of the rigid boundary below the fluid. The impulsive and harmonic motions of the boundary lead to the first and second kinds of problems, respectively (see Figure 1(a)). The related momentum equation, boundary, and initial conditions are

$$u_t = \nu u_{yy}, \tag{2.1a}$$

$$u(y = 0,\ t > 0) = u_{0,} \tag{2.1b}$$

$$u(y \longrightarrow \infty,\ t > 0) = 0, \tag{2.1c}$$

$$u(y > 0,\ t = 0) = 0, \tag{2.1d}$$

for the first problem and

$$
\begin{aligned}
u_t &= \nu u_{yy}, \\
u(y=0,\ t>0) &= u_0 \cos(\sigma t + \theta), \\
u(y \longrightarrow \infty,\ t>0) &= 0, \\
u(y>0,\ t=0) &= 0,
\end{aligned}
\tag{2.2}
$$

for the second problem in which u is the velocity, ν the kinematic viscosity, σ the angular frequency, and the subscripts the differentiations. Solution to the first problem can be obtained through two main ways, the direct integral transform and the similarity method (for detailed discussion, see Wang [7]). The solution is

$$
u = u_0 \operatorname{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right), \tag{2.3}
$$

and the stress on the boundary (wall stress) is

$$
\tau_b = -\frac{u_0 \mu}{\sqrt{\pi \nu t}}. \tag{2.4}
$$

It is clear that no steady-state solution exists since the velocity profile will gradually develop by acquiring energy from the moving boundary. As for the second problem, the dimensionless velocity profile and wall stress are

$$
\begin{aligned}
U = \frac{1}{2}\mathbf{Re}\Bigg\{ & \exp\left(-\frac{Y}{\sqrt{2}}\right)\cdot \exp\left[i\left(T - \frac{Y}{\sqrt{2}} + \theta\right)\right]\cdot \operatorname{erfc}\left(\frac{Y}{2\sqrt{T}} - \sqrt{iT}\right) \\
& + \exp\left(\frac{Y}{\sqrt{2}}\right)\cdot \exp\left[i\left(T + \frac{Y}{\sqrt{2}} + \theta\right)\right]\cdot \operatorname{erfc}\left(\frac{Y}{2\sqrt{T}} + \sqrt{iT}\right)\Bigg\},
\end{aligned}
\tag{2.5}
$$

$$
\mathrm{T}_b = -\mathbf{Re}\left\{\frac{1}{\sqrt{\pi T}}e^{i\theta} + \sqrt{2}\cdot e^{i(T+\theta)}\left[\mathrm{S}_F\left(\sqrt{\frac{2T}{\pi}}\right) + i\mathrm{C}_F\left(\sqrt{\frac{2T}{\pi}}\right)\right]\right\}, \tag{2.6}
$$

in which

$$
Y = \sqrt{\frac{\sigma}{\nu}}y, \qquad T = \sigma t, \qquad U = \frac{u}{u_0}, \qquad \mathrm{T}_b = \frac{\tau_b\sqrt{\nu}}{\mu u_0 \sqrt{\sigma}}, \tag{2.7}
$$

and **Re** denotes the real part of the bracket and S_F and C_F express the sine and cosine Fresnel integrals [3]. The steady-state solutions for large times are

$$U_s = \exp\left(-\frac{Y}{\sqrt{2}}\right)\cos\left(T - \frac{Y}{\sqrt{2}} + \theta\right),$$
$$\mathrm{T}_{bs} = -\cos\left(T + \theta + \frac{\pi}{4}\right). \tag{2.8}$$

Erdogan [2] pointed out that the transient solution ($U_t = U - U_s$) of the cosine oscillation decays more rapidly than that of the sine case. The wall stress also decays with time. Furthermore, it is found that there exists a phase difference between the steady-state velocity and wall stress.

3. Stokes' problems for a finite-depth fluid

In a mathematical viewpoint, a solution to an infinite-depth fluid cannot exactly describe the flow of a finite depth, specially the shallow-water configuration. Therefore, a viscous flow of a finite depth h shown in Figure 1(b) is considered in this section.

