Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2010, Article ID 132186, 12 pages
doi:10.1155/2010/132186

Research Article

The Calderón Reproducing Formula Associated with the Heisenberg Group $\mathbf{H}^d$

Jinsen Xiao and Jianxun He

Department of Mathematics, School of Mathematics and Information Sciences, Guangzhou University, Guangzhou 510006, China

Correspondence should be addressed to Jianxun He, h_jianxun@hotmail.com

Received 9 October 2009; Accepted 20 May 2010

Academic Editor: Victoria Vampa

We establish the Calderón reproducing formula for functions in L^2 on the Heisenberg group $\mathbf{H}^d$. Also, we develop this formula in $L^p(\mathbf{H}^d)$ with $1 < p < \infty$.

1. Introduction

The classical Calderón reproducing formula reads

$$f = \int_0^{+\infty} f * \phi_t * \psi_t \frac{dt}{t}, \tag{1.1}$$

where $\phi_t(x) = t^{-1}\phi(x/t)$, $\psi_t(x) = t^{-1}\psi(x/t)$, and $*$ denotes the convolution on $\mathbf{R}$. The Calderón reproducing formula is a useful tool in pure and applied mathematics (see [1–4]), particularly in wavelet theory (see [5, 6]). We always call (1.1) an inverse formula of wavelet transform. In [7], the authors generalized (1.1) to $\mathbf{R}^n$ when ϕ and ψ are sufficiently nice normalized radial wavelet functions. The generalization of (1.1) involving nonradial wavelets ϕ and ψ can be written in the following form:

$$f = \int_{SO(n)} d\gamma \int_0^{+\infty} f * \phi_{\gamma,t} * \psi_{\gamma,t} \frac{dt}{t}, \tag{1.2}$$

where $\phi_{\gamma,t}$ and $\psi_{\gamma,t}$ are rotated versions of ϕ and ψ on $\mathbf{R}^n$. The authors in [8, 9] established (1.1) for $f \in L^p(\mathbf{R})$. Holschneider [10] studied the formula (1.2) in case $f \in L^p(\mathbf{R}^2)$ and gave an inversion formula of the Radon transform in L^p-space by using wavelets. Furthermore, Rubin [4] developed the Calderón reproducing formula, windowed X-ray transforms, the Radon transforms, and k-plane transforms in L^p-spaces on $\mathbf{R}^n$.

It is a remarkable fact that the Heisenberg group, denoted by $\mathbf{H}^d$, arises in two fundamental but different setting in analysis. On the one hand, it can be realized as the boundary of the unit ball in several complex variables. On the other hand, an important aspect of the study of the Heisenberg group is the background of physics, namely, the mathematical ideas connected with the fundamental nations of quantum mechanics. In other words, there is its genesis in the context of quantum mechanics which emphasizes its symplectic role in connection with the Fourier transform, pseudodifferential operators, and related matters (see [11]). Due to this reason, many interesting works were devoted to the theory of harmonic analysis on $\mathbf{H}^d$ in [11–13] and the references therein. Also, the researches of wavelet analysis on $\mathbf{H}^d$ are concerned increasingly; for this we refer readers to [14–16]. And the inversion formula of the Radon transform by using inverse wavelet transform on $\mathbf{H}^d$ was established in [17]. Our goal of the present article is to study the Calderón reproducing formula on the Heisenberg group in L^p-space with $1 < p < \infty$. In the sequel we will develop the theory of inverse Radon transform on $\mathbf{H}^d$.

The Heisenberg group $\mathbf{H}^d$ is a Lie group with the underlying manifold $\mathbf{C}^d \times \mathbf{R}$, and the multiplication law is given by

$$(z,t)(z',t') = \left(z + z', t + t' + 2\operatorname{Im} z\overline{z'}\right), \tag{1.3}$$

where $z\overline{z'} = \sum_{j=1}^{d} z_j \overline{z'_j}$. The dilation of $\mathbf{H}^d$ is defined by $\rho(z,t) = (\rho z, \rho^2 t)$ with $\rho > 0$. For $(z,t) \in \mathbf{H}^d$, the homogeneous norm of (z,t) is given by (see [11])

$$|(z,t)| = \left|(z,t)^{-1}\right| = \max\left\{|z|, |t|^{1/2}\right\}. \tag{1.4}$$

Notice that $|\rho(z,t)| = \max\{|\rho z|, |\rho^2 t|^{1/2}\} = \rho|(z,t)|$. In addition, $|\cdot|$ satisfies the quasitriangle inequality:

$$\left|(z,t)(z',t')\right| \le |(z,t)| + \left|(z',t')\right|. \tag{1.5}$$

The homogeneous dimension of $\mathbf{H}^d$ is $2d+2$, and the volume of a ball $\mathcal{B}((z,t),r) = \{(z',t') \in \mathbf{H}^d : |(z,t)^{-1}(z',t')| \le r\}$ is $c' r^{2d+2}$, where c' is a constant.

