Research Article

Solution of Singular and Nonsingular Initial and Boundary Value Problems by Modified Variational Iteration Method

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We apply the modified variational iteration method (MVIM) for solving the singular and nonsingular initial and boundary value problems in this paper. The proposed modification is made by introducing Adomian's polynomials in the correct functional. The suggested algorithm is quite efficient and is practically well suited for use in such problems. The proposed iterative scheme finds the solution without any discretization, linearization, perturbation, or restrictive assumptions. Several examples are given to verify the efficiency and reliability of the suggested algorithm.

1. Introduction

Many problems in applied sciences can be modeled by singular and nonsingular boundary value problems. The application of these problems involve astrophysics, experimental and mathematical physics, nuclear charge in heavy atoms, thermal behavior of a spherical cloud of gas, thermodynamics, population models, chemical kinetics, and fluid mechanics, see [1–46] and the references therein. Several techniques including decomposition, variational iteration, finite difference, polynomial spline, and homotopy perturbation have been developed for solving such problems, see [1–51] and the references therein. Most of these methods have their inbuilt deficiencies coupled with the major drawback of huge computational work. These facts have motivated to develop other methods for solving these problems. Adomian's decomposition method [42–44, 51] was employed for finding solution of linear and nonlinear boundary value problems. He [12–18] developed the variational iteration method (VIM) for solving linear, nonlinear, initial, and boundary value problems. It is worth mentioning that the origin of variational iteration method can be traced back to Inokuti et al. [19]. In these

methods, the solution is given in an infinite series usually converging to an accurate solution, see [1–5, 12–19, 21, 23, 31–44, 46, 48, 51] and the references therein. In this paper, we apply the modified variational iteration method (MVIM), which is obtained by the elegant coupling of variational iteration method and the Adomian's polynomials for solving singular and nonsingular initial and boundary value problems. This idea has been used by Abbasbandy [1, 2] implicitly, and by Noor and Mohyud-Din [36, 38, 40] for the solution of nonlinear boundary value problems. The basic motivation of this paper is to apply this modified variational iteration method (MVIM) for finding the solution of singular and nonsingular initial and boundary value problems. It is shown that the MVIM provides the solution in a rapid convergent series with easily computable components. We write the correct functional for the boundary value problem and calculate the Lagrange multiplier optimally. The Adomian's polynomials are introduced in the correct functional and evaluated by using the specific algorithm [42–44] and the references therein. Finally, the approximants are calculated by employing the Lagrange multipliers and the Adomian's polynomial scheme simultaneously. The use of Lagrange multiplier reduces the successive application of the integral operator and minimizes the computational work. Moreover, the selection of the initial value is done by exploiting the concept of modified decomposition method. In the present study, we apply this technique to solve boundary layer problem, unsteady flow of gas, singularly perturbed sixth-order Boussinesq, third-order dispersive, and fourth-order parabolic equations. To make the work more concise and to get a better understanding of the solution behavior, in case of boundary layer problem and the unsteady flow of gas, we replace the series solutions by the powerful Pade approximants [22, 28, 34, 43, 44, 47]. The use of Pade approximants shows real promise in solving boundary value problems in an infinite domain. The proposed MVIM solves effectively, easily, and accurately a large class of linear, nonlinear, partial, deterministic, or stochastic differential equations with approximate solutions which converge very rapidly to accurate solutions. Our results can be viewed as important and significant improvement of the previously known results.

2. Variational iteration method

To illustrate the basic concept of the technique, we consider the following general differential equation:

$$Lu + Nu = g(x), \tag{2.1}$$

where L is a linear operator, N a nonlinear operator, and $g(x)$ is the inhomogeneous term. According to variational iteration method [1–5, 13–19, 21, 23, 31–41, 46, 48], we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s) \big(Lu_n(s) + N\tilde{u}_n(s) - g(s) \big) ds, \tag{2.2}$$

where $\lambda(s)$ is a Lagrange multiplier [13–18], which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation, that is, $\delta \tilde{u}_n = 0$. Relational (2.2) is called as a correct functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of variational iteration method and its

applicability for various kinds of differential equations are given in [13–18]. In this method, it is required first to determine the Lagrange multiplier λ optimally. The successive approximation u_{n+1}, $n \geq 0$ of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function. Consequently, the solution is given by $u = \lim_{n \to \infty} u_n$. For the convergence criteria and error estimates of variational iteration method, see Ramos [41].

3. Adomian's decomposition method

To convey an idea of the technique, we consider the differential equation [42–44] of the form

$$Lu + Ru + Nu = g, \tag{3.1}$$

where L is the highest-order derivative which is assumed to be invertible, R is a linear differential operator of order lesser than L, Nu represents the nonlinear terms, and g is the source term. Applying the inverse operator L^{-1} to both sides of (3.1) and using the given conditions, we obtain

$$u = f - L^{-1}(Ru) - L^{-1}(Nu), \tag{3.2}$$

where the function f represents the terms arising from integrating the source term g and by using the given conditions. Adomian's decomposition method [42–44] defines the solution by the series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{3.3}$$

where the components $u_n(x)$ are usually determined recurrently by using the relation

$$u_0 = f,$$
$$u_{k+1} = L^{-1}(Ru_k) - L^{-1}(Nu_k), \quad k \geq 0. \tag{3.4}$$

The nonlinear operator $N(u)$ can be decomposed into an infinite series of polynomials given by

$$N(u) = \sum_{n=0}^{\infty} A_n, \tag{3.5}$$

where A_n are the so-called Adomian's polynomials that can be generated for various classes of nonlinearities according to the specific algorithm developed in [42–44] which yields

$$A_n = \left(\frac{1}{n!}\right)\left(\frac{d^n}{d\lambda^n}\right)N\left(\sum_{i=0}^{n}(\lambda^i u_i)\right)_{\lambda=0}, \quad n = 0, 1, 2, \ldots. \tag{3.6}$$

4. Modified variational iteration method (MVIM)

To illustrate the basic concept of the variational decomposition method, we consider the following general differential (4.1), we have

$$Lu + Nu = g(x),\tag{4.1}$$

where L is a linear operator, N is a nonlinear operator, and $g(x)$ is the forcing term.

According to variational iteration method [1–5, 13–19, 21, 23, 31–41, 46, 48], we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda \left(Lu_n(s) + N\tilde{u}_n(s) - g(s) \right) ds,\tag{4.2}$$

where λ is a Lagrange multiplier [13–18], which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation, that is, $\delta\tilde{u}_n = 0$ (4.2) is called as a correct functional. We define the solution $u(x)$ by the series

$$u(x) = \sum_{i=0}^{\infty} u_i(x),\tag{4.3}$$

and the nonlinear term

$$N(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, \dots, u_i),\tag{4.4}$$

where A_n are the so-called Adomian's polynomials and can be generated for all types of nonlinearities according to the algorithm developed, in [42–44] which yields the following:

$$A_n = \left(\frac{1}{n!}\right)\left(\frac{d^n}{d\lambda^n}\right) N(u(\lambda)).\tag{4.5}$$

Hence, we obtain the following iterative scheme for finding the approximate solution

$$u_{n+1}(x) = u_n(x) + \int_0^t \lambda \left(Lu_n(x) + \sum_{n=0}^{\infty} A_n - g(x) \right) dx,\tag{4.6}$$

which is called the modified variational iteration method (MVIM) and is formulated by the elegant coupling of variational iteration method and the Adomian's polynomials.

