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Research Article

On the Solution n-Dimensional of the Product $\otimes^k$ Operator and Diamond Bessel Operator

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Firstly, we studied the solution of the equation $\otimes^k \Diamond_B^k u(x) = f(x)$ where $u(x)$ is an unknown unknown function for $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $f(x)$ is the generalized function, k is a positive integer. Finally, we have studied the solution of the nonlinear equation $\otimes^k \Diamond_B^k u(x) = f(x, \Box^{k-1} L^k \Delta_B^k \Box_B^k u(x))$. It was found that the existence of the solution $u(x)$ of such an equation depends on the condition of f and $\Box^{k-1} L^k \Delta_B^k \Box_B^k u(x)$. Moreover such solution $u(x)$ is related to the inhomogeneous wave equation depending on the conditions of p, q, and k.

1. Introduction

The operator $\Diamond^k$ has been first introduced by Kananthai (see [1]), is named as the Diamond operator iterated $k-$times, and is defined by

$$\Diamond^k = \left(\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right)^k, \quad p + q = n. \tag{1.1}$$

n is the dimension of the space $\mathbb{R}^n$, for $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and k is a nonnegative integer. The operator $\Diamond^k$ can be expressed in the form $\Diamond^k = \Delta^k \Box^k = \Box^k \Delta^k$, where Δ^k is the Laplacian operator itrerated $k-$times defined by

$$\Delta^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)^k, \tag{1.2}$$

and $\Box^k$ is the ultrahyperbolic operator iterated k−times defined by

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k. \tag{1.3}$$

Kananthai (see [1, Theorem 3.1, page 33]) has shown that the convolution $(-1)^k R_{2k}^e(x) * R_{2k}^H(x)$ is an elementary solution of the operator $\Diamond^k$, that is,

$$\Diamond^k \left((-1)^k R_{2k}^e(x) * R_{2k}^H(x) \right) = \delta(x). \tag{1.4}$$

Next, Kananthai (see [2]) has studied the linear equation

$$\Diamond^k u(x) = f(x). \tag{1.5}$$

This equation is the generalization of the ultrahyperbolic equation and it can be applied to the wave equation. We obtain $u(x) = (-1)^k M_{2k,2k}(x) * f(x)$ as a solution of such an equation (1.5) where

$$M_{2k,2k} = R_{2k}^H(x) * R_{2k}^e(x). \tag{1.6}$$

The function $R_{2k}^H(x)$ is called the ultrahyperbolic kernel defined by (2.2) and $R_{2k}^e(x)$ is called the elliptic kernel defined by (2.8), with $\alpha = 2k$.

Furthermore, Yıldırım et al. (see [3]) first introduced the $\Diamond_B^k$ operator that is named as Diamond Bessel operator, where $\Diamond_B^k$ is defined by

$$\Diamond_B^k = \left[\left(\sum_{i=1}^{p} B_{x_i} \right)^2 - \left(\sum_{j=p+1}^{p+q} B_{x_j} \right)^2 \right]^k, \tag{1.7}$$

and $B_{x_i} = \partial^2/\partial x_i^2 + (2v_i/x_i)(\partial/\partial x_i), 2v_i = 2\alpha_i + 1, \alpha_i > -1/2, x_i > 0$. The operator $\Diamond_B^k$ can be expressed by $\Diamond_B^k = \Delta_B^k \Box_B^k = \Box_B^k \Delta_B^k$, where

$$\Delta_B^k = \left(\sum_{i=1}^{n} B_{x_i} \right)^k. \tag{1.8}$$

$$\Box_B^k = \left(\sum_{i=1}^{p} B_{x_i} - \sum_{j=p+1}^{p+q} B_{x_j} \right)^k. \tag{1.9}$$

Next, W. Satsanit has first introduced $\otimes^k$ operator and $\otimes^k$ is defined by

$$
\begin{aligned}
\otimes^k &= \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^3 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^3\right]^k \\
&= \left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^k \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2 + \left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)\cdot\left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right) + \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k \\
&= \Box^k\left(\Delta^2 - \frac{1}{4}(\Delta+\Box)(\Delta-\Box)\right)^k \\
&= \left(\frac{3}{4}\Diamond\Delta + \frac{1}{4}\Box^3\right)^k,
\end{aligned}
\tag{1.10}
$$

where $\Diamond$, Δ, and $\Box$ are defined by (1.1), (1.2), and (1.3) with $k = 1$, respectively.

