Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2008, Article ID 942391, 10 pages
doi:10.1155/2008/942391

Research Article

A Modified Levenberg-Marquardt Method for Nonsmooth Equations with Finitely Many Maximum Functions

Shou-qiang Du[1, 2] and Yan Gao[1]

[1] *School of Management, University of Shanghai for Science and Technology, Shanghai 200093, China*
[2] *College of Mathematics, Qingdao University, Qingdao 266071, China*

Correspondence should be addressed to Shou-qiang Du, dsq8933@163.com

Received 2 August 2008; Accepted 26 November 2008

Recommended by Shijun Liao

For solving nonsmooth systems of equations, the Levenberg-Marquardt method and its variants are of particular importance because of their locally fast convergent rates. Finitely many maximum functions systems are very useful in the study of nonlinear complementarity problems, variational inequality problems, Karush-Kuhn-Tucker systems of nonlinear programming problems, and many problems in mechanics and engineering. In this paper, we present a modified Levenberg-Marquardt method for nonsmooth equations with finitely many maximum functions. Under mild assumptions, the present method is shown to be convergent Q-linearly. Some numerical results comparing the proposed method with classical reformulations indicate that the modified Levenberg-Marquardt algorithm works quite well in practice.

Copyright © 2008 S.-q. Du and Y. Gao. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

In the past few years, there has been a growing interest in the study of nonlinear equations (see, e.g., [1, 2]) and nonsmooth equations, which have been proposed in the study of the nonlinear complementarity problem, the variational inequality problem, equilibrium problem and engineering mechanics (see, e.g., [3–10]).

Finitely many maximum functions systems are very useful in the study of nonlinear complementarity problems, variational inequality problems, Karush-Kuhn-Tucker systems of nonlinear programming problems, and many problems in mechanics and engineering. In the present paper, we study a new method for nonsmooth equations with finitely many maximum functions system proposed in [11]

$$\begin{gathered}\max_{j\in J_1} f_{1j}(x)=0,\\ \vdots\\ \max_{j\in J_n} f_{nj}(x)=0,\end{gathered}\tag{1.1}$$

where $f_{ij} : R^n \to R$ for $j \in J_i$, $i = 1, \ldots, n$ are continuously differentiable, J_i for $i = 1, \ldots, n$ are finite index sets. Denote

$$H(x) = \left(f_1(x), \ldots, f_n(x)\right)^T, \quad x \in R^n, \tag{1.2}$$

where

$$\begin{gathered} f_i(x) = \max_{j \in J_i} f_{ij}(x), \quad x \in R^n,\ i = 1, \ldots, n, \\ J_i(x) = \left\{ j_i \in N \mid f_{ij}(x) = f_i(x) \right\}, \quad x \in R^n,\ i = 1, \ldots, n. \end{gathered} \tag{1.3}$$

Then (1.1) can be rewritten as follows:

$$H(x) = 0, \tag{1.4}$$

where $F : R^n \to R^n$ is a nonsmooth function. By using the following subdifferential for the function $H(x)$ given in (1.2),

$$\partial_\star H(x) = \left\{ \left(\nabla f_{1j_1}, \ldots, \nabla f_{nj_n}\right)^T \mid j_1 \in J_1(x), \ldots, j_n \in J_n(x) \right\}, \quad x \in R^n, \tag{1.5}$$

Gao gave Newton method for (1.4) with the superlinear convergence in [11].

Based on [5, 11], we present a modification of the Levenberg-Marquardt method for solving nonsmooth equations. In Section 2, we recall some results of generalized Jacobian and semismoothness. In Section 3, we give the Levenberg-Marquardt method which has been proposed in [5] and the new modified Levenberg-Marquardt method for the system of nonsmooth equations with finitely many maximum functions. The convergence of the modified Levenberg-Marquardt algorithm is also given. In Section 4, some numerical tests comparing the proposed modified Levenberg-Marquardt algorithm with the original method show that our algorithm works quite well.

2. Preliminaries

We start with some notions and propositions, which can be found in [8–11].