Let $\mathbf{P} = \{(z,t,\rho) : (z,t) \in \mathbf{H}^d,\ \rho > 0\}$; then $\mathbf{P}$ is a locally compact nonunimodular group with the group law

$$(z,t,\rho)(z',t',\rho') = \left(z + \rho z', t + \rho^2 t' + 2\rho \operatorname{Im} z\overline{z'}, \rho\rho'\right). \tag{1.6}$$

The left and right Haar measures on $\mathbf{P}$ are given by

$$d\mu_l(z,t,\rho) = \frac{dz\,dt\,d\rho}{\rho^{2d+3}}, \qquad d\mu_r(z,t,\rho) = \frac{dz\,dt\,d\rho}{\rho}, \tag{1.7}$$

where dz denotes the Lebesgue measure on $\mathbf{C}^d$.

Let $L^p(\mathbf{H}^d)$ be the space of measurable functions f on $\mathbf{H}^d$, such that

$$\begin{gathered}\|f\|_{L^p(\mathbf{H}^d)} = \left(\int_{\mathbf{H}^d} |f(z,t)|^p dz\,dt\right)^{1/p} < +\infty, \quad \text{if } p \in [1,+\infty), \\ \|f\|_{L^p(\mathbf{H}^d)} = \operatorname{ess}\sup_{(z,t)\in\mathbf{H}^d} |f(z,t)| < +\infty, \quad \text{if } p = +\infty.\end{gathered} \tag{1.8}$$

Let $\mathbf{Z}^+ = \{0,1,2,\ldots\}$ and $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_d) \in (\mathbf{Z}^+)^d$. The Fock space $\mathscr{H}$ is the space of holomorphic functions F on $\mathbf{C}^d$ such that

$$\|F\|_{\mathscr{H}}^2 = \left(\frac{2}{\pi}\right)^d \int_{\mathbf{C}^d} |F(\zeta)|^2 e^{-2|\zeta|^2} d\zeta < +\infty. \tag{1.9}$$

From [18] we know that $\{E_\alpha(\zeta) = (\sqrt{2}\zeta)^\alpha/\sqrt{\alpha!} : \alpha \in (\mathbf{Z}^+)^d\}$ is an orthonormal basis of the Hilbert space $\mathscr{H}$. For $\lambda \in \mathbf{R} \setminus \{0\}$, let π_λ be the Bargmann-Fock representation of $\mathbf{H}^d$ which acts on $\mathscr{H}$ by

$$\pi_\lambda(z,t)F(\zeta) = \begin{cases} e^{-i\lambda t-\lambda|z|^2+2\sqrt{\lambda}\zeta\overline{z}} F\left(\zeta - \sqrt{\lambda}z\right), & \text{if } \lambda > 0, \\ e^{-i\lambda t+\lambda|z|^2-2\sqrt{|\lambda|}\zeta z} F\left(\zeta + \sqrt{\lambda}\overline{z}\right), & \text{if } \lambda < 0. \end{cases} \tag{1.10}$$

The group Fourier transform of a function $f \in L^1(\mathbf{H}^d)$ is defined by

$$\hat{f}(\lambda) = \int_{\mathbf{H}^d} f(z,t)\pi_\lambda(z,t)dz\,dt. \tag{1.11}$$

Let $S_p(\mathscr{H})$ $(1 \le p < +\infty)$ be the classes of Schatten-von Neumann operators on Hilbert space $\mathscr{H}$, and let $S_\infty(\mathscr{H})$ denote the algebra of all bounded operators, that is, $S_\infty(\mathscr{H}) = B(\mathscr{H})$. For $T \in S_p(\mathscr{H})$, let $\|T\|_p = (\operatorname{tr}(T^*T)^{1/p})^{p/2}$ denote the S_p-norm of T. If $p = 2$, $\|T\|_2$ is just the Hilbert-Schmit norm of T, that denotes $\|T\|_{\mathrm{HS}}$. Let $\|T\|_\infty$ denote the usual operator norm of T in $S_\infty(\mathscr{H})$. For $1 \le p \le +\infty$, let $\mathscr{L}^p$ be the Banach space consisting of all weak measurable operator value functions F, which also satisfy $F(\lambda) \in S_p(\mathscr{H}_{|\lambda|})$, a.e. $\lambda \in \mathbf{R} \setminus \{0\}$, and

$$\begin{gathered}\|F\|_{\mathscr{L}^p} = \left(\frac{1}{\pi^{d+1}} \int_{\mathbf{R}\setminus\{0\}} \|F(\lambda)\|_p^p |\lambda|^d d\lambda\right)^{1/p} < +\infty, \quad \text{if } 1 \le p < +\infty, \\ \|F\|_{\mathscr{L}^p} = \operatorname{ess}\sup_{\lambda\in\mathbf{R}\setminus\{0\}} \|F(\lambda)\|_\infty < +\infty, \quad \text{if } p = \infty.\end{gathered} \tag{1.12}$$