Now, firstly, the purpose of this work is to study the equation

$$\otimes^k \Diamond_B^k u(x) = f(x), \tag{1.11}$$

where the operator $\otimes^k$ is defined by (1.10) and $\Diamond_B^k$ defined by (1.7), $f(x)$ is a generalized function and $u(x)$ is an unknown function. Finally we study the equation

$$\otimes^k \Diamond_B^k u(x) = f\left(x, \Box^{k-1} L^k \Delta_B^k \Box_B^k u(x)\right) \tag{1.12}$$

with f having a continuous first derivative for all $x \in \Omega \cup \partial\Omega$, where Ω is an open subset of R^n, and $\partial\Omega$ denotes the boundary of Ω, f is bounded on Ω, that is, $|f| \le N$, N is constant, as well as $\Box^{k-1}$, L^k, Δ_B^k, and $\Box_B^k$ are defined by (1.3), (2.46), (1.8) and (1.9), respectively.

We can find the solution $u(x)$ of (1.12) that is unique under the boundary condition $\Box^{k-1} L^k \Delta_B^k \Box_B^k u(x) = 0$ for $x \in \partial\Omega$. By [4, page 369] there exists a unique solution $W(x)$ of the equation $\Box W(x) = f(x, W(x))$ for all $x \in \Omega$ with the boundary condition $W(x) = 0$ for all $x \in \partial\Omega$ where $W(x) = \Box^{k-1} L^k \Delta_B^k \Box_B^k u(x)$.

Moreover, if we put $p = k = 1$ in $\Box^k \Box_B^k M(x) = W(x)$, then we found that $M(x) = I_2^H(x) * I_2(x) * W(x)$ is a solution of the inhomogeneous equation where $I_2^H(x)$ and $I_2(x)$ are defined by (2.6) and (2.20) with $\alpha = 2$, $\gamma = 2$, respectively.

Before going into details, the following definitions and some important concepts are needed.

2. Preliminaries

Definition 2.1. Let $x = (x_1, x_2, \ldots, x_n)$ be a point of the n-dimensional Euclidean space $\mathbb{R}^n$, denoted by

$$\upsilon = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2. \tag{2.1}$$

The nondegenerated quadratic form $p+q=n$ is the dimension of the space $\mathbb{R}^n$. Let $\Gamma_+=\{x\in\mathbb{R}^n : x_1>0 \text{ and } u>0\}$ be the interior of forward cone and let $\overline{\Gamma}_+$ denote its closure. For any complex number α, define the function

$$R_\alpha^H(\upsilon)=\begin{cases}\dfrac{\upsilon^{(\alpha-n)/2}}{K_n(\alpha)}, & \text{for } x\in\Gamma_+,\\ 0, & \text{for } x\notin\Gamma_+,\end{cases} \tag{2.2}$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha)=\frac{\pi^{(n-1)/2}\Gamma((2+\alpha-n)/2)\Gamma((1-\alpha)/2)\Gamma(\alpha)}{\Gamma((2+\alpha-p)/2)\Gamma((p-\alpha)/2)}. \tag{2.3}$$

The function $R_\alpha^H(\upsilon)$ is called the ultrahyperbolic kernel of Marcel Riesz and was introduced by Nozaki (see [5]).

It is well known that $R_\alpha^H(\upsilon)$ is an ordinary function if $Re(\alpha)\geq n$ and is a distribution of α if $Re(\alpha)<n$. Let supp $R_\alpha^H(\upsilon)$ denote the support of $R_\alpha^H(\upsilon)$ and suppose that supp $R_\alpha^H(\upsilon)\subset\overline{\Gamma}_+$, that is, supp $R_\alpha^H(\upsilon)$ is compact.

From Trione (see [6, page 11]), $R_{2k}^H(\upsilon)$ is an elementary solution of the operator $\square^k$, that is,

$$\square^k R_{2k}^H(\upsilon)=\delta(x). \tag{2.4}$$

By putting $p=1$ in $R_{2k}^H(\upsilon)$ and taking into account Legendre's duplication formula for $\Gamma(z)$, that is

$$\Gamma(2z)=2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \tag{2.5}$$

we obtain

$$I_\alpha^H(\upsilon)=\frac{\upsilon^{(\alpha-n)/2}}{H_n(\alpha)}, \tag{2.6}$$

and $\upsilon=x_1^2-x_2^2-x_3^2\cdots-x_n^2$ where

$$H_n(\alpha)=\pi^{(n-2)/2}2^{\alpha-1}\Gamma\left(\frac{\alpha+2-n}{2}\right)\Gamma\left(\frac{\alpha}{2}\right). \tag{2.7}$$

$I_\alpha^H(\upsilon)$ is the hyperbolic kernel of Marcel Riesz.