3.1. Solution to the first problem

The governing equation and associated conditions are

$$u_t = \nu u_{yy}, \tag{3.1a}$$
$$u(y = 0,\ t > 0) = u_0, \tag{3.1b}$$
$$u_y(y = h,\ t > 0) = 0, \tag{3.1c}$$
$$u(y > 0,\ t = 0) = 0, \tag{3.1d}$$

in which (3.1c) indicates that no stress is allowed at the free-surface. Applying the Laplace transform to (3.1a)–(3.1d) leads to

$$s\hat{u} = \nu\hat{u}_{yy},$$
$$\hat{u}(y = 0, s) = \frac{u_0}{s},$$
$$\hat{u}_y(y = h, s) = 0, \tag{3.2}$$

where $\hat{u}(y, z, s) = \int_0^\infty u(y, z, t) \cdot e^{-st}\, ds$. The solution to (3.2) is

$$\hat{u} = \frac{u_0}{s}\cosh\left[\sqrt{\frac{s}{\nu}}(y - h)\right]\operatorname{sech}\left[\sqrt{\frac{s}{\nu}} \cdot h\right]. \tag{3.3}$$

With the help of the inverse transform equations in the first row of Table 1, the inversion of (3.3) is obtained

$$u = u_0 - u_0\frac{\sqrt{\nu}}{h}\int_0^{y/\sqrt{\nu}} \theta_2\left(\frac{\sqrt{\nu}\chi}{2h}\middle|\frac{\nu t}{h^2}\right)d\chi, \tag{3.4}$$

Table 1

$\hat{u}(s)$	$u(t)$
$\frac{1}{s}\cosh(\beta\sqrt{s})\operatorname{sech}(\alpha\sqrt{s}),\ -\alpha \le \beta \le \alpha$	$1-\frac{1}{\alpha}\int_0^{\alpha+\beta}\theta_2\left(\frac{\chi}{2\alpha}\middle\vert\frac{t}{\alpha^2}\right)d\chi$
$\frac{1}{s}\exp\left(-\sqrt{\beta s}\right)$	$\operatorname{erfc}\left(\frac{\sqrt{\beta}}{2\sqrt{t}}\right)$
$\exp\left(-\beta\sqrt{s}\right)$	$\frac{\beta}{2\sqrt{\pi t^3}}\exp\left(-\frac{\beta^2}{4t}\right)$

which can be expressed in a dimensionless form in terms of infinite series (with the help of (A.2))

$$U = 1-\sum_{n=0}^{\infty}\frac{2}{K}\exp(-K^2T)\sin(KY), \tag{3.5}$$

where

$$Y=\frac{y}{h},\qquad T=\frac{\nu t}{h^2},\qquad U=\frac{u}{u_0},\qquad K=\frac{2n+1}{2}\pi. \tag{3.6}$$

It is noted that the above solution excludes the state at $T = 0$.

3.2. Solution to the second problem

The momentum equation and conditions for the second problem is the same as (3.1a), (3.1c), (3.1d) except that (3.1b) has to be replaced by the oscillating boundary condition. Hence, the momentum equation and conditions are

$$\begin{gathered} u_t = \nu u_{yy}, \\ u(y=0,\ t>0) = u_0\cos(\sigma t+\theta), \\ u_y(y=h,\ t>0)=0, \\ u(y>0,\ t=0)=0. \end{gathered} \tag{3.7}$$

Using the technique of Laplace transform and solving $\hat{u}$ with boundary conditions, we have

$$\hat{u} = u_0\left[\frac{s}{s^2+\sigma^2}\cos\theta-\frac{\sigma}{s^2+\sigma^2}\sin\theta\right]\cosh\left(\sqrt{\frac{s}{\nu}}(y-h)\right)\operatorname{sech}\left(\sqrt{\frac{s}{\nu}}\cdot h\right). \tag{3.8}$$

For obtaining the inverse transform of (3.8), one can rewrite it as

$$\hat{u} = u_0\left[\frac{s^2}{s^2+\sigma^2}\cos\theta-\frac{s\sigma}{s^2+\sigma^2}\sin\theta\right]\cdot\left[\frac{1}{s}\cosh\left(\sqrt{\frac{s}{\nu}}(y-h)\right)\operatorname{sech}\left(\sqrt{\frac{s}{\nu}}\cdot h\right)\right]. \tag{3.9}$$

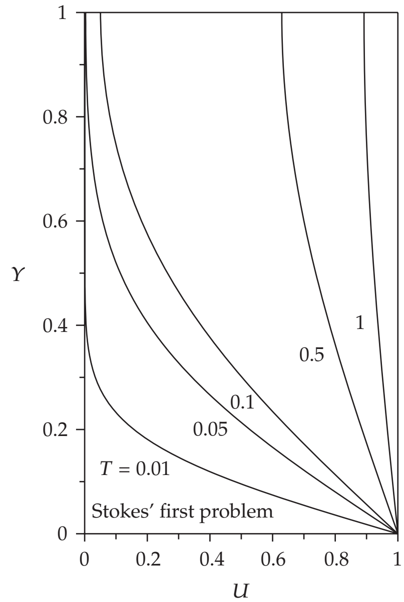

Figure 2: Velocity profiles of the first problem for a finite-depth fluid.