5. Pade approximants

A Pade approximant is the ratio of two polynomials constructed from the coefficients of the Taylor series expansion of a function $u(x)$. The $[L/M]$ Pade approximants to a function $y(x)$ are given by [22, 28, 34, 43, 44, 47]

$$\left[\frac{L}{M}\right] = \frac{P_L(x)}{Q_M(x)},\tag{5.1}$$

where $P_L(x)$ is polynomial of degree at most L and $Q_M(x)$ is a polynomial of degree at most M. The formal power series

$$y(x) = \sum_{i=1}^{\infty} a_i x^i,$$

$$y(x) - \frac{P_L(x)}{Q_M(x)} = O(x^{L+M+1}),$$

(5.2)

determine the coefficients of $P_L(x)$ and $Q_M(x)$ by the equation. Since we can clearly multiply the numerator and denominator by a constant and leave $[L/M]$ unchanged, we imposed the normalization condition

$$Q_M(0) = 1.0.$$

(5.3)

Finally, we require that $P_L(x)$ and $Q_M(x)$ have non-common factors. If we write the coefficient of $P_L(x)$ and $Q_M(x)$ as

$$P_L(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_L x^L,$$

$$Q_M(x) = q_0 + q_1 x + q_2 x^2 + \cdots + q_M x^M.$$

(5.4)

Then by (5.3) and (5.4), we may multiply (5.5) by $Q_M(x)$, which linearizes the coefficient equations. We can write out (5.5) in more details as

$$a_{L+1} + a_L q_1 + \cdots + a_{L-M} q_M = 0,$$

$$q_{L+2} + q_{L+1} q_1 + \cdots + a_{L-M+2} q_M = 0,$$

$$\vdots$$

$$a_{L+M} + a_{L+M-1} q_1 + \cdots + a_L q_M = 0,$$

(5.5)

$$a_0 = p_0,$$

$$a_0 + a_0 q_1 + \cdots + = p_1,$$

$$\vdots$$

$$a_L + a_{L-1} q_1 + \cdots + a_0 q_L = p_L.$$

(5.6)

To solve these equations, we start with (5.5), which is a set of linear equations for all the unknown q's. Once the q's are known, then (5.6) gives an explicit formula for the unknown p's, which complete the solution. If (5.5) and (5.6) are nonsingular, then we can solve them directly and obtain (5.7) [22], where (5.7) holds, and if the lower index on a sum exceeds the upper, the sum is replaced by zero:

$$\left[\frac{L}{M}\right] = \frac{\det \begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ \sum_{j=M}^{L} a_{j-M} x^j & \sum_{j=M-1}^{L} a_{j-M+1} x^j & \cdots & \sum_{j=0}^{L} a_j x^j \end{bmatrix}}{\det \begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \vdots & \ddots & \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ x^M & x^{M-1} & \cdots & 1 \end{bmatrix}}.$$

(5.7)

To obtain diagonal Pade approximants of different order such as $[2/2]$, $[4/4]$, or $[6/6]$, we can use the symbolic calculus software Maple.

6. Numerical applications

In this section, we apply the modified variational iteration method (MVIM) for solving the singular and nonsingular boundary value problems. We write the correct functional for the boundary value problem and carefully select the initial value because the approximants are heavily dependant on the initial value. The Adomian's polynomials are introduced in the correct functional for the nonlinear terms. The results are very encouraging indicating the reliability and efficiency of the proposed method. We apply the MVIM for solving the boundary layer problem; unsteady flow of gas through a porous medium; Boussinesq equations, third-order dispersive, and fourth-order parabolic singular partial differential equations. The powerful Pade approximants are applied in case of boundary-layer problem and unsteady flow in order to make the work more concise and for better understanding of the solution behavior.

Example 6.1 (see [43]). Consider the following nonlinear third-order boundary layer problem which appears mostly in the mathematical modeling of physical phenomena in fluid mechanics [43, 45]:

$$f'''(x) + (k-1)f(x)f''(x) - 2n(f'(x))^2 = 0, \quad k > 0, \tag{6.1}$$

with boundary conditions

$$f(0) = 0, \qquad f'(0) = 1, \qquad f'(\infty) = 0, \qquad k > 0. \tag{6.2}$$

The correct functional is given as

$$f_{n+1}(x) = f_n(x) + \int_0^x \lambda(s)\left(f_n'''(s) + (k-1)\widetilde{f}_n(x)\widetilde{f}_n''(s) - 2n(\widetilde{f}_n'(s))^2\right)ds = 0, \quad k > 0. \tag{6.3}$$

Making the correct functional stationary, the Lagrange multipliers can be identified as $\lambda(s) = -(1/2!)(s-x)^2$, consequently, we have

$$f_{n+1}(x) = f_n(x) - \int_0^x \frac{1}{2!}(s-x)^2\left(f_n'''(s) + (k-1)\widetilde{f}_n(s)\widetilde{f}_n''(x) - 2n(\widetilde{f}_n'(s))^2\right)ds = 0, \quad k > 0, \tag{6.4}$$

where $f''(0) = \alpha < 0$. Applying the modified variational iteration method, we have

$$f_{n+1}(x) = f_n(x) + \int_0^x \frac{1}{2!}(s-x)^2\left(f_n'''(s) + (k-1)\sum_{n=0}^{\infty} A_n - 2n\sum_{n=0}^{\infty} B_n\right)ds = 0, \tag{6.5}$$

Table 1: Numerical values for $\alpha = f''(0)$ for $0 < k < 1$ by using diagonal Pade approximants [43].

n	[2/2]	[3/3]	[4/4]	[5/5]	[6/6]
0.2	−0.3872983347	−0.3821533832	−0.3819153845	−0.3819148088	−0.3819121854
1/3	−0.5773502692	−0.5615999244	−0.5614066588	−0.5614481405	−0.561441934
0.4	−0.6451506398	−0.6397000575	−0.6389732578	−0.6389892681	−0.6389734794
0.6	−0.8407967591	−0.8393603021	−0.8396060478	−0.8395875381	−0.8396056769
0.8	−1.007983207	−1.007796981	−1.007646828	−1.007646828	−1.007792100

where A_n and B_n are the so-called Adomian's polynomials and can be generated for all types of nonlinearities according to the algorithm defined in [42–44]. Consequently the following approximants are made:

$$f_0(x) = x,$$

$$f_1(x) = x + \frac{1}{2}\alpha x^2 + \frac{1}{3}x^3,$$

$$f_2(x) = x + \frac{1}{2}\alpha x^2 + \frac{1}{3}x^3 + \frac{1}{24}\alpha(3n+1)x^4 + \frac{1}{30}n(n+1)x^5,$$

$$f_3(x) = x + \frac{1}{2}\alpha x^2 + \frac{1}{3}x^3 + \frac{1}{24}\alpha(3n+1)x^4 + \frac{1}{30}n(n+1)x^5 + \frac{1}{120}\alpha^2(3n+1)x^5$$
$$+ \frac{1}{720}\alpha(19n^2+18n+3)x^6 + \frac{1}{315}n(2n^2+2n+1)x^7, \tag{6.6}$$