Definition 2.2. Let $x=(x_1,x_2,\ldots,x_n)$ be a point of $\mathbb{R}^n$ and $\omega=x_1^2+x_2^2+\cdots+x_n^2$, then the function $R_\alpha^e(\omega)$ denoted the elliptic kernel of Marcel Riesz and is defined by

$$R_\alpha^e(\omega)=\frac{\omega^{(\alpha-n)/2}}{W_n(\alpha)}, \tag{2.8}$$

where

$$\omega = x_1^2 + x_2^2 + \cdots + x_n^2, \tag{2.9}$$

$$W_n(\alpha) = \frac{\pi^{n/2} 2^{\alpha} \Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)} \tag{2.10}$$

where α is a complex parameter and n is the dimension of $\mathbb{R}^n$.

It can be shown that $R^e_{-2k}(x) = (-1)^k \Delta^k \delta(x)$ where Δ^k is defined by (1.2). It follows that $R^e_0(x) = \delta(x)$, (see [7, page 118]).

Moreover, we obtain $(-1)^k R^e_{2k}(x)$ is an elementary solution of the operator Δ^k (see [8, Lemma 2.4, page 31]). That is

$$\Delta^k \left((-1)^k R^e_{2k}(x) \right) = \delta(x). \tag{2.11}$$

By (2.2) and (2.3) with $q = 0$, then $\upsilon^{(\alpha-n)/2}$ reduces to $\omega_p^{(\alpha-p)/2}$ where $\omega_p = x_1^2 + x_2^2 + \cdots + x_p^2$ and $K_n(\alpha)$ reduces to $K_p(\alpha) = (\pi^{(p-1)/2}\Gamma((1-\alpha)/2)\Gamma(\alpha))/\Gamma((p-\alpha)/2)$. By using the formula

$$\begin{gathered} \Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right), \\ \Gamma\left(\frac{1}{2} + z\right)\Gamma\left(\frac{1}{2} - z\right) = \pi \sec(\pi z), \end{gathered} \tag{2.12}$$

we obtain

$$K_p(\alpha) = \frac{1}{2}\sec\left(\frac{\pi\alpha}{2}\right) W_p(\alpha), \tag{2.13}$$

where $W_p(\alpha)$ is defined by (2.10) with $n = p$. Thus, for $q = 0$,

$$R^H_\alpha(\upsilon) = \frac{\upsilon^{(\alpha-p)/2}}{K_p(\alpha)} = 2\cos\left(\frac{\pi\alpha}{2}\right)\frac{\upsilon^{(\alpha-p)/2}}{W_p(\alpha)} = 2\cos\left(\frac{\pi\alpha}{2}\right) R^e_\alpha(\omega_p), \tag{2.14}$$

where $\omega_p = x_1^2 + x_2^2 + \cdots + x_p^2$. Thus, if $\alpha = 2k$, then

$$R^H_{2k}(\omega_p) = 2(-1)^k R^e_{2k}(\omega_p) \tag{2.15}$$

for $q = 0$ and $\omega_p = x_1^2 + x_2^2 + \cdots + x_p^2$.

Definition 2.3. Let $x = (x_1, x_2, \ldots, x_n), \nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{R}^+_n$. For any complex number α, we define the function $S_\alpha(x)$ by

$$S_\alpha(x) = \frac{2^{n+2|\nu|-2\alpha}\Gamma((n+2|\nu|-\alpha)/2)|x|^{\alpha-n-2|\nu|}}{\prod_{i=1}^n 2^{\nu_i-1/2}\Gamma(\nu_i+1/2)}. \tag{2.16}$$

Definition 2.4. Let $x = (x_1, x_2, \ldots, x_n), \nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{R}_n^+$, and $V = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2$ the nondegenerated quadratic form. Denote the interior of the forward cone by $\Gamma_+ = \{x \in \mathbb{R}_n^+ : x_1 > 0, x_2 > 0, \ldots, x_n > 0, V > 0\}$. The function $R_\gamma(x)$ is defined by

$$R_\gamma(x) = \frac{V^{(\gamma-n-2|\nu|)/2}}{K_n(\gamma)}, \tag{2.17}$$

where

$$K_n(\gamma) = \frac{\pi^{(n+2|\nu|-1)/2}\Gamma((2+\gamma-n-2|\nu|)/2)\Gamma((1-\gamma)/2)\Gamma(\gamma)}{\Gamma((2+\gamma-p-2|\nu|)/2)\Gamma((p-2|\nu|-\gamma)/2)}, \tag{2.18}$$

and γ is a complex number. By putting $p = 1$ in $R_\gamma(x)$ and taking into account Legendre's duplication formula for $\Gamma(z)$, that is,