For $f, g \in L^2(\mathbf{H}^d)$, the Parseval formula is

$$\langle f, g\rangle_{L^2(\mathbf{H}^d)} = \frac{2^{d-1}}{\pi^{d+1}} \int_{-\infty}^{+\infty} \operatorname{tr}\left(\hat{g}(\lambda)^* \hat{f}(\lambda)\right) |\lambda|^d d\lambda, \tag{1.13}$$

where $\widehat{g}(\lambda)^*$ denotes the adjoint of $\widehat{g}(\lambda)$. The Plancherel formula is

$$\|f\|_{L^2(\mathbf{H}^d)}^2 = \frac{2^{d-1}}{\pi^{d+1}} \int_{-\infty}^{+\infty} \left\|\widehat{f}(\lambda)\right\|_{\mathrm{HS}}^2 |\lambda|^d d\lambda. \tag{1.14}$$

As a consequence of (1.13), one has the inversion of the Fourier transform:

$$f(z,t) = \frac{2^{d-1}}{\pi^{d+1}} \int_{-\infty}^{+\infty} \operatorname{tr}\left(\pi_\lambda(z,t)^* \widehat{f}(\lambda)\right) |\lambda|^d d\lambda. \tag{1.15}$$

Suppose $\rho > 0$, and let

$$f_\rho(z,t) = \rho^{-(2d+2)} f\left(\rho^{-1}(z,t)\right) = \rho^{-(2d+2)} f\left(\frac{z}{\rho}, \frac{t}{\rho^2}\right). \tag{1.16}$$

By a direct computation, we have

$$\widehat{f_\rho}(\lambda) = \widehat{f}(\rho\lambda). \tag{1.17}$$

Let $f * g$ be the convolution of f and g, that is,

$$f * g(z,t) = \int_{\mathbf{H}^d} f(z',t') g\left((z',t')^{-1}(z,t)\right) dz' dt'. \tag{1.18}$$

Then

$$\widehat{f * g}(\lambda) = \widehat{f}(\lambda)\widehat{g}(\lambda). \tag{1.19}$$

We should notice the following facts: if $\widetilde{f}(z,t) = \overline{f((z,t)^{-1})} = \overline{f(-z,-t)}$, then

$$\widehat{\widetilde{f}}(\lambda) = \widehat{f}(\lambda)^*. \tag{1.20}$$

And if $g(z,t) = f(-\overline{z},-t)$, then

$$\widehat{g}(\lambda) = \widehat{f}(-\lambda). \tag{1.21}$$

The further detail of harmonic analysis on $\mathbf{H}^d$ can be found in [11, 12].

2. Calderón Reproducing Formula

The authors in [14, 15, 18] studied the theory of continuous wavelet associated with the concept of square integrable group representations. The unitary representation of $\mathbf{P}$ on $L^2(\mathbf{H}^d)$ is defined by

$$U_{(z,t,\rho)}f(z',t') = \rho^{-(d+1)}f\left(\frac{z'-z}{\rho}, \frac{t'-t-2\operatorname{Im} z\overline{z'}}{\rho^2}\right). \tag{2.1}$$

Let $\mathbf{R}^+$ denote the set of all positive real numbers, $\mathbf{R}^- = -\mathbf{R}^+$. Let P_α ($\alpha \in (Z^+)^d$) be the projection from $L^2(\mathbf{R}^d)$ to 1-dimensional subspace spanned by E_α, and let $\sigma = +$ or $-$,

$$H_\alpha^\sigma = \left\{ f \in L^2\left(\mathbf{H}^d\right) : \widehat{f}(\lambda) = \widehat{f}(\lambda)P_\alpha, \text{ and } \widehat{f}(\lambda) = 0 \text{ if } \lambda \notin \mathbf{R}^\sigma \right\}. \tag{2.2}$$

From [15, Theorem 1], we have

$$L^2\left(\mathbf{H}^d\right) = \bigoplus_{\alpha \in (Z^+)^d} \left(H_\alpha^+ \oplus H_\alpha^-\right). \tag{2.3}$$

Let $\alpha \in (\mathbf{Z}^+)^d$ and $\sigma = +$ or $-$, $\phi \in H_\alpha^\sigma$; if $\phi \neq 0$ and satisfies

$$C_\phi = \left\langle \int_{\mathbf{R}^\sigma} \widehat{\phi}(\lambda)^* \widehat{\phi}(\lambda) \frac{d\lambda}{|\lambda|} E_\alpha, E_\alpha \right\rangle_{\mathscr{H}} < +\infty, \tag{2.4}$$

then we call ϕ an admissible wavelet and write $\phi \in \mathrm{AW}_\alpha^\sigma$. Let $\phi \in \mathrm{AW}_\alpha^\sigma$, $f \in H_\alpha^\sigma$; the continuous wavelet transform of f with respect to ϕ is defined by

$$W_\phi f(z,t,\rho) = \left\langle f, U_{(z,t,\rho)}\phi \right\rangle_{L^2(\mathbf{H}^d)}. \tag{2.5}$$

And the following Calderón reproducing formula holds in the weak sense:

$$f(z',t') = C_\phi^{-1} \int_0^{+\infty} \int_{\mathbf{H}^d} W_\phi f(z,t,\rho) U_{(z,t,\rho)}\phi(z',t') \frac{dz\,dt\,d\rho}{\rho^{2d+3}}. \tag{2.6}$$