Let $F(x)$ be locally Lipschitzian. Then, $F(x)$ is almost everywhere F-differentiable. Let the set of points where $F(x)$ is F-differentiable be denoted by D_F. Then for $x \in R^n$,

$$\partial_B F(x) = \left\{ V \in R^{n \times n} \mid \exists \{x_k\} \in D_F,\ x_k \longrightarrow x,\ F'(x_k) \longrightarrow V \right\}. \tag{2.1}$$

The general Jacobian of $F(x) : R^n \to R^n$ at x in the sense of Clarke is defined as

$$\partial F(x) = \operatorname{conv} \partial_B F(x). \tag{2.2}$$

Proposition 2.1. *$\partial_B F(x)$ is a nonempty and compact set for any x; the point to set B-subdifferential map is upper semicontinuous.*

Proposition 2.2. *$\partial_\star H(x)$ is a nonempty and compact set for any x and upper semicontinuous.*

Proof. From the fact that $\partial_\star H(x)$ is a finite set of points in $R^{n \times n}$ and can be calculated by determining the index sets $J_i(x)$, $i = 1, \ldots, n$ and evaluating the gradients $\nabla f_{ij_i}(x)$, $j_i \in J_i(x)$, $i = 1, \ldots, n$. □

Definition 2.3. $F(x)$ is semismooth at x if $F(x)$ is locally Lipschitz at x and

$$\lim_{\substack{V\in\partial F(x+th')\\ h'\to h,t\downarrow 0}} Vh' \tag{2.3}$$

exists for all $h \in R^n$. If $F(x)$ is semismooth at x, one knows $Vh - F'(x;h) = o(\|h\|)$, $\forall V \in \partial F(x+h)$, $h \to 0$. If for all $V \in \partial F(x+h)$, $h \to 0$, $Vh - F'(x;h) = o(\|h\|^2)$, one calls the function $F(x)$ is strongly semismooth at x.

Proposition 2.4. (I) *If $F(x) : R^n \to R^n$ is locally Lipschitz continuous and semismooth at x, then*

$$\lim_{\substack{V\in\partial F(x+th)\\ h\to 0}} \frac{\|F(x+h)-F(x)-Vh\|}{\|h\|} = 0. \tag{2.4}$$

(II) *If $F(x) : R^n \to R^n$ is locally Lipschitz continuous, strongly semismooth at x, and directionally differentiable in a neighborhood of x, then*

$$\limsup_{\substack{V\in\partial F(x+th)\\ h\to 0}} \frac{\|F(x+h)-F(x)-Vh\|}{\|h\|^2} < \infty. \tag{2.5}$$

Lemma 2.5. *Equation of maximum functions* (1.4) *is a system of semismooth equations.*

In the study of algorithms for the local solution of semismooth systems of equations, similar to [11], one also has the following lemmas.

Lemma 2.6. *Suppose that $H(x)$ and $\partial_\star H(x)$ are defined by* (1.4) *and by* (1.5)*, respectively, and all $V \in \partial_\star H(x)$ are nonsingular. Then there exists a constant c such that*

$$\|V^{-1}\| \le c, \quad \forall V \in \partial_\star H(x). \tag{2.6}$$

The proof is similar to [11, Lemma 2.1], from the fact that $\partial_\star H(x)$ is a finite set of points.

Lemma 2.7. *Suppose that $x^\star$ is a solution of* (1.1)*, then*

$$\|\mathrm{diag}(\lambda_i^{(k)} f_i(x_k))\| \le M, \tag{2.7}$$

for all x in some neighborhood of $x^\star$ and $\lambda_i^{(k)} \in R$ and $0 < |\lambda_i^{(k)}| < +\infty$ for $i = 1,\dots,n$, $k = 0,1,2,\dots$.

Since each f_{ij} of (1.1) is continuous, one gets the lemma immediately.

3. Modified Levenberg-Marquardt method and its convergence

In this section, we briefly recall some results on the Levenberg-Marquardt-type method for the solution of nonsmooth equations and their local convergence (see, e.g., [5, 9]). We also give the modified Levenberg-Marquardt method and analyze its local behavior. Now we consider exact and inexact versions of Levenberg-Marquardt method.

Table 1: $x_0 = (1,1)^T \lambda_1 = 0.01, \lambda_2 = 1$ and computes d_k by (3.4).