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) \tag{2.19}$$

we obtain

$$I_\gamma(x) = \frac{V^{(\gamma-n-2|\nu|)/2}}{N_n(\gamma)}, \tag{2.20}$$

and $V = x_1^2 - x_2^2 - x_3^2 \cdots - x_n^2$ where

$$N_n(\gamma) = \pi^{(n+2|\nu|-1)/2} 2^{2k-1}\Gamma\left(\frac{2+\gamma-n-2|\nu|}{2}\right)\Gamma\left(\frac{\gamma}{2}\right). \tag{2.21}$$

Lemma 2.5. *Given the equation $\Delta_B^k u(x) = \delta(x)$ for $x \in \mathbb{R}_n^+$, where Δ_B^k is defined by* (1.8)*, then*

$$u(x) = (-1)^k S_{2k}(x), \tag{2.22}$$

where $S_{2k}(x)$ is defined by (2.16)*, with $\alpha = 2k$.*

Proof. (See [3, page 379] and [9]). □

Lemma 2.6. *Given the equation $\Box_B^k u(x) = \delta(x)$ for $x \in \mathbb{R}_n^+$, where $\Box_B^k$ is defined by* (1.9)*, then*

$$u(x) = R_{2k}(x), \tag{2.23}$$

where $R_{2k}(x)$ is defined by (2.17)*, with $\gamma = 2k$.*

Proof. (See [3, page 379] and [9]). □

Lemma 2.7. *Given that P is a hyperfunction, then*

$$P\delta^{k}(p)+k\delta^{(k-1)}(p)=0, \tag{2.24}$$

where $\delta^{(k)}$ is the Dirac-delta distribution with k-derivatives.

Proof. (See [8, page 233]). □

Lemma 2.8. *Given the equation*

$$\Box^{k}u(x)=0, \tag{2.25}$$

where $\Box^k$ is defined by (1.3) *and $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$, then $u(x)=(R^H_{2(k-1)}(\upsilon))^{(m)}$ is a solution of* (2.25) *with $m=(n-4)/2, n\geq 4$ and n is even dimension. The function $(R^H_{2(k-1)}(\upsilon))^{(m)}$ is defined by* (2.2) *with m-derivatives, $\alpha=2(k-1)$, and υ being defined by* (2.1).

Proof. We first show the generalized function $\delta^{(m)}(r^2-s^2)$ where $r^2=x_1^2+x_2^2+\cdots+x_p^2$ and $s^2=x_{p+1}^2+x_{p+2}^2+\cdots+x_{p+q}^2, p+q=n$, is a solution of the equation

$$\Box u(x)=0, \tag{2.26}$$

where $\Box$ is defined by (1.3) with $k=1$ and $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$,

$$\begin{aligned}
\frac{\partial}{\partial x_i}\delta^{(m)}\left(r^2-s^2\right)&=2x_i\delta^{(m+1)}\left(r^2-s^2\right),\\
\frac{\partial^2}{\partial x_i^2}\delta^{(m)}\left(r^2-s^2\right)&=2\delta^{(m+1)}\left(r^2-s^2\right)+4x_i^2\delta^{(m+2)}\left(r^2-s^2\right),\\
\Box\delta^{(m)}\left(r^2-s^2\right)&=\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\delta^{(m)}\left(r^2-s^2\right)\\
&=2p\delta^{(m+1)}\left(r^2-s^2\right)+4r^2\delta^{(m+2)}\left(r^2-s^2\right)\\
&=2p\delta^{(m+1)}\left(r^2-s^2\right)+4\left(r^2-s^2\right)\delta^{(m+2)}\left(r^2-s^2\right)+4s^2\delta^{(m+2)}\left(r^2-s^2\right)\\
&=2p\delta^{(m+1)}\left(r^2-s^2\right)-4(m+2)\delta^{(m+1)}\left(r^2-s^2\right)+4s^2\delta^{(m+2)}\left(r^2-s^2\right)\\
&=(2p-4(m+2))\delta^{(m+1)}\left(r^2-s^2\right)+4s^2\delta^{(m+2)}\left(r^2-s^2\right).
\end{aligned} \tag{2.27}$$