As the inversion of the second bracket is the same as that of (3.3), the inversion of (3.9) can be expressed by the convolution integral. After some algebra, the solution is

$$U = 1 - \cos\theta + \cos(T+\theta) + 2\cdot\sum_{n=0}^{\infty}\frac{\sin NY}{N}\left\{-\exp\left(-\lambda N^2 T\right)\cos\theta + \frac{1}{1+\lambda^2 N^4}\cdot\left[\lambda N^2\sin(T+\theta) - \cos(T-\theta) + \left(\cos\theta - \lambda N^2\sin\theta\right)\exp\left(-\lambda N^2 T\right)\right]\right\}, \tag{3.10}$$

where and the dimensionless variables are defined as

$$Y = \frac{y}{h}, \qquad T = \sigma t, \qquad U = \frac{u}{u_0}, \qquad \lambda = \frac{\nu}{\sigma h^2}, \qquad N = \frac{2n+1}{2}\pi. \tag{3.11}$$

By ignoring the exponential terms in (3.10), one can obtain the steady-state solution.

3.3. Results

Figure 2 displays the velocity profiles at various values of T for the first problem. It is evident that the velocity profile will gradually develop, and eventually be consistent with the wall speed. This result is quite similar to that of the traditional solution [5]. The significant difference between them is that the similarity phenomenon only exists in the traditional problem.

Results for the second problem are shown in Figures 3 and 4. It is noted that without loss of generality, the initial phase θ in all figures throughout this paper is set to be zero. The development of the velocity in the first oscillating cycle is displayed in Figure 3. As the upper boundary is free (no stress exists), the oscillation of the profile is less obvious than that of the traditional problem [5]. The comparison between the surface velocity (solid curve) and the wall velocity (dash curve) in the duration $T = 0$ to $T = 4\pi$ is shown in Figure 4. It is found that the maximum of the surface velocity is less than that of the wall velocity according to the viscous dissipation. Moreover, there is a phase difference between the surface velocity and the wall velocity.

4. Stokes' problems motivated by relatively moving planes

A flow generated by relatively moving planes is studied in this section. Unlike one-dimensional problems investigated in previous sections, present problem is a two-dimensional problem, as shown in Figure 1(c). Though this problem has been theoretically solved by Zeng and Weinbaum [6], it requires more improvements. Firstly, their solution to the second problem is a steady-state solution in which no decaying terms exist. This may be applicable for a long-term analysis, but it fails to describe the transient flow which strongly influences the total flow at the early stage. Secondly, a more direct transform rather than the semicylindrical transform used in their derivation seems necessary to standardize the derivation. Above weaknesses will be improved herein.

4.1. Solution to the first problem

The momentum equation and associated conditions are

$$
\begin{gathered}
u_t = \nu(u_{yy} + u_{zz}), \\
u(y = 0,\ z > 0,\ t > 0) = u_0, \\
u(y = 0,\ z < 0,\ t > 0) = 0, \\
u(y \longrightarrow \infty,\ z, t > 0) = 0, \\
u(y, z \longrightarrow \pm\infty, t) \text{ is finite}, \\
u(y > 0,\ z, t = 0) = 0.
\end{gathered} \tag{4.1}
$$

As (4.1) is purely linear, it can be decomposed into two subproblems ($u = u_1 + u_2$)

$$
\begin{gathered}
u_{1,t} = \nu u_{1,yy}, \\
u_1(y = 0,\ t > 0) = \frac{u_0}{2}, \\
u_1(y \longrightarrow \infty,\ t > 0) = 0, \\
u_1(y > 0,\ t = 0) = 0,
\end{gathered} \tag{4.2}
$$

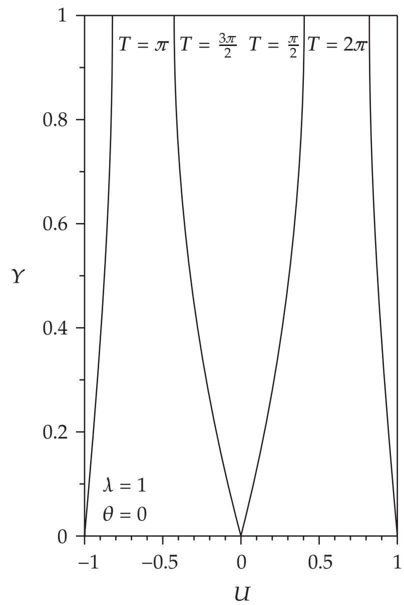

Figure 3: Velocity profiles of the second problem for a finite-depth fluid.

and

$$\begin{aligned} u_{2,t} &= \nu(u_{2,yy} + u_{2,zz}), \\ u_2(y = 0,\ z > 0,\ t > 0) &= \frac{u_0}{2}, \\ u_2(y = 0,\ z < 0,\ t > 0) &= -\frac{u_0}{2}, \\ u_2(y \longrightarrow \infty,\ z, t > 0) &= 0, \\ u_2(y, z \longrightarrow \pm\infty, t) &\text{ is finite}, \\ u_2(y > 0,\ z, t = 0) &= 0. \end{aligned} \tag{4.3}$$