Step	$(x_1, x_2)^T$	$F(x)$
1	$(1.0000, 1.0000)^T$	$(1.0000, 1.0000)^T$
2	$(0.5006, 1.0000)^T$	$(0.2506, 0.2506)^T$
3	$(0.2506, 1.0000)^T$	$(0.0628, 0.0628)^T$
4	$(0.1255, 1.0000)^T$	$(0.0157, 0.0157)^T$
5	$(0.0628, 1.0000)^T$	$(0.0039, 0.0039)^T$
6	$(0.0314, 1.0000)^T$	$1.0e - 003{*}(0.9888, 0.9888)^T$
7	$(0.0157, 1.0000)^T$	$1.0e - 003{*}(0.2478, 0.2478)^T$
8	$(0.0079, 1.0000)^T$	$1.0e - 004{*}(0.6211, 0.6211)^T$
9	$(0.0039, 1.0000)^T$	$1.0e - 004{*}(0.1557, 0.1557)^T$
10	$(0.0019, 1.0000)^T$	$1.0e - 005{*}(0.3901, 0.3901)^T$
11	$(0.00098, 1.0000)^T$	$1.0e - 006{*}(0.97777, 0.97777)^T$
12	$(0.000495, 1.0000)^T$	$1.0e - 006{*}(0.24505, 0.24505)^T$
13	$(0.00024, 1.0000)^T$	$1.0e - 007{*}(0.61416, 0.61416)^T$
14	$(0.00012, 1.0000)^T$	$1.0e - 007{*}(0.15392, 0.15392)^T$
15	$(0.000062, 1.0000)^T$	$1.0e - 008{*}(0.38577, 0.38577)^T$

Given a starting vector $x_0 \in R^n$, let

$$x_{k+1} = x_k + d_k, \tag{3.1}$$

where d_k is the solution of the system

$$\left((V_k)^T V_k + \sigma_k I\right)d = -(V_k)^T H(x_k), \quad V_k \in \partial_B H(x_k),\ \sigma_k \geq 0. \tag{3.2}$$

In the inexact versions of this method d_k can be given by the solution of the system

$$\left((V_k)^T V_k + \sigma_k I\right)d = -(V_k)^T H(x_k) + r_k, \quad V_k \in \partial_B H(x_k),\ \sigma_k \geq 0, \tag{3.3}$$

where r_k is the vector of residuals and we can assume $\|r_k\| \leq \alpha_k \|(V_k)^T H(x_k)\|$ for some $\alpha_k \geq 0$.

We now give the modified Levenberg-Marquardt method for (1.1) as follows.

Modified Levenberg-Marquardt Method

Step 1. Given x_0, $\epsilon > 0$, $\lambda_i^k \in R^n$, $0 < |\lambda_i^k| < +\infty$.

Step 2. Solve the system to get d_k,

$$\left((V_k)^T V_k + \operatorname{diag}\left(\lambda_i^{(k)} f_i(x_k)\right)\right)d_k = -(V_k)^T H(x_k) + r_k, \quad V_k \in \partial_\star H(x), \tag{3.4}$$

for $i = 1, \ldots, n$ and r_k is the vector of residuals

$$\|r_k\| \leq \alpha_k \|(V_k)^T H(x_k)\|, \quad \alpha_k \geq 0. \tag{3.5}$$

Step 3. Set $x_{k+1} = x_k + d_k$, if $\|H(x_k)\| \leq \epsilon$, terminate. Otherwise, let $k := k+1$, and go to Step 2.

Based upon the above analysis, we give the following local convergence result.

Table 2: $x_0 = (1,1)^T \sigma_1 = 0.01$, $\sigma_2 = 1$ and computes d_k by (3.3).