$$f_4(x) = x + \frac{1}{2}\alpha x^2 + \frac{1}{3}x^3 + \frac{1}{24}\alpha(3n+1)x^4 + \frac{1}{30}n(n+1)x^5 + \frac{1}{120}\alpha^2(3n+1)x^5$$
$$+ \frac{1}{720}\alpha(19n^2+18n+3)x^6 + \frac{1}{315}n(2n^2+2n+1)x^7, \frac{1}{5040}\alpha^2(27n^2+42n+11)x^7$$
$$+ \frac{1}{40320}\alpha(167n^3+297n^2+161n+15)x^8 + \frac{1}{22680}n(13n^3+38n^2+23n+6)x^9,$$

$$\vdots$$

The series solution is given as

$$f(x) = x + \frac{\alpha x^2}{2} + \frac{nx^3}{3} + \left(\frac{1}{8}n\alpha + \frac{1}{24}\alpha\right)x^4 + \left(\frac{1}{30}n^2 + \frac{1}{40}n\alpha^2 + \frac{1}{120}\alpha^2 + \frac{1}{30}n\right)x^5$$

$$+ \left(\frac{19}{720}n^2\alpha + \frac{1}{240}\alpha + \frac{1}{40}n\alpha\right)x^6 + \left(\frac{1}{120}n\alpha^2 + \frac{1}{315}n + \frac{2}{315}n^3 + \frac{11}{5040}\alpha^2 + \frac{3}{560}n^2\alpha^2 + \frac{2}{315}n^2\right)x^7$$

$$+ \left(\frac{11}{40320}\alpha^3 + \frac{33}{4480}n^2\alpha + \frac{3}{4480}\alpha^3n^2 + \frac{23}{5760}n\alpha + \frac{1}{2688}\alpha + \frac{167}{40320}n^3\alpha + \frac{1}{960}\alpha^3n\right)x^8$$

$$+ \left(\frac{1}{3780}n + \frac{527}{362880}n^3\alpha^2 + \frac{19}{11340}n^3 + \frac{709}{362880}n\alpha^2 + \frac{23}{8064}n^2\alpha^2 + \frac{23}{22680}n^2 + \frac{13}{22680}n^4\right.$$

$$\left. + \frac{43}{120960}\alpha^2\right)x^9 + \cdots. \tag{6.7}$$

Example 6.2 (see [34, 44]). Consider the following nonlinear differential equation which governs the unsteady flow of gas through a porous medium:

$$y''(x) + \frac{2x}{\sqrt{1-\alpha y}}y'(x) = 0, \quad 0 < \alpha < 1. \tag{6.8}$$

Table 2: Numerical values for $\alpha = f''(0)$ for $k > 1$ by using diagonal Pade approximants [43].

n	α
4	−2.483954032
10	−4.026385103
100	−12.84334315
1000	−40.65538218
5000	−104.8420672
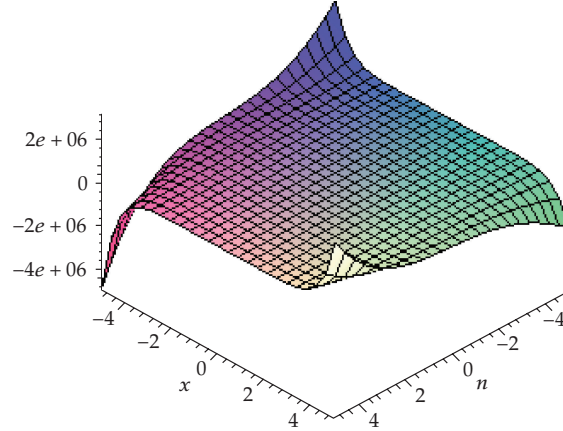
Figure 1: ($\alpha = -2.483954032$).

With the following typical boundary conditions imposed by the physical properties [34, 44],

$$y(0) = 1, \qquad \lim_{x \to \infty} y(x) = 0. \tag{6.9}$$

The correct functional is given as

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s)\left(y''(s) + \frac{2x}{\sqrt{1 - \alpha y}} y'(s) \right) ds, \quad 0 < \alpha < 1. \tag{6.10}$$

Making the correct functional stationary, using $\lambda = x - s$, as the Lagrange multiplier, we get the following iterative formula:

$$y_{n+1}(x) = y_n(x) + \int_0^x (s - x)\left(y''(s) + \frac{2x}{\sqrt{1 - \alpha y}} y'(s) \right) ds, \quad 0 < \alpha < 1, \tag{6.11}$$

where

$$A = y'(0). \tag{6.12}$$

Applying the modified variational iteration method, we have

$$y_{n+1}(x) = y_n(x) + \int_0^x (s - x)\left(y''(s) + 2x \sum_{n=0}^{\infty} A_n \right) ds, \quad 0 < \alpha < 1, \tag{6.13}$$

where A_n are the so-called Adomian's polynomials and can be generated for all types of nonlinearities according to the algorithm defined in [42–44]. First few Adomian's polynomials are as under:

$$A_0 = (1 - \alpha y_0)^{-1/2} y_0',$$
$$A_1 = (1 - \alpha y_0)^{-1/2} y_1' + \frac{\alpha}{2}(1 - \alpha y_0)^{-3/2} y_0' y_1,$$
$$A_2 = (1 - \alpha y_0)^{-1/2} y_2' + \frac{\alpha}{2}(1 - \alpha y_0)^{-3/2} y_1' y_1 + \frac{\alpha}{2}(1 - \alpha y_0)^{-3/2} y_0' y_2 + \frac{3}{8}\alpha^2 (1 - \alpha y_0)^{-5/2} y_0' y_1^2,$$
$$A_3 = (1 - \alpha y_0)^{-1/2} y_3' + \frac{\alpha}{2}(1 - \alpha y_0)^{-3/2} y_2' y_1 + \frac{\alpha}{2}(1 - \alpha y_0)^{-3/2} y_1' y_2 + \frac{\alpha}{2}(1 - \alpha y_0)^{-3/2} y_0' y_3$$
$$+ \frac{3}{8}\alpha^2 (1 - \alpha y_0)^{-5/2} y_1' y_1^2 + \frac{3}{4}\alpha^2 (1 - \alpha y_0)^{-5/2} y_0' y_1 y_2 + \frac{5}{16}\alpha^3 (1 - \alpha y_0)^{-7/2} y_0' y_1^3,$$
$$\vdots$$

$$(6.14)$$

Consequently, the following approximants are obtained:

$$y_0(x) = 1,$$
$$y_1(x) = 1 + Ax,$$
$$y_2(x) = 1 + Ax - \frac{A}{3\sqrt{1-\alpha}}x^3,$$
$$y_3(x) = 1 + Ax - \frac{A}{3\sqrt{1-\alpha}}x^3 - \frac{\alpha A^2}{12(1-\alpha)^{3/2}}x^4 + \frac{A}{10(1-\alpha)}x^5,$$
$$y_4(x) = 1 + Ax - \frac{A}{3\sqrt{1-\alpha}}x^3 - \frac{\alpha A^2}{12(1-\alpha)^{3/2}}x^4 + \frac{A}{10(1-\alpha)}x^5 - \frac{3\alpha^2 A^3}{80(1-\alpha)^{5/2}}x^5$$
$$+ \frac{\alpha A^2}{15(1-\alpha)^2}x^6 + O(x^7),$$
$$y_5(x) = 1 + Ax - \frac{A}{3\sqrt{1-\alpha}}x^3 - \frac{\alpha A^2}{12(1-\alpha)^{3/2}}x^4 + \frac{A}{10(1-\alpha)}x^5 - \frac{3\alpha^2 A^3}{80(1-\alpha)^{5/2}}x^5 + \frac{\alpha A^2}{15(1-\alpha)^2}x^6$$
$$- \frac{\alpha^3 A^4}{48(1-\alpha)^{7/2}}x^6 + O(x^7),$$
$$\vdots$$