By Lemma 2.5 with $P=r^2-s^2$, similarly,

$$\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\delta^{(m)}\left(r^2-s^2\right)=(-2q+4(m+2))\delta^{(m+1)}\left(r^2-s^2\right)+4r^2\delta^{(m+2)}\left(r^2-s^2\right). \tag{2.28}$$

Thus

$$\begin{aligned}\Box\delta^{(m)}\left(r^2-s^2\right) &= \sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\delta^{(m)}\left(r^2-s^2\right)-\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_i^2}\delta^{(m)}\left(r^2-s^2\right)\\ &= (2(p+q)-8(m+2))\delta^{(m+1)}\left(r^2-s^2\right)-4\left(r^2-s^2\right)\delta^{(m+2)}\left(r^2-s^2\right)\\ &= (2n-8(m+2))\delta^{(m+1)}\left(r^2-s^2\right)+4(m+2)\delta^{(m+1)}\left(r^2-s^2\right)\\ &= (2n-4(m+2))\delta^{(m+1)}\left(r^2-s^2\right).\end{aligned} \tag{2.29}$$

If $2n-4(m+2)=0$, then we have $\Box\delta^{(m)}(r^2-s^2)=0$. That is, $u(x)=\delta^{(\mathrm{m})}(r^2-s^2)$ is a solution of (2.26) with $m=(n-4)/2,\ n\geq 4$ and n is even dimension. We write

$$^{k}u(x)=\Box\left(^{k-1}u(x)\right)=0, \tag{2.30}$$

and from the above proof we have $\Box^{k-1}u(x)=\delta^{(m)}(r^2-s^2)$ with $m=(n-4)/2, n\geq 4$ and n is even dimension. Convolving the above equation by $R^H_{2(k-1)}(\upsilon)$, we obtain

$$\begin{gathered}R^H_{2(k-1)}(\upsilon)*\Box^{k-1}u(x)=R^H_{2(k-1)}(\upsilon)*\delta^{(m)}\left(r^2-s^2\right)\\ \Box^{k-1}\left(R^H_{2(k-1)}(\upsilon)\right)*u(x)=\left(R^H_{2(k-1)}(\upsilon)\right)^{(m)},\quad \text{where } \upsilon=\left(r^2-s^2\right)\\ \delta*u(x)=u(x)=\left(R^H_{2(k-1)}(\upsilon)\right)^{(m)}\end{gathered} \tag{2.31}$$

by (2.2), and $\upsilon=r^2-s^2$ is defined by Definition (2.1).

Thus $u(x)=(R^H_{2(k-1)}(\upsilon))^{(m)}$ is a solution of (2.25) with $m=(n-4)/2, n\geq 4$ and n is even dimension. □

Lemma 2.9. *Given the equation*

$$\otimes^k G(x)=\delta(x), \tag{2.32}$$

then

$$G(x)=\left(R^H_{6k}(x)*(-1)^{2k}R^e_{4k}(x)\right)*\left(O^{*k}(x)\right)^{*-1} \tag{2.33}$$

is an elementary solution for the $\otimes^k$ *operator iterated* k*–times where* $\otimes^k$ *is defined by* (1.10), *and*

$$O(x)=\frac{3}{4}R^H_4(x)+\frac{1}{4}(-1)^2R^e_4(x) \tag{2.34}$$

where $O^{*k}(x)$ *denotes the convolution of* $O(x)$ *itself* k*–times and* $(O^{*k}(x))^{*-1}$ *denotes the inverse of* $O^{*k}(x)$ *in the convolution algebra. Moreover* $G(x)$ *is a tempered distribution.*

Proof. From (3.1), we have

$$\otimes^k G(x) = \left(\frac{3}{4}\Diamond\Delta + \frac{1}{4}^3\right)^k G(x) = \delta(x), \tag{2.35}$$

or we can write

$$\left(\frac{3}{4}\Diamond\Delta + \frac{1}{4}\Box^3\right)\left(\frac{3}{4}\Diamond\Delta + \frac{1}{4}\Box^3\right)^{k-1} G(x) = \delta(x). \tag{2.36}$$