It is evident that the former problem governed by (4.2) is the traditional Stokes' first problem, and the solution to u_1 is a half of (2.3). As for the latter problem, the flow satisfies the condition

$$u_2(y, z > 0, t) = -u_2(y, z < 0, t), \tag{4.4}$$

which further leads to

$$u_2(y > 0,\ z = 0, t) = 0. \tag{4.5}$$

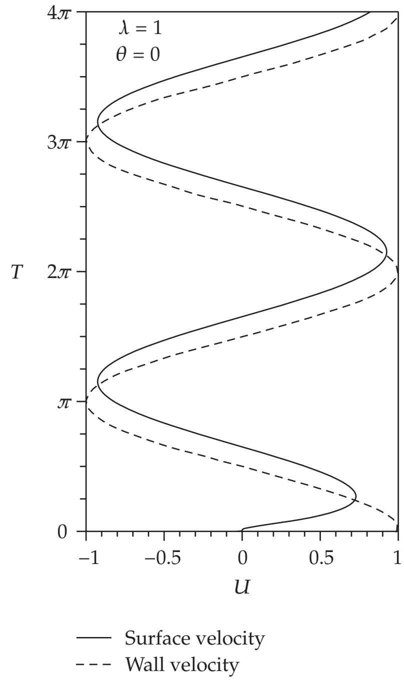

Figure 4: Comparison between the surface velocity (solid curve) and the wall velocity (dash curve) of the second problem for a finite-depth fluid.

Since the flow is antisymmetrical with respect to $z = 0$, one only needs to solve u_2 for the domain of $z \geq 0$ only. The resulting equation and conditions are

$$
\begin{gathered}
u_{2,t} = \nu(u_{2,yy} + u_{2,zz}), \\
u_2(y = 0,\ z > 0,\ t > 0) = \frac{u_0}{2}, \\
u_2(y > 0,\ z = 0,\ t > 0) = 0, \\
u_2(y \longrightarrow \infty,\ z \geq 0,\ t > 0) = 0, \\
u_2(y, z \longrightarrow +\infty, t) \text{ is finite}, \\
u_2(y > 0,\ z \geq 0,\ t = 0) = 0.
\end{gathered} \tag{4.6}
$$

Applying the Laplace transform to (4.6) leads to

$$
\begin{gathered}
s\widehat{u}_2 = \nu(\widehat{u}_{2,yy} + \widehat{u}_{2,zz}), \\
\widehat{u}_2(y = 0,\ z > 0, s) = \frac{u_0}{2s}, \\
\widehat{u}_2(y > 0,\ z = 0, s) = 0, \\
\widehat{u}_2(y \longrightarrow 0,\ z \geq 0, s) = 0.
\end{gathered} \tag{4.7}
$$

Now one further applies the Fourier sine transform to (4.7). It yields

$$\tilde{u}_{zz} - \left(\omega^2 + \frac{s}{\nu}\right)\tilde{u} = -\frac{u_0\omega}{2s}, \tag{4.8}$$

where $\tilde{u}(\omega, z, s) = \int_0^\infty \hat{u}_2(y, z, s)\sin(\omega y)\mathrm{d}y$. From the boundedness of $\tilde{u}$ as z approaches infinity and the boundary condition at $z = 0$, the solution to $\tilde{u}$ is

$$\tilde{u} = -\frac{u_0\omega\nu}{2s(\omega^2\nu + s)}\exp\left(-\sqrt{\omega^2 + s/\nu}\cdot z\right) + \frac{u_0\omega\nu}{2s(\omega^2\nu + s)}. \tag{4.9}$$

The solution of u_2 can be obtained by inverting (4.9) twice

$$u_2 = \frac{u_o}{2}\operatorname{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right) - \frac{u_0 y}{4\sqrt{\pi\nu}}\int_0^t \operatorname{erfc}\left(\frac{z}{2\sqrt{\nu t'}}\right)(t')^{-1.5}\exp\left(-\frac{y^2}{4\nu t'}\right)\mathrm{d}t', \tag{4.10}$$

which is equivalent to

$$u_2 = \frac{u_o}{2}\operatorname{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right) + \frac{u_0}{2\sqrt{\pi}}\int_{t'=0}^{t'=t}\operatorname{erfc}\left(\frac{z}{2\sqrt{\nu t'}}\right)\exp\left(-\frac{y^2}{4\nu t'}\right)\mathrm{d}\left(\frac{y}{\sqrt{\nu t'}}\right). \tag{4.11}$$