$$(6.15)$$

The series solution is given as

$$y(x) = 1 + Ax - \frac{A}{3\sqrt{1-\alpha}}x^3 - \frac{\alpha A^2}{12(1-\alpha)^{3/2}}x^4 + \left(\frac{A}{10(1-\alpha)} - \frac{3\alpha^2 A^3}{80(1-\alpha)^{5/2}}\right)x^5$$
$$+ \left(\frac{\alpha A^2}{15(1-\alpha)^2} - \frac{\alpha^3 A^4}{48(1-\alpha)^{7/2}}\right)x^6 + O(x^7).$$

$$(6.16)$$

Now, we investigate the mathematical behavior of the solution $y(x)$ in order to determine the

Table 3: The initial slopes $A = y'(0)$ for various values of α.

α	$B_{[2/2]} = y'(0)$	$B_{[3/3]} = y'(0)$
0.1	−3.556558821	−1.957208953
0.2	−2.441894334	−1.786475516
0.3	−1.928338405	−1.478270843
0.4	−1.606856838	−1.231801809
0.5	−1.373178096	−1.025529704
0.6	−1.185519607	−0.8400346085
0.7	−1.021411309	−0.6612047893
0.8	−0.8633400217	−0.4776697286
0.9	−0.6844600642	−0.2772628386

initial slope $y'(0)$. This goal can be achieved by forming diagonal Pade approximants [34, 44, 47] which have the advantage of manipulating the polynomial approximation into a rational function to gain more information about $y(x)$. It is well-known that Pade approximants will converge on the entire real axis [20, 22, 28, 34, 43, 44, 47], if $y(x)$ is free of singularities on the real axis. It is of interest to note that Pade approximants give results with no greater error bounds than approximation by polynomials. More importantly, the diagonal approximant is the most accurate approximant; therefore we will construct only the diagonal approximants in the following discussions. Using the boundary condition $y(\infty) = 0$, the diagonals approximant $[M/M]$ vanishes if the coefficient of x with the highest power in the numerator vanishes. The computational work can be performed by using the mathematical software MAPLE. The physical behavior indicates that $y(x)$ is a decreasing function, hence $y'(0) < 0$. Using this fact, and following [20, 34, 44], complex roots and nonphysical positive roots should be excluded. Based on this, the [2/2] Pade approximant produced the slope A to be

$$A = -\frac{2(1-\alpha)^{1/4}}{\sqrt{3\alpha}}, \tag{6.17}$$

and using [3/3] Pade approximants we find

$$A = -\frac{\sqrt{(-4674\alpha + 8664)\sqrt{1-\alpha} - 144\gamma}}{57\alpha}, \tag{6.18}$$

where

$$\gamma = \sqrt{5(1-\alpha)(1309\alpha^2 - 2280\alpha + 1216)}. \tag{6.19}$$

Using (6.17)–(6.19) gives the values of the initial slope $A = y'(0)$ listed in Table 1. The formulas (6.17) and (6.19) suggest that the initial slope $A = y'(0)$ depends mainly on the parameter α, where $0 < \alpha < 1$. Table 3 shows that the initial slope $A = y'(0)$ increases with the increase of α. The mathematical structure of $y(x)$ was successfully enhanced by using the Pade approximants. Table 4 indicates the values of $y(x)$ [34, 44] and by using the [2/2] and [3/3] approximants for specific value of $\alpha = 0.5$.

Table 4: The values of $y(x)$ for $\alpha = 0.5$ for $x = 0.1$ to 1.0.

X	y kidder	$y_{[2/2]}$	$y_{[3/3]}$
0.1	0.8816588283	0.8633060641	0.8979167028
0.2	0.7663076781	0.7301262261	0.7985228199
0.3	0.6565379995	0.6033054140	0.7041129703
0.4	0.5544024032	0.4848898717	0.6165037901
0.5	0.4613650295	0.3761603869	0.5370533796
0.6	0.3783109315	0.2777311628	0.4665625669
0.7	0.3055976546	0.1896843371	0.4062426033
0.8	0.2431325473	0.1117105165	0.3560801699
0.9	0.1904623681	0.04323673236	0.3179966614
1.0	0.1587689826	0.01646750847	0.2900255005

Example 6.3 (see [32, 51]). Consider the following singularly perturbed sixth-order Boussinesq equation:

$$u_{tt} = u_{xx} + \left(p(u)\right)_{xx} + \alpha u_{xxxx} + \beta u_{xxxxxx}, \tag{6.20}$$

taking $\alpha = 1$, $\beta = 0$, and $p(u) = 3u^2$, the model equation is given as

$$u_{tt} = u_{xx} + 3\left(u^2\right)_{xx} + u_{xxxx}, \tag{6.21}$$

with initial conditions

$$u(x,0) = \frac{2ak^2 e^{kx}}{\left(1 + ae^{kx}\right)^2}, \qquad u_t(x,0) = \frac{2ak^3\sqrt{1+k^2}\left(1 - ae^{kx}\right)e^{kx}}{\left(1 + ae^{kx}\right)^3}, \tag{6.22}$$

where a and k are arbitrary constants. The exact solution $u(x,t)$ of the problem is given as [32, 51]

$$u(x,t) = 2\frac{ak^2 \exp\left(kx + k\sqrt{1+k^2}t\right)}{\left(1 + a\exp\left(kx + k\sqrt{1+k^2}t\right)\right)^2}. \tag{6.23}$$

The correct functional is given as

$$
\begin{aligned}
u_{n+1}(x,t) = {} & \frac{2ak^2 e^{kx}}{\left(1 + ae^{kx}\right)^2} + \frac{2ak^3\sqrt{1+k^2}\left(1 - ae^{kx}\right)e^{kx}}{\left(1 + ae^{kx}\right)^3}t \\
& + \int_0^t \lambda\left(\frac{\partial^2 u_n}{\partial t^2} - \left(\tilde{u}_n\right)_{xx} - \left(\tilde{u}_n\right)_{xxxx} - 3\sum_{n=0}^{\infty} B_n\right)dt,
\end{aligned}
\tag{6.24}
$$

where B_n are Adomian's polynomials for nonlinear operator $F(u) = u^2(x)$ and can be generated for all types of nonlinearities according to the algorithm developed in [42–44, 51] which yields the following:

$$
\begin{aligned}
B_0 &= \left(u_0^2\right)_{xx}, \\
B_1 &= 2u_0 u_{1xx} + 4u_{0x}u_{1x} + 2u_{0xx}u_1, \\
B_2 &= 2u_0 u_{2xx} + 4u_{0x}u_{2x} + 2u_{0xx}u_2 + 2u_{1x}u_{1xx} + 2\left(u_{1x}\right)^2 \\
B_3 &= \left(2u_0 u_3 + 2u_1 u_2\right)_{xx}, \\
&\qquad\qquad \vdots
\end{aligned}
\tag{6.25}
$$