Convolving both sides of the above equation by $R_6^H(x) * (-1)^2 R_4^e(x)$,

$$\left(\frac{3}{4}\Diamond\Delta + \frac{1}{4}\Box^3\right) * \left(R_6^H(x) * (-1)^2 R_4^e(x)\right)\left(\frac{3}{4}\Diamond\Delta + \frac{1}{4}^3\right)^{k-1} G(x) = \delta(x) * R_6^H(x) * (-1)^2 R_4^e(x), \tag{2.37}$$

or

$$\begin{aligned}&\left(\frac{3}{4}\Box\left(R_2^H(x)\right) * \Delta^2(-1)^2 R_4^e(x) * R_4^H(x) + \frac{1}{4}\Box^3 R_6^H(x) * (-1)^2 R_4^e(x)\right)\\ &\quad * \left(\frac{3}{4}\Diamond\Delta + \frac{1}{4}\Box^3\right)^{k-1} G(x) = \delta(x) * R_6^H(x) * (-1)^2 R_4^e(x).\end{aligned} \tag{2.38}$$

By (2.4) and (2.8), we obtain

$$\left(\frac{3}{4}\delta * \delta * R_4^H(x) + \frac{1}{4}\delta * (-1)^2 R_4^e(x)\right) * \left(\frac{3}{4}\Diamond\Delta + \frac{1}{4}\Box^3\right)^{k-1} G(x) = \delta(x) * R_6^H(x) * (-1)^2 R_4^e(x). \tag{2.39}$$

Thus

$$\left(\frac{3}{4}R_4^H(x) + \frac{1}{4}(-1)^2 R_4^e(x)\right) * \left(\frac{3}{4}\Diamond\Delta + \frac{1}{4}^3\right)^{k-1} G(x) = R_6^H(x) * (-1)^2 R_4^e(x). \tag{2.40}$$

Keeping on convolving both sides of the above equation by $R_6^H(x) * (-1)^2 R_4^e(x)$ up to $k-1$ times, we obtain

$$O^{*k}(x) * G(x) = \left(R_6^H(x) * (-1)^2 R_4^e(x)\right)^{*k} \tag{2.41}$$

where the symbol $*k$ denotes the convolution of itself k–times. By properties of $R_\alpha(x)$, we have

$$\left(R_6^H(x) * (-1)^2 R_4^e(x)\right)^{*k} = R_{6k}^H(x) * (-1)^{2k} R_{4k}^e(x). \tag{2.42}$$

Thus,

$$O^{*k}(x) * G(x) = R_{6k}^{H}(x) * (-1)^{2k} R_{4k}^{e}(x). \tag{2.43}$$

Now, consider the function $O^{*k}(x)$, since $R_6^H(x)*(-1)^2R_4^e(x)$ is a tempered distribution. Thus $O(x)$ defined by (2.34) is a tempered distribution, and we obtain that $O^{*k}(x)$ is a tempered distribution and $R_{6k}^{H}(x) * (-1)^{2k} R_{4k}^{e}(x) \in \mathcal{S}'$ is the space of tempered distribution. Choose $\mathcal{S}' \subset \mathfrak{D}'_{\mathcal{R}}$ where $\mathfrak{D}'_{\mathcal{R}}$ is the right-side distribution which is a subspace of $\mathfrak{D}'$ of distribution.

Thus $R_{6k}^{H}(x) * (-1)^{2k} R_{4k}^{e}(x) \in \mathfrak{D}'_{\mathcal{R}}$. It follows that $R_{6k}^{H}(x) * (-1)^{2k} R_{4k}^{e}(x)$ is an element of convolution algebra, since $\mathfrak{D}'_{\mathcal{R}}$ is a convolution algebra. Hence by the method of Zemanian (see [10]), (2.33) has a unique solution

$$G(x) = \left(R_{6k}^{H}(x) * (-1)^{2k} R_{4k}^{e}(x)\right) * \left(O^{*k}(x)\right)^{*-1}, \tag{2.44}$$

where $(O^{*k}(x))^{*-1}$ is an inverse of $O^{*k}(x)$ in the convolution algebra and $G(x)$ is called the Green function of the $\otimes^k$ operator. □

Lemma 2.10. *Given the equation*

$$L^k K(x) = \delta(x), \tag{2.45}$$

where L^k is the operator defined by

$$L^k = \left(\frac{3}{4}\Delta^2 + \frac{1}{4}\Box^2\right)^k \tag{2.46}$$

and Δ and $\Box$ are defined by (1.2) and (1.3) with $k = 1$, respectively, one obtains that $K(x)$ is an elementary solution of the L^k operator where

$$K(x) = \left(R_{4k}^{H}(x) * (-1)^{2k} R_{4k}^{e}(x)\right) * \left(O^{*k}(x)\right)^{*-1},$$
$$O(x) = \frac{3}{4}R_4^H(x) + \frac{1}{4}(-1)^2 R_4^e(x), \tag{2.47}$$

where $O^{*k}(x)$ denotes the convolution of $O(x)$ itself $k-$times and $(O^{*k}(x))^{*-1}$ denotes the inverse of $O^{*k}(x)$ in the convolution algebra. Moreover $K(x)$ is a tempered distribution.