The dimensionless form of (4.11) is

$$U_2 = \frac{1}{2}\operatorname{erfc}(Y) + \frac{1}{\sqrt{\pi}}\int_\infty^Y \operatorname{erfc}\left(\frac{Z}{Y}Y'\right)\exp\left(-Y'^2\right)\mathrm{d}Y', \tag{4.12}$$

where

$$U_2 = \frac{u_2}{u_0}, \qquad Y = \frac{y}{2\sqrt{\nu t}}, \qquad Z = \frac{z}{2\sqrt{\nu t}}. \tag{4.13}$$

Finally, (4.12) can be further simplified to be

$$U_2 = \frac{1}{\sqrt{\pi}}\int_Y^\infty \operatorname{erf}\left(\frac{Z}{Y}\cdot Y'\right)\exp\left(-Y'^2\right)\mathrm{d}Y'. \tag{4.14}$$

As for the flow in the left domain ($Z < 0$), one can readily derive the solution by following the above processes.

4.2. Solution to the second problem

The techniques for solving the second problem are analogous to those for the first problem. Hence the following governing equation, boundary and initial conditions,

$$\begin{gathered} u_t = \nu(u_{yy} + u_{zz}), \\ u(y = 0,\ z > 0,\ t > 0) = u_0 \cos(\sigma t + \theta), \\ u(y = 0,\ z < 0,\ t > 0) = 0, \\ u(y \longrightarrow \infty,\ z, t > 0) = 0, \\ u(y, z \longrightarrow \pm\infty, t)\ \text{is finite}, \\ u(y > 0,\ z, t = 0) = 0, \end{gathered} \tag{4.15}$$

are decomposed to be

$$\begin{gathered} u_{1,t} = \nu(u_{1,yy} + u_{1,zz}), \\ u_1(y = 0,\ z, t > 0) = \frac{u_0}{2}\cos(\sigma t + \theta), \\ u_1(y \longrightarrow \infty,\ z, t > 0) = 0, \\ u_1(y > 0,\ z, t = 0) = 0, \end{gathered} \tag{4.16}$$

and

$$\begin{gathered} u_{2,t} = \nu(u_{2,yy} + u_{2,zz}), \\ u_2(y = 0,\ z > 0,\ t > 0) = \frac{u_0}{2}\cos(\sigma t + \theta), \\ u_2(y = 0,\ z < 0,\ t > 0) = -\frac{u_0}{2}\cos(\sigma t + \theta), \\ u_2(y \longrightarrow \infty,\ z, t > 0) = 0, \\ u_2(y, z \longrightarrow +\infty, t)\ \text{is finite}, \\ u_2(y > 0,\ z, t = 0) = 0. \end{gathered} \tag{4.17}$$

The solution to the former system is a half of (2.5), and the solution to the latter case is solved to be shown in the dimensionless form

$$\begin{aligned} U_2 = {} & \frac{Y}{4\sqrt{\pi}} \int_0^T \alpha^{-1.5} \cos(T - \alpha + \theta) \exp\left(-\frac{Y^2}{4\alpha} \right) \mathrm{d}\alpha \\ & + \frac{Z}{2\sqrt{\pi^3}} \int_0^\infty \int_0^T \sin(\omega Y) \cdot G_1(\omega, T - \alpha) \cdot G_2(\omega, \alpha, Z) \mathrm{d}\alpha\, \mathrm{d}\omega, \end{aligned} \tag{4.18}$$

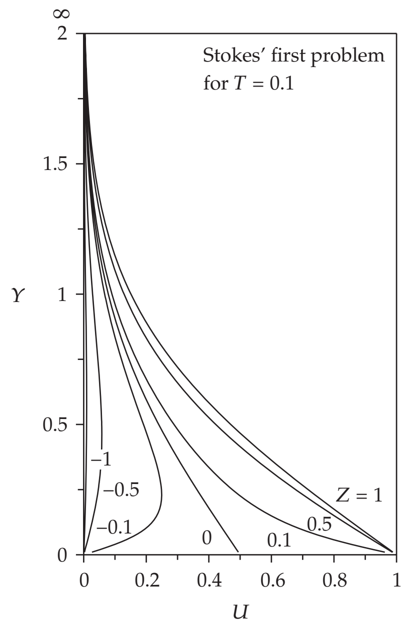

Figure 5: Velocity profiles of the first problem for an infinite-depth fluid motivated by relatively moving planes.

where

$$T = \sigma t, \qquad Y = \sqrt{\frac{\sigma}{\nu}}y, \qquad Z = \sqrt{\frac{\sigma}{\nu}}z, \qquad U_2 = \frac{u_2}{u_0}, \tag{4.19}$$

$$G_1(\omega, T) = \frac{\omega}{\omega^4 + 1}\left[\sin\theta\left(e^{-\omega^2 T} - \cos T + \omega^2 \sin T\right) - \cos\theta\left(\sin T + \omega^2 \cos T - \omega^2 e^{-\omega^2 T}\right)\right],$$
$$G_2(\omega, T, Z) = T^{-1.5}\exp\left(-\omega^2 T - \frac{Z^2}{4T}\right). \tag{4.20}$$