Step	$(x_1, x_2)^T$	$F(x)$
1	$(1.0000, 1.0000)^T$	$(1.0000, 1.0000)^T$
2	$(0.5006, 1.0000)^T$	$(0.2506, 0.2506)^T$
3	$(0.2516, 1.0000)^T$	$(0.0633, 0.0633)^T$
4	$(0.1282, 1.0000)^T$	$(0.0164, 0.0164)^T$
5	$(0.0686, 1.0000)^T$	$(0.0047, 0.0047)^T$
6	$(0.0415, 1.0000)^T$	$(0.0017, 0.0017)^T$
7	$(0.0295, 1.0000)^T$	$1.0e - 003*(0.8693, 0.8693)^T$
8	$(0.0234, 1.0000)^T$	$1.0e - 003*(0.5493, 0.5493)^T$
9	$(0.0199, 1.0000)^T$	$1.0e - 003*(0.3944, 0.3944)^T$
10	$(0.0175, 1.0000)^T$	$1.0e - 003*(0.3055, 0.3055)^T$
11	$(0.0158, 1.0000)^T$	$1.0e - 003*(0.2484, 0.2484)^T$
12	$(0.0145, 1.0000)^T$	$1.0e - 003*(0.2089, 0.2089)^T$
13	$(0.0134, 1.0000)^T$	$1.0e - 003*(0.1801, 0.1801)^T$
14	$(0.0126, 1.0000)^T$	$1.0e - 003*(0.1581, 0.1581)^T$
15	$(0.0119, 1.0000)^T$	$1.0e - 003*(0.1409, 0.1409)^T$
16	$(0.0112, 1.0000)^T$	$1.0e - 003*(0.12696, 0.12696)^T$
17	$(0.0107, 1.0000)^T$	$1.0e - 003*(0.1155, 0.1155)^T$
18	$(0.0103, 1.0000)^T$	$1.0e - 003*(0.10596, 0.10596)^T$

Table 3: $x_0 = (1,1)^T \sigma_1 = 0.01$, $\sigma_2 = 1$ and computes d_k by (3.3).

Step	$(x_1, x_2)^T$	$F(x)$
19	$(0.00989, 1.0000)^T$	$1.0e - 004*(0.9784, 0.9784)^T$
20	$(0.0095, 1.0000)^T$	$1.0e - 004*(0.9087, 0.9087)^T$
21	$(0.0092, 1.0000)^T$	$1.0e - 004*(0.8481, 0.8481)^T$
22	$(0.0089, 1.0000)^T$	$1.0e - 004*(0.7951, 0.7951)^T$
23	$(0.0087, 1.0000)^T$	$1.0e - 004*(0.7483, 0.7483)^T$
24	$(0.0084, 1.0000)^T$	$1.0e - 004*(0.7066, 0.7066)^T$
25	$(0.0082, 1.0000)^T$	$1.0e - 004*(0.6693, 0.6693)^T$
26	$(0.00797, 1.0000)^T$	$1.0e - 004*(0.6357, 0.6357)^T$
27	$(0.0078, 1.0000)^T$	$1.0e - 004*(0.6053, 0.6053)^T$
28	$(0.0076, 1.0000)^T$	$1.0e - 004*(0.5777, 0.5777)^T$
29	$(0.0074, 1.0000)^T$	$1.0e - 004*(0.5525, 0.5525)^T$
30	$(0.0073, 1.0000)^T$	$1.0e - 004*(0.5293, 0.5293)^T$
31	$(0.0071, 1.0000)^T$	$1.0e - 004*(0.5080, 0.5080)^T$
32	$(0.00699, 1.0000)^T$	$1.0e - 004*(0.4884, 0.4884)^T$
33	$(0.0069, 1.0000)^T$	$1.0e - 004*(0.4702, 0.4702)^T$
34	$(0.0067, 1.0000)^T$	$1.0e - 004*(0.4533, 0.4533)^T$
35	$(0.0066, 1.0000)^T$	$1.0e - 004*(0.4376, 0.4376)^T$
36	$(0.0065, 1.0000)^T$	$1.0e - 004*(0.4229, 0.4229)^T$
37	$(0.0064, 1.0000)^T$	$1.0e - 004*(0.4092, 0.4092)^T$
38	$(0.0063, 1.0000)^T$	$1.0e - 004*(0.3963, 0.3963)^T$
39	$(0.0062, 1.0000)^T$	$1.0e - 004*(0.3842, 0.3842)^T$

Theorem 3.1. *Suppose that $\{x_k\}$ is a sequence generated by the above method and there exist constants $a > 0$, $\alpha_k \leq a$ for all k. Let $x^\star$ be a solution of $H(x) = 0$, and let all $V \in \partial_\star H(x^\star)$ be nonsingular. Then the sequence $\{x_k\}$ converges Q-linearly to $x^\star$ for $\|x_0 - x^\star\| \leq \epsilon$.*