Proof. The proof of Lemma 2.10 is similar to the proof of Lemma 2.9. □

Lemma 2.11. *Given the equation*

$$\Box u(x) = f(x, u(x)), \tag{2.48}$$

where f is defined and has continuous first derivatives for all $x \in \Omega \cup \partial\Omega$, where Ω is an open subset of R^n and $\partial\Omega$ is the boundary of Ω, assume that f is bounded, that is, $|f(x, u(x))| \leq N$ for all $x \in \Omega$.

Then one obtains a continuous function $u(x)$ as unique solution of (2.48) *with the boundary condition $u(x) = 0$ for $x \in \partial\Omega$.*

Proof. We can prove the existence of the solution $u(x)$ of (2.48) by the method of iterations and Schuder's estimates. The details of the proof are given by Courant and Hilbert; (see [4, pages 369–372]). □

Lemma 2.12. *The function $R^H_{-2k}(x)$ and $S_{-2k}(x)$ are the inverse of the convolution algebra of R^H_{2k} and S_{2k}, respectively, that is,*

$$R^H_{-2k}(x) * R^H_{2k}(x) = R^H_{-2k+2k}(x) = R^H_0(x) = \delta,$$
$$S_{-2k}(x) * S_{2k}(x) = S_{-2k+2k}(x) = S_0(x) = \delta. \tag{2.49}$$

Proof. (See [7, page 158] and [11]). □

3. Main Results

Theorem 3.1. *Given the equation*

$$\otimes^k \Diamond^k_B u(x) = 0, \tag{3.1}$$

where $\otimes^k$ is the Otimes operator iterated $k-$times and $\Diamond^k_B$ is Diamond Bessel operator iterated $k-$ times defined by (1.10) *and* (1.7)*, respectively, and $u(x)$ is an unknown function, one obtains that $u(x)$ is a solution of* (3.1) *where*

$$u(x) = K(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * (-1)^{k-1}(R^H_{2(k-1)}(\upsilon))^{(m)} \tag{3.2}$$

where $K(x)$ is defined by (2.47)*, as well as $S_{2k}(x)$, $R_{2k}(x)$, and $(R^H_{2(k-1)}(\upsilon))^m$ are defined by* (2.16),(2.17)*, and* (2.2) *with $\alpha = 2k$, $\gamma = 2k$ and $\alpha = 2(k-1)$, respectively.*

Proof. Since

$$\otimes^k = \left(\frac{3}{4}\Diamond\Delta + \frac{1}{4}\Box^3\right)^k, \qquad \Diamond^k_B = \Delta^k_B \Box^k_B. \tag{3.3}$$

Consider the homogeneous equation

$$\otimes^k \Diamond^k_B u(x) = 0. \tag{3.4}$$

The above equation can be written as

$$\left(\frac{3}{4}\Diamond\Delta + \frac{1}{4}\Box^3\right)^k \Delta^k_B \Box^k_B u(x) = 0, \tag{3.5}$$

or

$$\Box^k \left(\frac{3}{4}\Delta^2 + \frac{1}{4}\Box^2\right)^k \Delta^k_B \Box^k_B u(x) = 0. \tag{3.6}$$

That is,

$$\square^k L^k \Delta_B^k \square_B^k u(x) = 0, \tag{3.7}$$

where $\square^k, L^k, \Delta_B^k$, and $\square_B^k$ are defined by (1.3), (2.46), (1.8), and (1.9), respectively. By Lemma 2.8, we obtain

$$L^k \Delta_B^k \square_B^k u(x) = \left(R_{2(k-1)}^H(\upsilon)\right)^{(m)}. \tag{3.8}$$

Since $(-1)^k S_{2k}(x), R_{2k}(x)$ are the elementary solution of the operators Δ_B^k and $\square_B^k$, respectively, and by Lemma 2.10, we have that $K(x)$ is an elementary of the operator L^k defined by (2.46), that is,

$$\Delta_B^k(-1)^k S_{2k}(x) = \delta(x), \qquad \square_B^k R_{2k}(x) = \delta(x),$$
$$L^k K(x) = \delta(x). \tag{3.9}$$