2.1. Calderón Reproducing Formula in $L^2(\mathbf{H}^d)$

By (1.16) and (1.18), we can rewrite (2.5) and (2.6) as follows:

$$W_\phi f(z,t,\rho) = \rho^{d+1} f * \widetilde{\phi_\rho}(z,t), \tag{2.7}$$

$$f(z,t) = C_\phi^{-1} \int_0^{+\infty} f * \phi_\rho * \widetilde{\phi_\rho}(z,t) \frac{d\rho}{\rho}. \tag{2.8}$$

For $0 < \varepsilon < \eta < +\infty$, let

$$f_{\varepsilon,\eta}(z,t) = C_\phi^{-1} \int_\varepsilon^\eta f * \phi_\rho * \widetilde{\phi_\rho}(z,t) \frac{d\rho}{\rho}, \tag{2.9}$$

$$\Phi_{\varepsilon,\eta}(z,t) = C_\phi^{-1} \int_\varepsilon^\eta \phi_\rho * \widetilde{\phi_\rho}(z,t) \frac{d\rho}{\rho}, \tag{2.10}$$

then $f_{\varepsilon,\eta}(z,t) = f * \Phi_{\varepsilon,\eta}(z,t)$. We are now in a position to show that $f_{\varepsilon,\eta}$ converges to f in L^2-space when $\varepsilon \to 0$ and $\eta \to \infty$. The result in this paper is an extension of that of Mourou and Trimèche [19].

Lemma 2.1. *Suppose that* $\phi \in AW_\alpha^\sigma$ *and* $\phi \in H_\alpha^\sigma$ *satisfies* $\widehat{\phi}(\lambda) \in S_\infty(\mathscr{H})$. *Let* $\Phi_{\varepsilon,\eta}$ *be defined by* (2.10). *Then one has* $\Phi_{\varepsilon,\eta} \in L^2(\mathbf{H}^d)$.

Proof. By Hölder's inequality, we have

$$\left|\Phi_{\varepsilon,\eta}\right|^2 \leq C_\phi^{-2} \int_\varepsilon^\eta \left|\phi_\rho * \widetilde{\phi_\rho}(z,t)\right|^2 \frac{d\rho}{\rho} \int_\varepsilon^\eta \frac{d\rho}{\rho}. \tag{2.11}$$

Thus

$$\int_{\mathbf{H}^d} \left|\Phi_{\varepsilon,\eta}\right|^2 dz\,dt \leq C_\phi^{-2} \int_\varepsilon^\eta \int_{\mathbf{H}^d} \left|\phi_\rho * \widetilde{\phi_\rho}(z,t)\right|^2 \frac{dz\,dt\,d\rho}{\rho} \int_\varepsilon^\eta \frac{d\rho}{\rho}. \tag{2.12}$$

By (1.14), (1.19), and (1.20), we have

$$\begin{aligned} \int_{\mathbf{H}^d} \left|\phi_\rho * \widetilde{\phi_\rho}(z,t)\right|^2 dz\,dt &= \frac{2^{d-1}}{\pi^{d+1}} \int_{-\infty}^{+\infty} \left\|\widehat{\phi}(\rho\lambda)\widehat{\phi}(\rho\lambda)^*\right\|_{\mathrm{HS}}^2 |\lambda|^d d\lambda \\ &= \frac{2^{d-1}}{\pi^{d+1}} \int_{-\infty}^{+\infty} \operatorname{tr}\left(\widehat{\phi}(\rho\lambda)^*\widehat{\phi}(\rho\lambda)\widehat{\phi}(\rho\lambda)^*\widehat{\phi}(\rho\lambda)\right)|\lambda|^d d\lambda. \end{aligned} \tag{2.13}$$

Noticing that

$$\operatorname{tr}\left(\widehat{\phi}(\rho\lambda)^*\widehat{\phi}(\rho\lambda)\widehat{\phi}(\rho\lambda)^*\widehat{\phi}(\rho\lambda)\right) = \sum_\alpha \left\langle \widehat{\phi}(\rho\lambda)^*\widehat{\phi}(\rho\lambda)E_\alpha, \widehat{\phi}(\rho\lambda)^*\widehat{\phi}(\rho\lambda)E_\alpha \right\rangle_{\mathscr{H}}, \tag{2.14}$$

we obtain

$$\operatorname{tr}\left(\widehat{\phi}(\rho\lambda)^*\widehat{\phi}(\rho\lambda)\widehat{\phi}(\rho\lambda)^*\widehat{\phi}(\rho\lambda)\right) \leq \left\|\widehat{\phi}(\rho\lambda)\right\|_\infty^2 \operatorname{tr}\left(\widehat{\phi}(\rho\lambda)^*\widehat{\phi}(\rho\lambda)\right). \tag{2.15}$$