Proof. By Lemma 2.6 and the continuously differentiable of $f_i(x)$, there is a constant $C > 0$ such that for all x_k sufficiently close to $x^\star (V_k)^T V_k + \mathrm{diag}(\lambda_i^{(k)} f_i(x_k))$ are nonsingular with

$$\left\| \left[(V_k)^T V_k + \mathrm{diag}(\lambda_i^{(k)} f_i(x_k)) \right]^{-1} \right\| \leq C. \tag{3.6}$$

Furthermore, by Proposition 2.4, there exists $\delta > 0$, which can be taken arbitrarily small, such that

$$\| H(x_k) - H(x^\star) - V_k(x_k - x^\star) \| \leq \delta \| x_k - x^\star \| \tag{3.7}$$

for all x_k in a sufficiently small neighborhood of $x^\star$ depending on δ. By Proposition 2.2 the upper semicontinuity of the $\partial_\star H(x)$, we also know

$$\| (V_k)^T \| \leq c_1, \tag{3.8}$$

for all $V_k \in \partial_\star H(x)$ and all x_k sufficiently close to $x^\star$, with $c_1 > 0$ being a suitable constant. From the locally Lipschitz continuous of $H(x)$, we have

$$\| (V_k)^T H(x_k) \| \leq \| (V_k)^T \| \, \| H(x_k) - H(x^\star) \| \leq c_1 L \| x_k - x^\star \|, \tag{3.9}$$

for all x_k in a sufficiently small neighborhood of $x^\star$ and a constant $L > 0$. From (3.4), we also know

$$\begin{aligned} &\left[(V_k)^T V_k + \mathrm{diag}(\lambda_i^{(k)} f_i(x_k)) \right] (x_{k+1} - x^\star) \\ &\quad = \left[(V_k)^T V_k + \mathrm{diag}(\lambda_i^{(k)} f_i(x_k)) \right] (x_k - x^\star) - (V_k)^T H(x_k) + r_k \\ &\quad = (V_k)^T \left[H(x^\star) - H(x_k) + V_k(x_k - x^\star) \right] + \mathrm{diag}(\lambda_i^{(k)} f_i(x_k)) (x_k - x^\star) + r_k. \end{aligned} \tag{3.10}$$

Multiply the above equation by $[(V_k)^T V_k + \mathrm{diag}(\lambda_i^{(k)} f_i(x_k))]^{-1}$ and taken into account Lemma 2.7, and (3.6), (3.7), (3.8), and (3.9), we get

$$\begin{aligned} \| x_{k+1} - x^\star \| &\leq C \Big(\| (V_k)^T \| \, \| H(x_k) - H(x^\star) - V_k(x_k - x^\star) \| \\ &\qquad + \| \mathrm{diag}(\lambda_i^{(k)} f_i(x_k)) \| \, \| x_k - x^\star \| + a \| (V_k)^T H(x_k) \| \Big) \\ &\leq C \left(c_1 \delta \| x_k - x^\star \| + M \| x_k - x^\star \| + a c_1 L \| x_k - x^\star \| \right) \\ &= C (c_1 \delta + M + a c_1 L) \| x_k - x^\star \|. \end{aligned} \tag{3.11}$$

Let $\tau = C(c_1 \delta + M + a c_1 L)$, so

$$\| x_{k+1} - x^\star \| \leq \tau \| x_k - x^\star \|. \tag{3.12}$$

Since δ can be chosen arbitrarily small, by taking x_k sufficiently close to $x^\star$, there exist $M > 0$ and $a > 0$ such that $\tau < 1$, so that the Q-linear convergence of $\{x_k\}$ to $x^\star$ follows by taking $\|x_0 - x^\star\| \leq \epsilon$ for a small enough $\epsilon > 0$. Thus we complete the proof of the theorem. □

Table 4: $x_0 = (10, 1)^T \lambda_1 = 0.01$, $\lambda_2 = 1$ and computes d_k by (3.4).