4.3. Results

For the first problem, the velocity profiles for various Z-sections at $T = 0.1$ are drawn in Figure 5. At the far end of the right-hand side ($Z \gg 0$), the profile will approach that of the traditional problem. This implies that the influence of the still half plate will decay as the value of Z grows. Similarly, the velocity will approach zero at the other far end ($Z \ll 0$) where the effects of the oscillating half plate are much weaker. Results for the second problem are shown in Figure 6. The phenomena at the far ends are quite similar to those appearing in the first problem.

5. Stokes' problems of a finite-depth fluid motivated by relatively moving planes

A finite-depth flow motivated by relatively moving planes is studied in this section, as shown in Figure 1(d). In addition to techniques used previously, more mathematical techniques must be applied to solve current problems. The detailed derivation is shown below.

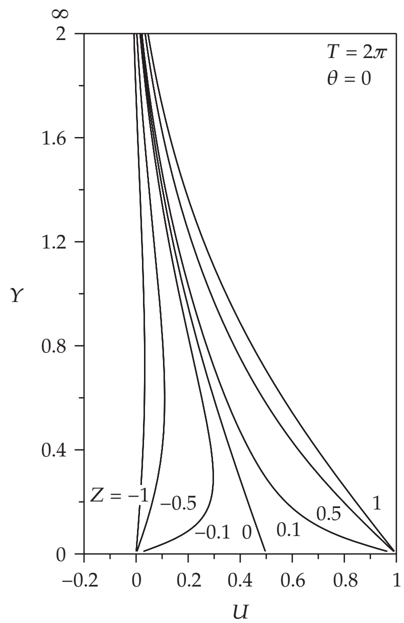

Figure 6: Velocity profiles of the second problem for an infinite-depth fluid motivated by relatively moving planes.

5.1. Solution to the first problem

The governing equation and associated conditions are expressed as

$$u_t = \nu(u_{yy} + u_{zz}), \tag{5.1a}$$

$$u(y = 0,\ z > 0,\ t > 0) = u_0, \tag{5.1b}$$

$$u(y = 0,\ z < 0,\ t > 0) = 0, \tag{5.1c}$$

$$u_y(y = h, z, t > 0) = 0, \tag{5.1d}$$

$$u(y, z \longrightarrow \pm\infty, t) \text{ is finite}, \tag{5.1e}$$

$$u(y > 0,\ z, t = 0) = 0. \tag{5.1f}$$

The direct integral transforms cannot work anymore in solving (5.1a), (5.1b), (5.1c), (5.1d), (5.1e), (5.1f) as this flow system is bounded by Dirichlet and Neumann conditions at$y = 0$and$y = h$, respectively. For the sake of eliminating different types of boundary conditions, the concept of symmetry is employed to generate a symmetrical flow. Accordingly, the boundary condition (5.1d) is replaced by

$$u(y = 2h, z, t > 0) = u_0, \tag{5.2}$$

where the flow domain is doubly expanded. Similar to the method introduced in Section 4, the flow system can be decomposed into two subsystems in which U_1 is a half of (3.5) and u_2

for the domain $z > 0$ is governed by

$$\begin{gathered}
u_{2,t} = \nu(u_{2,yy} + u_{2,zz}), \\
u_2(y = 0,\ z > 0,\ t > 0) = \frac{u_0}{2}, \\
u_2(y = 2h,\ z > 0,\ t > 0) = \frac{u_0}{2}, \\
u_2(y > 0,\ z = 0,\ t > 0) = 0, \\
u_2(y, z \longrightarrow +\infty, t) \text{ is finite}, \\
u_2(y > 0,\ z, t = 0) = 0.
\end{gathered} \tag{5.3}$$

By applying the Laplace transform to (5.3), it generates

$$\begin{gathered}
s\hat{u}_2 = \nu(\hat{u}_{2,yy} + \hat{u}_{2,zz}), \\
\hat{u}_2(y = 0,\ z > 0, s) = \frac{u_0}{2s}, \\
\hat{u}_2(y = 2h,\ z > 0, s) = \frac{u_0}{2s}, \\
\hat{u}_2(y, z \longrightarrow +\infty, t) \text{ is finite}, \\
\hat{u}_2(y > 0,\ z = 0, s) = 0.
\end{gathered} \tag{5.4}$$