Thus we get

$$\begin{aligned}\int_{\mathbf{H}^d}\left|\phi_\rho * \widetilde{\phi_\rho}(z,t)\right|^2 dz\,dt &\le \frac{2^{d-1}}{\pi^{d+1}}\int_{-\infty}^{+\infty}\left\|\widehat{\phi}\right\|_{\mathcal{L}^\infty}^2 \operatorname{tr}\left(\widehat{\phi}(\rho\lambda)^*\widehat{\phi}(\rho\lambda)\right)|\lambda|^d d\lambda \\ &= \frac{2^{d-1}}{\pi^{d+1}}\left\|\widehat{\phi}\right\|_{\mathcal{L}^\infty}^2\int_{-\infty}^{+\infty}\operatorname{tr}\left(\widehat{\phi}(\rho\lambda)^*\widehat{\phi}(\rho\lambda)\right)|\lambda|^d d\lambda \\ &= \left\|\widehat{\phi}\right\|_{\mathcal{L}^\infty}^2\left\|\phi_\rho\right\|_{L^2(\mathbf{H}^d)}^2.\end{aligned} \tag{2.16}$$

Therefore,

$$\begin{aligned}\int_{\mathbf{H}^d}\left|\Phi_{\varepsilon,\eta}\right|^2 dz\,dt &\le C_\phi^{-2}\int_\varepsilon^\eta\left\|\widehat{\phi}\right\|_{\mathcal{L}^\infty}^2\left\|\phi_\rho\right\|_{L^2(\mathbf{H}^d)}^2\frac{d\rho}{\rho}\int_\varepsilon^\eta\frac{d\rho}{\rho} \\ &= C_\phi^{-2}\left\|\widehat{\phi}\right\|_{\mathcal{L}^\infty}^2\left\|\phi_\rho\right\|_{L^2(\mathbf{H}^d)}^2\int_\varepsilon^\eta\rho^{-2(d+1)}\frac{d\rho}{\rho}\int_\varepsilon^\eta\frac{d\rho}{\rho} \\ &< +\infty.\end{aligned} \tag{2.17}$$

Then we complete the proof of this lemma. □

Theorem 2.2. *Let $\phi \in AW_\alpha^\sigma$ and $\phi \in H_\alpha^\sigma$ satisfy $\widehat{\phi}(\lambda) \in S_\infty(\mathcal{H})$. Then for $f \in H_\alpha^\sigma$, one has $f_{\varepsilon,\eta} \in L^2(\mathbf{H}^d)$ and $\lim_{\varepsilon\to 0,\eta\to\infty} f_{\varepsilon,\eta} = f$.*

Proof. Notice that

$$f_{\varepsilon,\eta}(z,t) = f * \Phi_{\varepsilon,\eta}(z,t), \tag{2.18}$$

and by (1.19) and Lemma 2.1 we deduce $f_{\varepsilon,\eta} \in L^2(\mathbf{H}^d)$. Then by (1.17) and (1.19), we have

$$\begin{aligned}&\lim_{\substack{\varepsilon\to 0\\ \eta\to\infty}}\left\langle\int_{\mathbf{H}^d} f_{\varepsilon,\eta}(z,t)\pi_\lambda(z,t)dz\,dt\,E_\alpha, E_\alpha\right\rangle_{\mathcal{H}} \\ &\quad= \lim_{\substack{\varepsilon\to 0\\ \eta\to\infty}}\left\langle\int_{\mathbf{H}^d} f * \Phi_{\varepsilon,\eta}(z,t)\pi_\lambda(z,t)dz\,dt\,E_\alpha, E_\alpha\right\rangle_{\mathcal{H}} \\ &\quad= \lim_{\substack{\varepsilon\to 0\\ \eta\to\infty}}\left\langle C_\phi^{-1}\int_\varepsilon^\eta \widehat{f}(\lambda)\widehat{\phi}(\rho\lambda)^*\widehat{\phi}(\rho\lambda)\frac{d\rho}{\rho}E_\alpha, E_\alpha\right\rangle_{\mathcal{H}} \\ &\quad= \lim_{\substack{\varepsilon\to 0\\ \eta\to\infty}} C_\phi^{-1}\left\langle\int_\varepsilon^\eta \widehat{\phi}_\alpha^\sigma(\lambda')^*\widehat{\phi}_\alpha^\sigma(\lambda')\frac{d\lambda'}{|\lambda'|}E_\alpha, E_\alpha\right\rangle_{\mathcal{H}}\left\langle\widehat{f}_\alpha^\sigma(\lambda)E_\alpha, E_\alpha\right\rangle_{\mathcal{H}} \\ &\quad= \left\langle\widehat{f}(\lambda)E_\alpha, E_\alpha\right\rangle_{\mathcal{H}},\end{aligned} \tag{2.19}$$

where $\sigma = +$ if $\lambda > 0$, otherwise $\sigma = -$. By (1.11) we get the desired result. □

2.2. Calderón Reproducing Formula in $L^p(\mathbf{H}^d)$ with $1 < p < \infty$

For $f \in L^p(\mathbf{H}^d)$ with $1 < p < \infty$, the continuous wavelet transform of f with respect to a wavelet ϕ can be defined by formula (2.7) under certain conditions on ϕ. In this part we will show that $f_{\varepsilon,\eta}$ converges to f.