Step	$(x_1, x_2)^T$	$F(x)$
1	$(10.0000, 1.0000)^T$	$(100.0000, 100.0000)^T$
2	$(5.0062, 1.0000)^T$	$(25.0625, 25.0625)^T$
3	$(2.5062, 1.0000)^T$	$(6.2813, 6.2813)^T$
4	$(1.2547, 1.0000)^T$	$(1.5742, 1.5742)^T$
5	$(0.6281, 1.0000)^T$	$(0.3945, 0.3945)^T$
6	$(0.3145, 1.0000)^T$	$(0.0989, 0.0989)^T$
7	$(0.1574, 1.0000)^T$	$(0.0248, 0.0248)^T$
8	$(0.0788, 1.0000)^T$	$(0.0062, 0.0062)^T$
9	$(0.0395, 1.0000)^T$	$(0.0016, 0.0016)^T$
10	$(0.0198, 1.0000)^T$	$1.0e - 003{*}(0.3901, 0.3901)^T$
11	$(0.0099, 1.0000)^T$	$1.0e - 004{*}(0.9778, 0.9778)^T$
12	$(0.00495, 1.0000)^T$	$1.0e - 004{*}(0.2451, 0.2451)^T$
13	$(0.0025, 1.0000)^T$	$1.0e - 005{*}(0.6142, 0.6142)^T$
14	$(0.0012, 1.0000)^T$	$1.0e - 005{*}(0.1539, 0.1539)^T$
15	$(0.0006, 1.0000)^T$	$1.0e - 006{*}(0.3858, 0.3858)^T$

Table 5: $x_0 = (10, 1)^T \lambda_1 = 0.01$, $\lambda_2 = 1$ and computes d_k by (3.4).

Step	$(x_1, x_2)^T$	$F(x)$
16	$(0.0003, 1.0000)^T$	$1.0e - 007{*}(0.9668, 0.9668)^T$
17	$(0.0002, 1.0000)^T$	$1.0e - 007{*}(0.2423, 0.2423)^T$
18	$(0.00008, 1.0000)^T$	$1.0e - 008{*}(0.6073, 0.6073)^T$

Theorem 3.2. *Suppose that $\{x_k\}$ is a sequence generated by the above method and there exist constants $a > 0$, $\alpha_k \le a$ for all k. $\|r_k\| \le \alpha_k \|H(x_k)\|$, $\alpha_k \ge 0$. Then the sequence $\{x_k\}$ converges Q-linearly to $x^\star$ for $\|x_0 - x^\star\| \le \epsilon$.*

The proof is similar to that of Theorem 3.1, so we omit it.

Following the proof of Theorem 3.1, the following statement holds.

Remark 3.3. Theorems 3.1 and 3.2 hold with $\|r_k\| = 0$ in (3.4).

4. Numerical test

In order to show the performance of the modified Levenberg-Marquardt method, in this section, we present numerical results and compare the Levenberg-Marquardt method and modified Levenberg-Marquardt method. The results indicate that the modified Levenberg-Marquardt algorithm works quite well in practice. All the experiments were implemented in Matlab 7.0.

Example 4.1.

$$\begin{aligned} &\max\{f_{11}(x_1, x_2), f_{12}(x_1, x_2)\} = 0, \\ &\max\{f_{21}(x_1, x_2), f_{22}(x_1, x_2)\} = 0, \end{aligned} \tag{4.1}$$

Table 6: $x_0 = (10, 1)^T \lambda_1 = 0.01$, $\lambda_2 = 1$ and computes d_k by (3.3).