The system of $\hat{u}_2$ is now shifted to the system of u^*

$$\begin{gathered}
su^* + \frac{u_0}{2} = \nu(u^*_{2,yy} + u^*_{2,zz}), \\
u^*(y = 0,\ z > 0, s) = 0, \\
u^*(y = 2h,\ z > 0, s) = 0, \\
u^*(y, z \longrightarrow +\infty, t) \text{ is finite}, \\
u^*(y > 0,\ z = 0, s) = -\frac{u_0}{2s},
\end{gathered} \tag{5.5}$$

in which $u^* = \hat{u} - u_0/2s$. After applying the finite sine transform, $\tilde{u}(s, n, y) = \int_0^{2h} u^* \sin(n\pi y/2h)\mathrm{d}y$, to (5.5), it yields

$$\tilde{u}_{zz} - \left(\frac{s}{\nu} + \frac{n^2\pi^2}{4h^2}\right)\tilde{u} = \frac{u_0 h}{n\pi\nu}(1 - (-1)^n), \tag{5.6}$$

with the boundedness condition for $z \to \infty$ and $\tilde{u}(z = 0) = -(u_0 h/n\pi\nu)(1 - (-1)^n)$. The solution is

$$\tilde{u} = (1 - (-1)^n) \cdot \left[\frac{u_0 h}{n\pi\nu\alpha}\left(\exp\left(-\sqrt{\alpha}z\right) - 1\right) - \frac{u_0 h}{n\pi s}\exp\left(-\sqrt{\alpha}z\right)\right], \tag{5.7}$$

where $\alpha = s/\nu + n^2\pi^2/4h^2$. Applying the inverse transforms with the help of the equations in the second and third rows of Table 1, the dimensionless solution is solved after some algebra

$$\begin{aligned} U_2 &= \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin\frac{n\pi Y}{2} \\ &\quad \cdot \left\{ -\exp\left(-\frac{n^2\pi^2 T}{4}\right)\operatorname{erf}\left(\frac{Z}{2\sqrt{T}}\right) - \int_0^T \frac{Z}{2\sqrt{\pi T'^3}}\exp\left(-\frac{n^2\pi^2 T'}{4} - \frac{Z^2}{4T'}\right)\mathrm{d}T' \right\}, \end{aligned} \tag{5.8}$$

in which $Z = z/h$ and the remaining dimensionless variables are identical to those shown in (3.6).

5.2. Solution to the second problem

Using the techniques described in previous subsections, the dimensionless solution to the second problem is the summation of a half of (3.10) and the following solution

$$\begin{aligned} U_2 &= \frac{1}{2}\cos(T+\theta) + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin\frac{n\pi Y}{2} \\ &\quad \cdot \left\{\cos\theta\int_0^T G_1(T-u)H(u)\mathrm{d}u + \sin\theta\int_0^T G_2(T-u)H(u)\mathrm{d}u - G_3(T)\cos\theta + G_4(T)\sin\theta\right\}, \end{aligned} \tag{5.9}$$

where

$$\begin{aligned} H(T) &= \frac{Z}{2\sqrt{\pi\lambda T^3}}\exp\left(-\frac{Z^2}{4\lambda T} - \lambda_n T\right), \\ G_1(T) &= \frac{\lambda_n^2\left(e^{-\lambda_n T} - \cos T\right) - \lambda_n \sin T}{1+\lambda_n^2}, \\ G_2(T) &= \frac{\lambda_n\left(e^{-\lambda_n T} - \cos T\right) + \lambda_n^2 \sin T}{1+\lambda_n^2}, \\ G_3(T) &= \frac{\lambda_n^2 e^{-\lambda_n T} - \lambda_n \sin T + \cos T}{1+\lambda_n^2}, \\ G_4(T) &= \frac{\lambda_n\left(\cos T - e^{-\lambda_n T}\right) + \sin T}{1+\lambda_n^2}, \end{aligned} \tag{5.10}$$

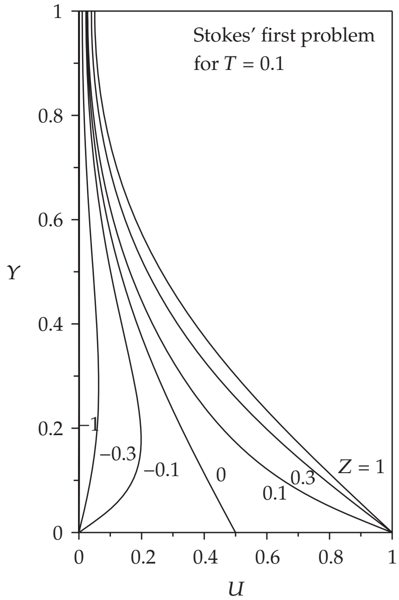

Figure 7: Velocity profiles of the first problem for a finite-depth fluid motivated by relatively moving planes.

and $\lambda_n = n^2\pi^2\lambda/4$, $Z = z/h$ and the remaining variables are identical to those defined in (3.11).