Let f be a measurable function on $\mathbf{H}^d$; for $(z,t) \in \mathbf{H}^d$, define

$$f^*(z,t) = \operatorname{ess\,sup}\left\{ \left| f(z',t') \right| : (z',t') \in \mathbf{H}^d,\ \left| (z',t') \right| > |(z,t)| \right\}. \tag{2.20}$$

It is easy to see that f^* is nonnegative and radially decreasing, that is,

$$0 \le f^*(z_1,t_1) \le f^*(z_2,t_2) \quad \text{only if } |(z_1,t_1)| > |(z_2,t_2)|, \tag{2.21}$$

and $f^* \ge |f|$ a.e. on $\mathbf{H}^d$.

Let $g \in L^1(\mathbf{R}^+)$. Then for any $(z,t) \in \mathbf{H}^d$, we define

$$G^*(z,t) = \sup\left\{ \rho^{-(2d+2)} \left| \int_\rho^{+\infty} g(s)ds \right| : \rho > |(z,t)| \right\}, \tag{2.22}$$

and thus we have the following lemma.

Lemma 2.3. *Let $g \in L^1(\mathbf{R}^+)$ and $G^*(z,t)$ be defined by* (2.22)*. Then one has*

$$\int_{\mathbf{H}^d \setminus \mathcal{B}} G^*(z,t)dz\,dt \le c \int_1^{+\infty} \left| g(s) \right| \ln s\,ds, \tag{2.23}$$

where c is a positive constant, $\mathcal{B} = \mathcal{B}((0,0),1)$.

Proof. First we let

$$G(z,t) = |(z,t)|^{-(2d+2)} \int_{|(z,t)|}^{+\infty} \left| g(s) \right| ds. \tag{2.24}$$

It is obvious that $G^*(z,t) \le G(z,t)$ for any $(z,t) \in \mathbf{H}^d \setminus \{(0,0)\}$. Since $G(z,t)$ is nonnegative and radially decreasing, from [11, page 542], we know that there exists a positive constant c such that

$$\int_{\mathbf{H}^d} G(z,t)dz\,dt = c \int_0^{+\infty} G(r) r^{2d+1} dr. \tag{2.25}$$

Thus,

$$
\begin{aligned}
\int_{\mathbf{H}^d\setminus\mathcal{B}} G^*(z,t)dz\,dt &\le \int_{\mathbf{H}^d\setminus\mathcal{B}} G(z,t)dz\,dt \\
&= c\int_1^{+\infty} G(r)r^{2d+1}dr \\
&= c\int_1^{+\infty} r^{-(2d+2)}\int_r^{+\infty} \left|g(s)\right|ds\, r^{2d+1}dr \\
&= c\int_1^{+\infty}\int_1^{s}\left|g(s)\right|ds\frac{dr}{r} \\
&= c\int_1^{+\infty}\left|g(s)\right|\ln s\, ds.
\end{aligned}
\tag{2.26}
$$

This completes the proof. □

Let k be a radial function in $L^p(\mathbf{H}^d)$ $(1<p<\infty)$ and let

$$
K(z,t) = |(z,t)|^{-(2d+2)}\int_0^{|(z,t)|} k(s\omega)s^{2d+1}ds \tag{2.27}
$$

for any $(z,t)\in\mathbf{H}^d\setminus\{(0,0)\}$, where ω is a unit vector of $\mathbf{H}^d$. Then we have the following lemma.

Lemma 2.4. *Let k be a radial function in $L^p(\mathbf{H}^d)$ $(1<p<\infty)$ and let $K(z,t)$ be defined by k in* (2.27)*. If k satisfies $\int_{\mathbf{H}^d} k(z,t)dz\,dt = 0$, and $k(z,t)\ln|(z,t)|\in L^1(\mathbf{H}^d)$, then $K\in L^1(\mathbf{H}^d)$.*

Proof. Analogous to (2.20) we define

$$
K^*(z,t) = \operatorname{ess\,sup}\left\{\left|K(z',t')\right| : (z',t')\in\mathbf{H}^d,\ \left|(z',t')\right| > |(z,t)|\right\}, \tag{2.28}
$$

and then $K^*\ge|K|$ a.e. on $\mathbf{H}^d$.

Let $(z,t),(z',t')\in\mathbf{H}^d$ and $0<|(z,t)|<|(z',t')|$; by (2.25) we have

$$\begin{aligned}
\left|K(z',t')\right| &= \frac{1}{|(z',t')|^{(2d+2)}}\left|\int_0^{|(z',t')|} k(s\omega)s^{2d+1}ds\right| \\
&\leq \frac{1}{|(z',t')|^{(2d+2)}}\int_0^{|(z',t')|}|k(s\omega)|s^{2d+1}ds \\
&= \frac{1}{c|(z',t')|^{(2d+2)}}\int_{\mathcal{B}((0,0),|(z',t')|)}\left|k(z'',t'')\right|' dz''dt'' \\
&\leq \frac{2^{2d+2}}{c(|(z,t)|+|(z',t')|)^{(2d+2)}}\int_{\mathcal{B}((z,t),|(z,t)|+|(z',t')|)}\left|k(z'',t'')\right|dz''dt'' \\
&\leq 2^{2d+2}(Mk)(z,t),
\end{aligned} \tag{2.29}$$

where Mk is the Hardy-Littlewood maximal function of k. From the definition of K^*, we have $K^*\leq 2^{2d+2}(Mk)$. Because $k\in L^p(\mathbf{H}^d)$, we get $K^*\in L^p(\mathbf{H}^d)$.