Making the correct functional stationary, the Lagrange multiplier can be identified as $\lambda = x - t$, consequently,

$$u_{n+1}(x,t) = \frac{2ak^2 e^{kx}}{(1 + ae^{kx})^2} + \frac{2ak^3\sqrt{1+k^2}(1 - ae^{kx})e^{kx}}{(1 + ae^{kx})^3} t$$

$$+ \int_0^t (x - t)\left(\frac{\partial^2 u_n}{\partial t^2} - (\tilde{u}_n)_{xx} - (\tilde{u}_n)_{xxxx} - 3\sum_{n=0}^{\infty} B_n \right) dt.$$

(6.26)

The following approximants are obtained:

$$u_0(x,t) = \frac{2e^x}{(1 + e^x)^2},$$

$$u_1(x,t) = \frac{2e^x}{(1 + e^x)^2} + \frac{2ak^3\sqrt{1+k^2}(1 - ae^{kx})e^{kx}}{(1 + ae^{kx})^3} t + \frac{2e^x(1 - 4e^x + e^{2x})}{(1 + e^x)^4} t^2,$$

$$u_2(x,t) = \frac{2e^x}{(1 + e^x)^2} + \frac{2ak^3\sqrt{1+k^2}(1 - ae^{kx})e^{kx}}{(1 + ae^{kx})^3} t + \frac{2e^x(1 - 4e^x + e^{2x})}{(1 + e^x)^4} t^2$$

$$- \frac{2\sqrt{2}e^x(-1 + e^x)(1 - 10e^x + e^{2x})}{3(1 + e^x)^5} t^3$$

$$+ \frac{e^x(1 - 4e^x + e^{2x})(1 - 44e^x + 78e^{2x} - 44e^{3x} + e^{4x})}{3(1 + e^x)^8} t^4,$$

$$u_3(x,t) = \frac{2e^x}{(1 + e^x)^2} + \frac{2ak^3\sqrt{1+k^2}(1 - ae^{kx})e^{kx}}{(1 + ae^{kx})^3} t + \frac{2e^x(1 - 4e^x + e^{2x})}{(1 + e^x)^4} t^2$$

(6.27)

$$- \frac{2\sqrt{2}e^x(-1 + e^x)(1 - 10e^x + e^{2x})}{3(1 + e^x)^5} t^3$$

$$+ \frac{e^x(1 - 4e^x + e^{2x})(1 - 44e^x + 78e^{2x} - 44e^{3x} + e^{4x})}{3(1 + e^x)^8} t^4$$

$$- \frac{\sqrt{2}e^x(-1 + e^x)(1 - 56e^x + 246e^{2x} - 56e^{3x} + e^{4x})}{15(1 + e^x)^7} t^5$$

$$+ \frac{e^x(1 - 452e^x + 19149e^{2x} - 207936e^{3x} + 807378e^{4x} - 1256568e^{5x})}{45(1 + e^x)^{12}} t^6$$

$$+ \frac{e^x(807378e^{6x} - 207936e^{7x} + 19149e^{8x} - 452e^{9x} + e^{10x})}{45(1 + e^x)^{12}} t^6,$$

$$\vdots$$

Table 5: (Error estimates.) The absolute error between the exact and the series solutions. Higher accuracy can be obtained by introducing some more components of the series solution.

x_i	t_j					
	0.01	0.02	0.04	0.1	0.2	0.5
−1	$2.80886\,E-14$	$1.79667\,E-12$	$1.15235\,E-10$	$2.83355\,E-8$	$1.83899\,E-6$	$4.74681\,E-4$
−0.8	$6.27276\,E-14$	$4.01362\,E-12$	$2.57471\,E-10$	$6.33178\,E-8$	$4.10454\,E-6$	$1.04489\,E-3$
−0.6	$6.08402\,E-14$	$3.90188\,E-12$	$2.25663\,E-10$	$6.18024\,E-8$	$4.02299\,E-6$	$1.03093\,E-3$
−0.4	$1.16573\,E-14$	$7.41129\,E-13$	$4.82756\,E-11$	$1.23843\,E-8$	$8.53800\,E-6$	$2.46302\,E-4$
−0.2	$5.53446\,E-14$	$3.53395\,E-12$	$2.25663\,E-10$	$5.47485\,E-8$	$3.47264\,E-6$	$8.35783\,E-4$
0	$8.63198\,E-14$	$5.53357\,E-12$	$2.54174\,E-10$	$8.65197\,E-8$	$5.54893\,E-6$	$1.37353\,E-3$
0.2	$5.56222\,E-14$	$3.55044\,E-12$	$2.27779\,E-10$	$5.60362\,E-8$	$3.63600\,E-6$	$9.29612\,E-4$
0.4	$1.14353\,E-14$	$7.14928\,E-13$	$4.49107\,E-11$	$1.03370\,E-8$	$5.93842\,E-7$	$9.61260\,E-5$
0.6	$6.06182\,E-14$	$3.87551\,E-12$	$2.47218\,E-10$	$5.97562\,E-8$	$3.76275\,E-6$	$8.79002\,E-4$
0.8	$6.23945\,E-14$	$3.99519\,E-12$	$2.55127\,E-10$	$6.18881\,E-8$	$3.92220\,E-6$	$9.36404\,E-4$
1	$2.79776\,E-14$	$1.78946\,E-12$	$1.14307\,E-10$	$2.77684\,E-8$	$1.76607\,E-6$	$4.28986\,E-4$

The series solution is given as

$$u(x,t) = \frac{2e^x}{(1+e^x)^2} + \frac{2ak^3\sqrt{1+k^2}\,(1-ae^{kx})e^{kx}}{(1+ae^{kx})^3}t + \frac{2e^x(1-4e^x+e^{2x})}{(1+e^x)^4}t^2$$
$$- \frac{2\sqrt{2}e^x(-1+e^x)(1-10e^x+e^{2x})}{3(1+e^x)^5}t^3$$
$$+ \frac{e^x(1-4e^x+e^{2x})(1-44e^x+78e^{2x}-44e^{3x}+e^{4x})}{3(1+e^x)^8}t^4$$
$$+ \frac{8e^{2x}(1-10e^x+20e^{2x}-10e^{3x}+e^{4x})}{(1+e^x)^8}t^4 \qquad (6.28)$$
$$- \frac{\sqrt{2}e^x(-1+e^x)(1-56e^x+246e^{2x}-56e^{3x}+e^{4x})}{15(1+e^x)^7}t^5$$
$$+ \frac{e^x(1-452e^x+19149e^{2x}-207936e^{3x}+807378e^{4x}-1256568e^{5x})}{45(1+e^x)^{12}}t^6$$
$$+ \frac{e^x(807378e^{6x}-207936e^{7x}+19149e^{8x}-452e^{9x}+e^{10x})}{45(1+e^x)^{12}}t^6 + \cdots.$$