5.3. Results

For the first problem, the velocity profiles for various Z-sections for the first and second problems are drawn in Figure 7 and Figure 8, respectively. The developments in these two figures are similar to those shown in previous section (Figures 5 and 6). The comparison between the surface velocity (solid curves) and the wall velocity (dash curves) is made in Figure 9 for $Z = 1$ and $Z = -1$. The phase lag and the smaller amplitude for the surface velocity profile at $Z = 1$ are found. Besides, the amplitude of the surface velocity profile for $Z = -1$ is much smaller than that for $Z = 1$. In conclusion, solutions in this section combine the characteristics of problems in Section 3 (for a finite-depth case) and Section 4 (for relatively moving planes).

6. Conclusions

One- and two-dimensional viscous flows of either a finite or infinite depth generated by moving plane(s) are theoretically examined. Traditional Stokes' problems are firstly revisited and three extended problems are subsequently examined. Mathematical techniques used in the derivation processes include the Laplace transform, the Fourier transform, and transforming the original flow into either a symmetrical or antisymmetrical flow. Complete solutions, that is, exact solutions, derived in this study can precisely capture the flow characteristics at any time. As for the transient solution, it can be obtained by subtracting the steady-state solution from the complete solution. The shear stress on the boundary, which

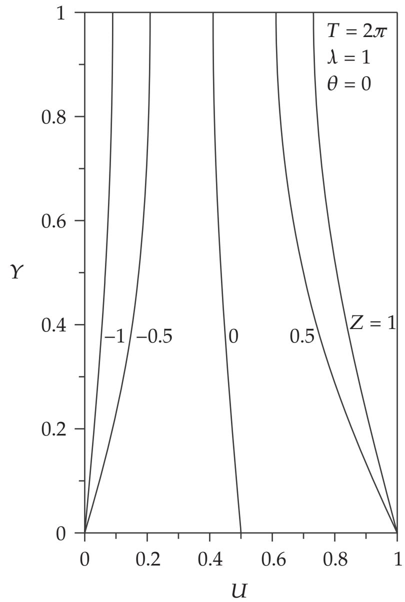

Figure 8: Velocity profiles of the second problem for a finite-depth fluid motivated by relatively moving planes.

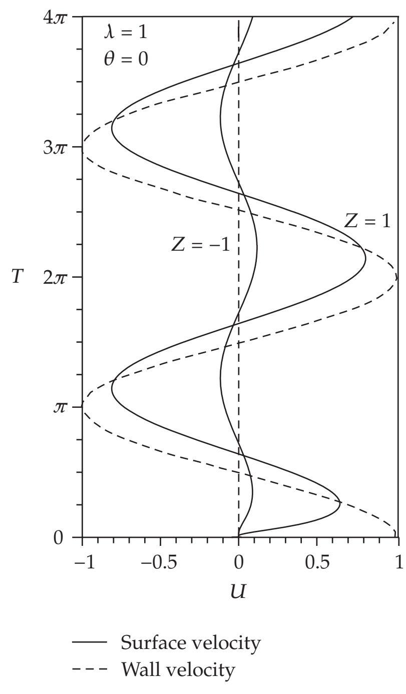

Figure 9: Comparison between the surface velocity (solid curve) and the wall velocity (dash curve) of the second problem for a finite-depth fluid motivated by relatively moving planes.

indicates external force driving the flow, is also readily obtained from the velocity profile. Results of Sections 3 to 5 are also depicted and discussed.

The field most related to the present study is the heat-conduction problem. For a traditional problem of a rod (only one-dimensional dependence in the y direction) heated at one side, it owns the diffusion-type differential equation (referred to (2.1a)), $T_t = \alpha T_{yy}$, in which T is the temperature and α the thermal diffusivity. If one end of the rod is suddenly heated with a constant temperature or harmonically heated, the solution is referred to the aforementioned Stokes' first or second problem. Based on techniques and results given in this paper, the two-dimensional heat-conduction problem for a plate can be studied through a purely mathematical way.

Current solutions are also valuable to other fields which include chemical engineering, mechanical manufacturing, and geophysical science. On the basis of the mathematical method provided in the present study, in the coming future, more conditions on the boundary could be considered for widely engineering applications.

Appendix

Tables of Laplace transforms

The Laplace transform is defined as

$$\widehat{u}(s) = \int_0^\infty u(t) \cdot e^{-st}\, \mathrm{d}t, \tag{A.1}$$

and the transform pairs used in the current paper are listed in Table 1 [8], where

$$\theta_2(z|t) = 2\sum_0^\infty \exp\left(-\frac{(2n+1)^2\pi^2 t}{4}\right)\cos(2n+1)\pi z. \tag{A.2}$$

Acknowledgment

The author appreciates the financial support from the National Science Council of Taiwan with Grant no. NSC 97-2221-E-270-013-MY3.

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