By the hypothesis k is radial and $\int_{\mathbf{H}^d}k(z,t)dz\,dt=0$; together with the definition of K, we have

$$|K(z,t)| = \frac{1}{|(z,t)|^{(2d+2)}}\left|\int_{|(z,t)|}^{+\infty}k(s\omega)s^{2d+1}ds\right|. \tag{2.30}$$

Since $k(z,t)(\ln|(z,t)|)\in L^1(\mathbf{H}^d)$, it follows from Lemma 2.3 that $\int_{\mathbf{H}^d\setminus\mathcal{B}}K^*(z,t)dz\,dt<+\infty$, that is, $K^*\in L^1_{\text{loc}}(\mathbf{H}^d)$. On the other hand, $K^*\in L^p(\mathbf{H}^d)\subset L^1_{\text{loc}}(\mathbf{H}^d)$, and thus $K^*\in L^1(\mathbf{H}^d)$, which implies that $K\in L^1(\mathbf{H}^d)$. □

Without loss of generality, we assume that $C_\phi=1$; then (2.9) and (2.10) can be written as

$$\Phi_{\varepsilon,\eta}(z,t)=\int_\varepsilon^\eta \phi_\rho*\widetilde{\phi_\rho}(z,t)\frac{d\rho}{\rho}, \tag{2.31}$$

$$f_{\varepsilon,\eta}(z,t)=\int_\varepsilon^\eta f*\phi_\rho*\widetilde{\phi_\rho}(z,t)\frac{d\rho}{\rho}. \tag{2.32}$$

In fact, $\Phi_{\varepsilon,\eta}$ is always stated under conditions on $k:=\phi*\widetilde{\phi}$ rather than under conditions on ϕ for convenience (see [4, 10]). By Lemma 2.4 we have the following theorem.

Theorem 2.5. *Let k be in the conditions of Lemma 2.4 and let K be defined by* (2.27)*. Suppose $\phi*\widetilde{\phi}=k$ and $\int_{\mathbf{H}^d}K(z,t)dz\,dt=1$. Then for $f\in L^p(\mathbf{H}^d)$, one has $\lim_{\varepsilon\to0,\eta\to\infty}f_{\varepsilon,\eta}=f$.*

Proof. From (2.31) we have

$$\Phi_{\varepsilon,\eta}(z,t)=\Phi_{\varepsilon,\infty}(z,t)-\Phi_{\eta,\infty}(z,t). \tag{2.33}$$

Since $k(z,t) = \phi * \widetilde{\phi}(z,t)$, and $\widetilde{\phi}(z,t) = \overline{\phi(-z,-t)}$, we deduce

$$\begin{aligned}
k_\rho(z,t) &= \rho^{-(2d+2)} \int_{\mathbf{H}^d} \phi(z',t')\overline{\phi}\Big(\left(\rho^{-1}z,\rho^{-2}t\right)^{-1}(z',t')\Big)dz'dt' \\
&= \rho^{-(4d+4)} \int_{\mathbf{H}^d} \phi(z',t')\overline{\phi}\Big(\rho^{-1}\Big((z,t)^{-1}\Big(\rho z',\rho^2 t'\Big)\Big)\Big)d(\rho z')d\Big(\rho^2 t'\Big) \\
&= \rho^{-(4d+4)} \int_{\mathbf{H}^d} \phi\Big(\rho^{-1}(z',t')\Big)\overline{\phi}\Big(\rho^{-1}\Big((z,t)^{-1}(z',t')\Big)\Big)dz'dt' \\
&= \phi_\rho * \widetilde{\phi_\rho}(z,t).
\end{aligned} \tag{2.34}$$

Then we have

$$\begin{aligned}
\Phi_{\varepsilon,\infty}(z,t) &= \int_\varepsilon^{+\infty} k_\rho(z,t)\frac{d\rho}{\rho} \\
&= \int_\varepsilon^{+\infty} k\left(\frac{(z,t)}{\rho}\right)\frac{d\rho}{\rho^{2d+3}} \\
&= \frac{1}{|(z,t)|^{2d+2}} \int_{\varepsilon/|z,t|}^{+\infty} k\left(\frac{(z,t)}{\rho|(z,t)|}\right)\frac{d\rho}{\rho^{2d+3}} \\
&= \frac{1}{|(z,t)|^{2d+2}} \int_0^{|z,t|/\varepsilon} k\left(\frac{\rho(z,t)}{|(z,t)|}\right)\rho^{2d+1}d\rho \\
&= \frac{\varepsilon^{-2(d+1)}}{|(z,t)/\varepsilon|^{2d+2}} \int_0^{|(z,t)|/\varepsilon} k(\rho\omega)\rho^{2d+1}d\rho \\
&= K_\varepsilon(z,t).
\end{aligned} \tag{2.35}$$