Example 6.4 (see [32, 51]). Consider the following singularly perturbed sixth-order Boussinesq equation:

$$u_{tt} = u_{xx} + \left(u^2\right)_{xx} - u_{xxxx} + \frac{1}{2}u_{xxxxxx}, \qquad (6.29)$$

with initial conditions

$$u(x,0) = -\frac{105}{169}\sec h^4\!\left(\frac{x}{\sqrt{26}}\right), \qquad u_t(x,0) = \frac{-210\sqrt{194/13}\,\sec h^4(x/\sqrt{26})\tanh(x/\sqrt{26})}{2197}.$$
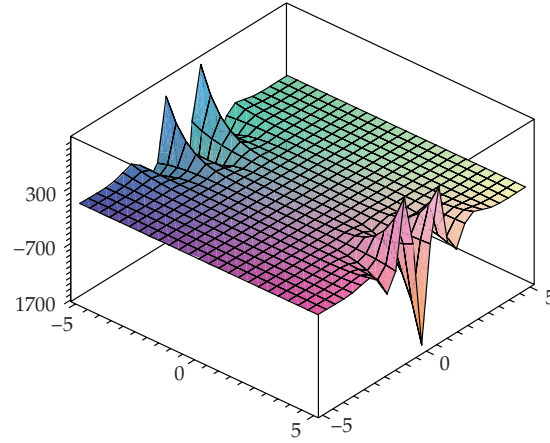$$(6.30)$$

Figure 2

The exact solution of the problem is given as

$$u(x,t) = -\frac{105}{169} \sec h^4\left[\sqrt{\frac{1}{26}}\left(x - \sqrt{\frac{97}{169}}t\right)\right]. \tag{6.31}$$

Applying the modified variational iteration method, we obtain

$$u_{n+1}(x,t) = -\frac{105}{169} \sec h^4\left(\frac{x}{\sqrt{26}}\right) + \frac{-210\sqrt{194/13}\,\sec h^4(x/\sqrt{26})\,\tanh(x/\sqrt{26})}{2197}t$$
$$+ \int_0^t \lambda\left(\frac{\partial^2 u_n}{\partial t^2} - (\tilde{u}_n)_{xx} + (\tilde{u}_n)_{xxxx} - \frac{1}{2}(\tilde{u}_n)_{xxxxxx} + \sum_{n=0}^{\infty} b_n\right)dt. \tag{6.32}$$

Making the correct functional stationary, the Lagrange multiplier can be identified as $\lambda = x - t$, consequently

$$u_{n+1}(x,t) = -\frac{105}{169} \sec h^4\left(\frac{x}{\sqrt{26}}\right) + \frac{-210\sqrt{194/13}\,\sec h^4(x/\sqrt{26})\,\tanh(x/\sqrt{26})}{2197}t$$
$$+ \int_0^t (x-t)\left(\frac{\partial^2 u_n}{\partial t^2} - (\tilde{u}_n)_{xx} + (\tilde{u}_n)_{xxxx} - \frac{1}{2}(\tilde{u}_n)_{xxxxxx} + \sum_{n=0}^{\infty} B_n\right)dt, \tag{6.33}$$

where B_n are Adomian's polynomials for nonlinear operator $F(u) = u^2(x)$ and can be generated for all types of nonlinearities according to the algorithm developed in [42–44, 51]. Consequently, the following approximants are obtained:

$$u_0(x,t) = -\frac{105}{169} \sec h^4\left(\frac{x}{\sqrt{26}}\right),$$

$$u_1(x,t) = -\frac{105}{169} \sec h^4\left(\frac{x}{\sqrt{26}}\right) - \frac{105\sqrt{194/13} \sec h^6(x/\sqrt{26}) \sinh(\sqrt{2}x/\sqrt{13})}{2197}t$$

$$- \frac{105}{371293}\left(-291 + 194\cosh\left(\frac{\sqrt{2}x}{\sqrt{13}}\right)\right) \sec h^6 \frac{x}{\sqrt{26}} t^2,$$

$$u_2(x,t) = -\frac{105}{169} \sec h^4\left(\frac{x}{\sqrt{26}}\right) - \frac{105\sqrt{194/13} \sec h^6(x/\sqrt{26}) \sinh(\sqrt{2}x/\sqrt{13})}{2197}t$$

$$- \frac{105}{371293}\left(-291 + 194\cosh\left(\frac{\sqrt{2}x}{\sqrt{13}}\right)\right) \sec h^6 \frac{x}{\sqrt{26}} t^2$$

$$+ \frac{395 \sec h^7(x/\sqrt{26})}{52206766144}\left(10816\sqrt{2522} \sinh \frac{x}{\sqrt{26}} - 1664\sqrt{2522} \sinh \frac{3x}{\sqrt{26}}\right)t^3$$

$$+ \left(-334200 \sec h^5\left(\frac{x}{\sqrt{26}}\right) + 354247\cosh\left(\frac{2}{\sqrt{13}}x\right) \sec h^5\left(\frac{x}{\sqrt{26}}\right)\right.$$

$$\left. - 47164\cosh\left(\frac{2\sqrt{2}}{\sqrt{13}}x\right) \sec h^5\left(\frac{x}{\sqrt{26}}\right)\right)t^4$$

$$+ \left(3201\cosh^3\left(\frac{3\sqrt{2}}{\sqrt{13}}x\right) \sec h^5\left(\frac{x}{\sqrt{26}}\right) - 388\cosh\left(\frac{4\sqrt{2}}{\sqrt{13}}x\right) \sec h^5\left(\frac{x}{\sqrt{26}}\right)\right)t^4 + \cdots,$$

$$\vdots$$

(6.34)

The series solution is obtained as

$$u(x,t) = -\frac{105}{169} \sec h^4\left(\frac{x}{\sqrt{26}}\right) - \frac{105\sqrt{194/13} \sec h^6(x/\sqrt{26}) \sinh(\sqrt{2}x/\sqrt{13})}{2197}t$$

$$- \frac{105}{371293}\left(-291 + 194\cosh\left(\frac{\sqrt{2}x}{\sqrt{13}}\right)\right) \sec h^6 \frac{x}{\sqrt{26}} t^2$$

$$+ \frac{395 \sec h^7(x/\sqrt{26})}{52206766144}\left(10816\sqrt{2522} \sinh \frac{x}{\sqrt{26}} - 1664\sqrt{2522} \sinh \frac{3x}{\sqrt{26}}\right)t^3$$

$$+ \left(-334200 \sec h^5\left(\frac{x}{\sqrt{26}}\right) + 354247\cosh\left(\frac{2}{\sqrt{13}}x\right) \sec h^5\left(\frac{x}{\sqrt{26}}\right)\right.$$

$$\left. - 47164\cosh\left(\frac{2\sqrt{2}}{\sqrt{13}}x\right) \sec h^5\left(\frac{x}{\sqrt{26}}\right)\right)t^4$$

$$+ \left(3201\cosh^3\left(\frac{3\sqrt{2}}{\sqrt{13}}x\right) \sec h^5\left(\frac{x}{\sqrt{26}}\right) - 388\cosh\left(\frac{4\sqrt{2}}{\sqrt{13}}x\right) \sec h^5\left(\frac{x}{\sqrt{26}}\right)\right)t^4 + \cdots.$$

(6.35)

Table 6: The absolute error between the exact and the series solutions. Higher accuracy can be obtained by introducing some more components of the series solution.