Step	$(x_1, x_2)^T$	$F(x)$
1	$(10.0000, 1.0000)^T$	$(100.0000, 100.0000)^T$
2	$(5.0001, 1.0000)^T$	$(25.0006, 25.0006)^T$
3	$(2.5002, 1.0000)^T$	$(6.2508, 6.2508)^T$
4	$(1.2503, 1.0000)^T$	$(1.5633, 1.5633)^T$
5	$(0.6257, 1.0000)^T$	$(0.3915, 0.3915)^T$
6	$(0.3138, 1.0000)^T$	$(0.0985, 0.0985)^T$
7	$(0.1589, 1.0000)^T$	$(0.0252, 0.0252)^T$
8	$(0.0832, 1.0000)^T$	$(0.0069, 0.0069)^T$
9	$(0.0479, 1.0000)^T$	$(0.0023, 0.0023)^T$
10	$(0.0324, 1.0000)^T$	$(0.0011, 0.0011)^T$
11	$(0.0250, 1.0000)^T$	$1.0e - 003{*}(0.6258, 0.6258)^T$
12	$(0.0208, 1.0000)^T$	$1.0e - 003{*}(0.4345, 0.4345)^T$
13	$(0.0182, 1.0000)^T$	$1.0e - 003{*}(0.3296, 0.3296)^T$
14	$(0.0163, 1.0000)^T$	$1.0e - 003{*}(0.2644, 0.2644)^T$
15	$(0.0148, 1.0000)^T$	$1.0e - 003{*}(0.2203, 0.2203)^T$
16	$(0.0137, 1.0000)^T$	$1.0e - 003{*}(0.1885, 0.1885)^T$
17	$(0.0128, 1.0000)^T$	$1.0e - 003{*}(0.1646, 0.1646)^T$
18	$(0.0121, 1.0000)^T$	$1.0e - 003{*}(0.1460, 0.1460)^T$
19	$(0.0115, 1.0000)^T$	$1.0e - 003{*}(0.1311, 0.1311)^T$
20	$(0.0109, 1.0000)^T$	$1.0e - 003{*}(0.11899, 0.11899)^T$
21	$(0.0104, 1.0000)^T$	$1.0e - 003{*}(0.1089, 0.1089)^T$
22	$(0.0100, 1.0000)^T$	$1.0e - 003{*}(0.1003, 0.1003)^T$
23	$(0.0096, 1.0000)^T$	$1.0e - 004{*}(0.9301, 0.9301)^T$
24	$(0.0093, 1.0000)^T$	$1.0e - 004{*}(0.8668, 0.8668)^T$
25	$(0.0090, 1.0000)^T$	$1.0e - 004{*}(0.8115, 0.8115)^T$
26	$(0.0087, 1.0000)^T$	$1.0e - 004{*}(0.7628, 0.7628)^T$
27	$(0.0085, 1.0000)^T$	$1.0e - 004{*}(0.7195, 0.7195)^T$
28	$(0.0083, 1.0000)^T$	$1.0e - 004{*}(0.6809, 0.6809)^T$
29	$(0.0080, 1.0000)^T$	$1.0e - 004{*}(0.6462, 0.6462)^T$
30	$(0.0078, 1.0000)^T$	$1.0e - 004{*}(0.6148, 0.6148)^T$

where

$$f_{11} = \frac{1}{5}x_1^2, \qquad f_{12} = x_1^2, \qquad f_{21} = \frac{1}{3}x_1^2, \qquad f_{22} = x_1^2. \tag{4.2}$$

From (1.1), we know

$$H(x) = \left(f_1(x), f_2(x)\right)^T, \tag{4.3}$$

where $f_1(x) = x_1^2$, $f_2(x) = x_1^2$, $x \in R^2$.

Our subroutine computes d_k such that (3.3) and (3.4) hold with $\alpha_k = 0$. We also use the condition $\|x_k - x_{k-1}\| \leq 10^{-4}$ as the stopping criterion. We can see that our method is good for Example 4.1.

Results for Example 4.1 with initial point x_0= $(1, 1)^T \lambda_1 = 0.01$, $\lambda_2 = 1$ and computes d_k by (3.4) are listed in Table 1.

Table 7: $x_0 = (10, 1)^T \sigma_1 = 0.01$, $\sigma_2 = 1$ and computes d_k by (3.3).

Step	$(x_1, x_2)^T$	$F(x)$
31	$(0.0077, 1.0000)^T$	$1.0e - 004{*}(0.5863, 0.5863)^T$
32	$(0.0075, 1.0000)^T$	$1.0e - 004{*}(0.5603, 0.5603)^T$
33	$(0.0073, 1.0000)^T$	$1.0e - 004{*}(0.5366, 0.5366)^T$
34	$(0.0072, 1.0000)^T$	$1.0e - 004{*}(0.5147, 0.5147)^T$
35	$(0.0070, 1.0000)^T$	$1.0e - 004{*}(0.4946, 0.4946)^T$
36	$(0.0069, 1.0000)^T$	$1.0e - 004{*}(0.4759, 0.4759)^T$
37	$(0.0068, 1.0000)^T$	$1.0e - 004{*}(0.4586, 0.4586)^T$
38	$(0.0067, 1.0000)^T$	$1.0e - 004{*}(0.4425, 0.4425)^T$
39	$(0.0065, 1.0000)^T$	$1.0e - 004{*}(0.4275, 0.4275)^T$
40	$(0.0064, 1.0000)^T$	$1.0e - 004{*}(0.4135, 0.4135)^T$
41	$(0.0063, 1.0000)^T$	$1.0e - 004{*}(0.4004, 0.4004)^T$
42	$(0.0062, 1.0000)^T$	$1.0e - 004{*}(0.3880, 0.3880)^T$

Results for Example 4.1 with initial point $x_0 = (1, 1)^T \sigma_1 = \lambda_1 = 0.01$, $\sigma_2 = \lambda_2 = 1$ and computes d_k by (3.3) are listed in Tables 2 and 3.