By Lemma 2.4 together with the approximation of the identity, we have

$$\lim_{\substack{\varepsilon\to 0\\ \eta\to\infty}} f_{\varepsilon,\eta}(z,t) = \lim_{\substack{\varepsilon\to 0\\ \eta\to\infty}} \Phi_{\varepsilon,\eta} * f(z,t) = \lim_{\varepsilon\to 0} K_\varepsilon * f(z,\mathrm{t}) - \lim_{\eta\to\infty} K_\eta * f(z,t) = f(z,t). \tag{2.36}$$

Then we complete the proof of this theorem. □

Acknowledgments

The work for this paper is supported by the National Natural Science Foundation of China (no.s 10671041 and 10971039) and the Doctoral Program Foundation of the Ministry of Education of China (no. 200810780002).

References

[1] V. S. Guliyev and E. J. Ibrahimov, "Calderón reproducing formula associated with the Gegenbauer operator on the half-line," *Journal of Mathematical Analysis and Applications*, vol. 335, no. 2, pp. 1079–1094, 2007.

[2] Y. Han, S. Lu, and D. Yang, "Inhomogeneous discrete Calderón reproducing formulas for spaces of homogeneous type," *Journal of Fourier Analysis and Applications*, vol. 7, no. 6, pp. 571–600, 2001.

[3] M. Holschneider and P. Tchamitchian, "Pointwise analysis of Riemann's "nondifferentiable" function," *Inventiones Mathematicae*, vol. 105, no. 1, pp. 157–175, 1991.

[4] B. Rubin, "The Calderón reproducing formula, windowed X-ray transforms, and Radon transforms in L^p-spaces," *Journal of Fourier Analysis and Applications*, vol. 4, no. 2, pp. 175–197, 1998.

[5] I. Daubechies, *Ten Lectures on Wavelets*, vol. 61 of *CBMS-NSF Regional Conference Series in Applied Mathematics*, SIAM, Philadelphia, Pa, USA, 1992.

[6] Y. Meyer, *Wavelets and Operators*, vol. 37 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, UK, 1992.

[7] M. Frazier, B. Jawerth, and G. Weiss, *Littlewood-Paley Theory and the Study of Function Spaces*, vol. 79 of *CBMS Regional Conference Series in Mathematics*, American Mathematical Society, Providence, RI, USA, 1991.

[8] M. A. Mourou and K. Trimèche, "Calderon's reproducing formula related to the Dunkl operator on the real line," *Monatshefte für Mathematik*, vol. 136, no. 1, pp. 47–65, 2002.

[9] B. Rubin and E. Shamir, "Calderón's reproducing formula and singular integral operators on a real line," *Integral Equations and Operator Theory*, vol. 21, no. 1, pp. 77–92, 1995.

[10] M. Holschneider, "Inverse Radon transforms through inverse wavelet transforms," *Inverse Problems*, vol. 7, no. 6, pp. 853–861, 1991.

[11] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, vol. 43 of *Princeton Mathematical Series*, Princeton University Press, Princeton, NJ, USA, 1993.

[12] D. Geller, "Fourier analysis on the Heisenberg group. I. Schwartz space," *Journal of Functional Analysis*, vol. 36, no. 2, pp. 205–254, 1980.

[13] R. S. Strichartz, "L^p harmonic analysis and Radon transforms on the Heisenberg group," *Journal of Functional Analysis*, vol. 96, no. 2, pp. 350–406, 1991.

[14] J. He and H. Liu, "Admissible wavelets associated with the affine automorphism group of the Siegel upper half-plane," *Journal of Mathematical Analysis and Applications*, vol. 208, no. 1, pp. 58–70, 1997.

[15] H. Liu and L. Peng, "Admissible wavelets associated with the Heisenberg group," *Pacific Journal of Mathematics*, vol. 180, no. 1, pp. 101–123, 1997.

[16] H. Liu, Y. Liu, and H. Wang, "Multiresolution analysis, self-similar tilings and Haar wavelets on the Heisenberg group," *Acta Mathematica Scientia B*, vol. 29, no. 5, pp. 1251–1266, 2009.

[17] J. He, "An inversion formula of the Radon transform on the Heisenberg group," *Canadian Mathematical Bulletin*, vol. 47, no. 3, pp. 389–397, 2004.

[18] J. He, "Continuous multiscale analysis on the Heisenberg group," *Bulletin of the Korean Mathematical Society*, vol. 38, no. 3, pp. 517–526, 2001.

[19] M. A. Mourou and K. Trimèche, "Calderón's formula associated with a differential operator on $(0, \infty)$ and inversion of the generalized Abel transform," *Journal of Fourier Analysis and Applications*, vol. 4, no. 2, pp. 229–245, 1998.