| x_i	t_j					
	0.01	0.02	0.04	0.1	0.2	0.5
−1	$7.77156\,E-16$	$1.36557\,E-14$	$8.57869\,E-13$	$2.09264\,E-10$	$1.33823\,E-8$	$3.25944\,E-6$
−0.8	$1.11022\,E-16$	$1.99840\,E-15$	$1.12688\,E-13$	$2.73880\,E-11$	$1.74288\,E-9$	$4.14094\,E-7$
−0.6	$2.22045\,E-16$	$1.09912\,E-14$	$7.28861\,E-13$	$1.78030\,E-10$	$1.14025\,E-8$	$2.79028\,E-6$
−0.4	$1.11022\,E-16$	$2.32037\,E-14$	$1.50302\,E-12$	$3.67002\,E-10$	$2.34944\,E-8$	$5.74091\,E-6$
−0.2	$6.66134\,E-16$	$3.23075\,E-14$	$2.04747\,E-12$	$4.99918\,E-10$	$3.19983\,E-9$	$7.81509\,E-6$
0	$4.44089\,E-16$	$3.49720\,E-14$	$2.24365\,E-12$	$5.47741\,E-10$	$3.50559\,E-8$	$8.55935\,E-6$
0.2	$5.55112\,E-16$	$3.19744\,E-14$	$2.04714\,E-12$	$4.99820\,E-10$	$3.19858\,E-8$	$7.80749\,E-6$
0.4	$3.33067\,E-16$	$2.32037\,E-14$	$1.50324\,E-12$	$3.66815\,E-10$	$2.34706\,E-8$	$5.72641\,E-6$
0.6	$3.33067\,E-16$	$1.12133\,E-14$	$7.28528\,E-12$	$1.77772\,E-10$	$1.13695\,E-8$	$2.77022\,E-6$
0.8	$3.33067\,E-16$	$1.99840\,E-15$	$1.13132\,E-13$	$2.76944\,E-11$	$1.78208\,E-9$	$4.41936\,E-7$
1	$7.77156\,E-16$	$1.38778\,E-14$	$8.58313\,E-13$	$2.09593\,E-10$	$1.34244\,E-8$	$3.28504\,E-6$
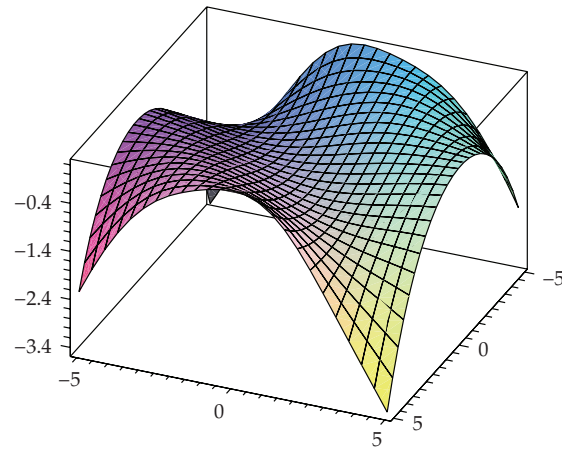
Figure 3

Example 6.5. Consider the following linear third-order dispersive KdV equation:

$$u_t + 2u_x + u_{xxx} = 0, \quad t > 0, \tag{6.36}$$

with initial condition

$$u(x,0) = \sin x. \tag{6.37}$$

The correct functional is given as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^x \lambda(s)\left(\frac{\partial u_n}{\partial t} + 2\frac{\partial \tilde{u}_n}{\partial x} + \frac{\partial^3 \tilde{u}_n}{\partial x^3}\right) ds. \tag{6.38}$$

Making the correct functional stationary, the Lagrange multiplier can be identified as $\lambda = -1$, consequently,

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^x \left(\frac{\partial u_n}{\partial t} + 2\frac{\partial \tilde{u}_n}{\partial x} + \frac{\partial^3 \tilde{u}_n}{\partial x^3} \right) ds. \tag{6.39}$$

Applying the modified variational iteration method,

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^x \left(\frac{\partial u_n}{\partial t} + 2\sum_{n=0}^\infty \frac{\partial \tilde{u}_n}{\partial x} + \sum_{n=0}^\infty \frac{\partial^3 \tilde{u}_n}{\partial x^3} \right) ds. \tag{6.40}$$

Consequently, the following approximants are obtained:

$$
\begin{aligned}
u_0(x,t) &= \sin x, \\
u_1(x,t) &= \sin x - t\cos x, \\
u_2(x,t) &= \sin x \left(1 - \frac{t^2}{2!} \right) - t\cos x, \\
u_3(x,t) &= \sin x \left(1 - \frac{t^2}{2!} \right) - \cos x \left(t - \frac{t^3}{3!} \right), \\
&\vdots
\end{aligned}
\tag{6.41}
$$

The series solution is given by

$$u(x,t) = \sin x \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots \right) - \cos x \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots \right), \tag{6.42}$$

and in a closed form by

$$u(x,t) = \sin(x - t). \tag{6.43}$$

If we change the initial condition as $u(x,0) = \cos x$, than the following closed-form solution will be obtained:

$$u(x,t) = \cos(x - t). \tag{6.44}$$

Example 6.6. Consider the following linear third-order dispersive KdV equation in a two-dimensional space:

$$u_t + u_{xxx} + u_{yyy} = 0, \quad t > 0, \tag{6.45}$$

with initial condition

$$u(x,y,0) = \cos(x + y). \tag{6.46}$$

The correct functional is given as

$$u_{n+1}(x,y,t) = u_n(x,y,t) + \int_0^x \lambda(s) \left(\frac{\partial u_n}{\partial t} + 2\frac{\partial^3 \tilde{u}_n}{\partial x^3} + \frac{\partial^3 \tilde{u}_n}{\partial y^3} \right) ds. \tag{6.47}$$
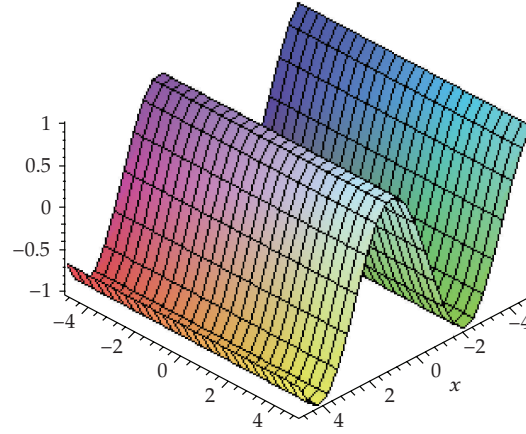
Figure 4

Making the correct functional stationary, the Lagrange multiplier can be identified as $\lambda = -1$, consequently,

$$u_{n+1}(x, y, t) = u_n(x, y, t) - \int_0^x \left(\frac{\partial u_n}{\partial t} + 2 \frac{\partial^3 \tilde{u}_n}{\partial x^3} + \frac{\partial^3 \tilde{u}_n}{\partial y^3} \right) ds. \tag{6.48}$$

Applying the modified variational iteration method,

$$u_{n+1}(x, y, t) = u_n(x, y, t) - \int_0^x \left(\frac{\partial u_n}{\partial t} + 2 \sum_{n=0}^{\infty} \frac{\partial^3 \tilde{u}_n}{\partial x^3} + \sum_{n=0}^{\infty} \frac{\partial^3 \tilde{u}_n}{\partial y^3} \right) ds. \tag{6.49}$$