Results for Example 4.1 with initial point $x_0 = (10, 1)^T \lambda_1 = 0.01$, $\lambda_2 = 1$ and computes d_k by (3.4) are listed in Tables 4 and 5.

Results for Example 4.1 with initial point $x_0 = (10, 1)^T \sigma_1 = \lambda_1 = 0.01$, $\sigma_2 = \lambda_2 = 1$ and computes d_k by (3.3) are listed in Tables 6 and 7.

Results are shown for Example 4.1 with initial point $x_0 = (100, 1)^T$. We also use the condition $\|x_k - x_{k-1}\| \leq 10^{-4}$ as the stopping criterion and computes d_k by (3.4) we get that by 21 steps $F(x) = 1.0e - 008{*}(0.9560, 0.9560)^T$. When we compute d_k by (3.3), we get that by 45 steps $F(x) = 1.0e - 004{*}(0.3919, 0.3919)^T$. We can test the method with other examples and will think the global convergence of the method in another paper.

Acknowledgments

This work was supported by National Science Foundation of China (under Grant: 10671126). The Innovation Fund Project for Graduate Student of Shanghai (JWCXSL0801) and key project for Fundamental Research of STCSM (Project no. 06JC14057) and Shanghai Leading Academic Discipline Project (S30501). The authors are also very grateful to referees for valuable suggestions and comments.

References

[1] C. Kanzow, N. Yamashita, and M. Fukushima, "Levenberg-Marquardt methods with strong local convergence properties for solving nonlinear equations with convex constraints," *Journal of Computational and Applied Mathematics*, vol. 172, no. 2, pp. 375–397, 2004.

[2] J.-Y. Fan and Y.-X. Yuan, "On the quadratic convergence of the Levenberg-Marquardt method without nonsingularity assumption," *Computing*, vol. 74, no. 1, pp. 23–39, 2005.

[3] L. Qi and P. Tseng, "On almost smooth functions and piecewise smooth functions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 3, pp. 773–794, 2007.

[4] E. G. Birgin and J. M. Martínez, "Structured minimal-memory inexact quasi-Newton method and secant preconditioners for augmented Lagrangian optimization," *Computational Optimization and Applications*, vol. 39, no. 1, pp. 1–16, 2008.

[5] F. Facchinei and C. Kanzow, "A nonsmooth inexact Newton method for the solution of large-scale nonlinear complementarity problems," *Mathematical Programming*, vol. 76, no. 3, pp. 493–512, 1997.

[6] D. Sun and J. Han, “Newton and quasi-Newton methods for a class of nonsmooth equations and related problems,” *SIAM Journal on Optimization*, vol. 7, no. 2, pp. 463–480, 1997.

[7] X. J. Chen and L. Q. Qi, “A parameterized Newton method and a quasi-Newton method for nonsmooth equations,” *Computational Optimization and Applications*, vol. 3, no. 2, pp. 157–179, 1994.

[8] L. Q. Qi and J. Sun, “A nonsmooth version of Newton’s method,” *Mathematical Programming*, vol. 58, no. 3, pp. 353–367, 1993.

[9] N. Yamashita and M. Fukushima, “Modified Newton methods for solving a semismooth reformulation of monotone complementarity problems,” *Mathematical Programming*, vol. 76, no. 3, pp. 469–491, 1997.

[10] Z. Yu, “The bounded smooth reformulation and a trust region algorithm for semidefinite complementarity problems,” *Applied Mathematics and Computation*, vol. 159, no. 1, pp. 157–170, 2004.

[11] Y. Gao, “Newton methods for solving two classes of nonsmooth equations,” *Applications of Mathematics*, vol. 46, no. 3, pp. 215–229, 2001.