Consequently, the following approximants are obtained:

$$\begin{aligned}
u_0(x, y, t) &= \cos(x + y), \\
u_1(x, y, t) &= \cos(x + y) - 2t \sin(x + y), \\
u_2(x, y, t) &= \cos(x + y) \left(1 - \frac{(2t)^2}{2!} \right) - (2t) \sin(x + y), \\
u_3(x, y, t) &= \cos(x + y) \left(1 - \frac{(2t)^2}{2!} \right) - \sin(x + y) \left((2t) - \frac{(2t)^3}{3!} \right),
\end{aligned} \tag{6.50}$$
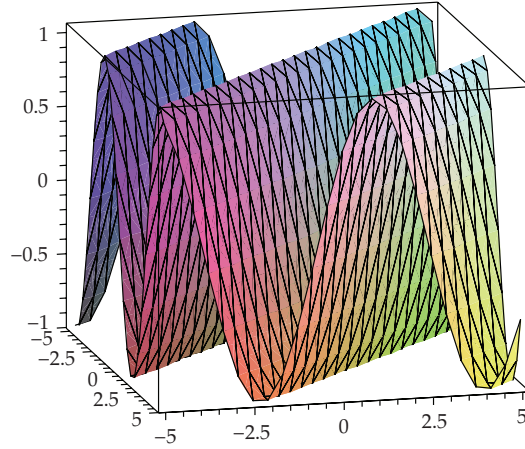
$$\vdots$$

The series solution is given by

$$u(x, y, t) = \cos(x + y) \left(1 - \frac{(2t)^2}{2!} + \cdots \right) - \sin(x + y) \left((2t) - \frac{(2t)^3}{3!} + \cdots \right), \tag{6.51}$$

and in a closed form by

$$u(x, y, t) = \cos(x + y + 2t). \tag{6.52}$$

If we change the initial condition as $u(x, y, 0) = \sin(x + y)$, than the following closed-form solution will be obtained:

$$u(x, y, t) = \sin(x + y + 2t). \tag{6.53}$$

Figure 5: $(t = 1)$.

Example 6.7 (see [42]). Consider the following singular fourth-order parabolic partial differential equation in two space variables:

$$\frac{\partial^2 u}{\partial t^2} + 2\left(\frac{1}{x^2} + \frac{x^4}{6!}\right)\frac{\partial^4 u}{\partial x^4} + 2\left(\frac{1}{y^2} + \frac{y^4}{6!}\right)\frac{\partial^4 u}{\partial y^4} = 0, \tag{6.54}$$

with initial conditions

$$u(x,y,0) = 0, \qquad \frac{\partial u}{\partial t}(x,y,0) = 2 + \frac{x^6}{6!} + \frac{y^6}{6!}, \tag{6.55}$$
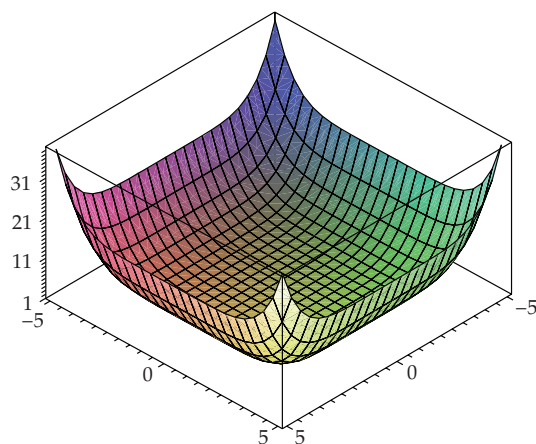
and the boundary conditions

$$u\left(\frac{1}{2},y,t\right) = \left(2 + \frac{(0.5)^6}{6!} + \frac{y^6}{6!}\right)\sin t, \qquad u(1,y,t) = \left(2 + \frac{1}{6!} + \frac{y^6}{6!}\right)\sin t,$$

$$\frac{\partial^2 u}{\partial x^2}\left(\frac{1}{2},y,t\right) = \frac{(0.5)^4}{24}\sin t, \qquad \frac{\partial^2 u}{\partial x^2}(1,y,t) = \frac{1}{24}\sin t, \tag{6.56}$$

$$\frac{\partial^2 u}{\partial y^2}\left(x,\frac{1}{2},t\right) = \frac{(0.5)^4}{24}\sin t, \qquad \frac{\partial^2 u}{\partial y^2}(x,1,t) = \frac{1}{24}\sin t.$$

The correct functional is given as

$$u_{n+1}(x,t) = u_0(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial^2 u_n}{\partial t^2} + 2\left(\frac{1}{x^2} + \frac{x^4}{6!}\right)\frac{\partial^4 \tilde{u}_n}{\partial x^4} + \left(2\left(\frac{1}{y^2} + \frac{y^4}{6!}\right)\frac{\partial^4 \tilde{u}_n}{\partial y^4}\right)\right)d\xi, \tag{6.57}$$

where $\tilde{u}_n$ is considered as a restricted variation. Making the above functional stationary, the Lagrange multiplier can be identified as $\lambda = \xi - t$, consequently,

$$u_{n+1}(x,t) = u_0(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 u_n}{\partial t^2} + 2\left(\frac{1}{x^2} + \frac{x^4}{6!}\right)\frac{\partial^4 \tilde{u}_n}{\partial x^4} + \left(2\left(\frac{1}{y^2} + \frac{y^4}{6!}\right)\frac{\partial^4 \tilde{u}_n}{\partial y^4}\right)\right)d\xi. \tag{6.58}$$

Figure 6: $(t = 1)$.

Applying the modified variational iteration method, we have

$$u_{n+1}(x,t) = u_0(x,t) + \int_0^t (\xi - t) \left(\frac{\partial^2 u_n}{\partial t^2} + 2 \left(\frac{1}{x^2} + \frac{x^4}{6!} \right) \sum_{n=0}^{\infty} \left(\frac{\partial^4 \tilde{u}_n}{\partial x^4} \right) + \left(2 \left(\frac{1}{y^2} + \frac{y^4}{6!} \right) \sum_{n=0}^{\infty} \left(\frac{\partial^4 \tilde{u}_n}{\partial y^4} \right) \right) \right) d\xi.$$

(6.59)

Consequently, the following approximants are obtained as:

$$u_0(x,t) = \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) t$$

$$u_1(x,t) = \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) \left(t - \frac{t^3}{3!} \right)$$

$$u_2(x,t) = \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} \right),$$

$$u_3(x,t) = \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} \right),$$

$$u_4(x,t) = \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} \right),$$

$$\vdots$$

(6.60)

The solution is given by

$$u(x,y,t) = \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} + \cdots \right) = \left(2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) \sin t. \quad (6.61)$$

Remark 6.8. It is worth mentioning that Ghorbani and Saberi-Nadjafi [49] and Ghorbani [50] introduced He polynomials which are compatible to Adomian's polynomials, are easier to calculate, and hence make the solution procedure simpler.

7. Conclusion

In this paper, we applied the modified variational iteration method (MVIM) for solving singular and nonsingular initial and boundary value problems. The proposed technique is applied on boundary layer problem, unsteady flow of gas, Boussinesq equations, third-order dispersive and fourth-order parabolic partial differential equations. The Pade approximants were employed in order to make the work more concise and for better understanding of the solution behavior. It may be concluded that the proposed frame work is very powerful and efficient in finding the analytical solutions for singular and nonsingular boundary value problems. The method gives more realistic series solutions that converge very rapidly in physical problems.

Acknowledgments

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