Convolving both sides of (3.8) by $K(x) * (-1)^k S_{2k}(x) * R_{2k}(x)$, we obtain

$$K(x)*(-1)^k S_{2k}(x) * R_{2k}(x) * Ł^k \Delta_B^k \square_B^k u(x) = K(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * \left(R_{2(k-1)}^H(\upsilon)\right)^{(m)}. \tag{3.10}$$

By properties of convolution

$$Ł^k K(x) * \Delta_B^k (-1)^k S_{2k}(x) * \square_B^k R_{2k}(x) * u(x) = K(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * \left(R_{2(k-1)}^H(\upsilon)\right)^{(m)}. \tag{3.11}$$

By Lemmas 2.10, 2.5, and 2.6, we obtain

$$\delta(x) * \delta(x) * \delta(x) * u(x) = K(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * \left(R_{2(k-1)}^H(\upsilon)\right)^{(m)}. \tag{3.12}$$

Thus

$$u(x) = K(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * \left(R_{2(k-1)}^H(\upsilon)\right)^{(m)} \tag{3.13}$$

is the solution of (3.1). □

Theorem 3.2. *Given the equation*

$$\otimes^k \Diamond_B^k u(x) = f(x), \tag{3.14}$$

where $\otimes^k$ is the Otimes operator iterated $k-$times defined by (1.10), and $\Diamond_B^k$ is the Diamond Bessel operator iterated $k-$times defined by (1.7), $f(x)$ is the generalized function, $u(x)$ is an unknown function, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and n is even,

One obtains that

$$u(x) = K(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * \left(R^H_{2(k-1)}(\upsilon)\right)^{(m)} + G(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * f(x) \tag{3.15}$$

is a general solution of (3.14) and $G(x)$ is defined by (2.33), $K(x)$ is defined by (2.47), as well as $S_{2k}(x)$ and $R_{2k}(x)$ are defined by (2.16) and (2.17) with $\alpha = 2k$ and $\gamma = 2k$, respectively.

Proof. Consider the equation

$$\otimes^k \Diamond_B^k u(x) = f(x) \tag{3.16}$$

or

$$\otimes^k \Delta_B^k \Box_B^k u(x) = f(x). \tag{3.17}$$

Convolving both sides of (3.14) by $G(x) * (-1)^k S_{2k}(x) * R_{2k}(x)$, we obtain

$$G(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * \otimes^k \Delta_B^k \Box_B^k u(x) = G(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * f(x). \tag{3.18}$$

By properties of convolution,

$$\otimes^k G(x) * \Delta_B^k (-1)^k S_{2k}(x) * \Box_B^k R_{2k}(x) * u(x) = G(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * f(x). \tag{3.19}$$

By Lemmas 2.9, 2.5, and 2.6, we obtain

$$\delta(x) * \delta(x) * \delta(x) * u(x) = G(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * f(x). \tag{3.20}$$

Thus

$$u(x) = G(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * f(x). \tag{3.21}$$

Consider the homogeneous equation

$$\otimes^k \Diamond_B^k u(x) = 0. \tag{3.22}$$

By Theorem 3.1, we have a homogeneous solution

$$u(x) = K(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * (R^H_{2(k-1)}(\upsilon))^{(m)}. \tag{3.23}$$

Thus, the general solution of (3.14) is

$$u(x) = K(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * \left(R^H_{2(k-1)}(\upsilon)\right)^{(m)} + G(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * f(x), \tag{3.24}$$

as required. □

Theorem 3.3. *Consider the nonlinear equation*

$$\otimes^k \Diamond^k_B u(x) = f\left(x, \Box^{k-1} L^k \Delta^k_B \Box^k_B u(x)\right) \tag{3.25}$$

where $\otimes^k, \Diamond^k_B, \Box^{k-1}, L^k, \Delta^k_B$, *and* $\Box^k_B$ *are defined by* (1.10), (1.7),(1.3),(2.44), *and* (1.9), *respectively. Let* f *be defined, and having continuous first derivative for all* $x \in \Omega \cup \partial\Omega, \Omega$ *is an open subset of* R^n *and* $\partial\Omega$ *denotes the boundary function, that is,*

$$\left| f\left(x, \Box^{k-1} L^k \Delta^k_B \Box^k_B u(x)\right) \right| \leq N \tag{3.26}$$

for all $x \in \Omega$ *and the boundary condition*

$$\Box^{k-1} L^k \Delta^k_B \Box^k_B u(x) = 0 \tag{3.27}$$

for all $x \in \partial\Omega$. *Then one obtains*

$$u(x) = R^H_{2(k-1)}(x) * G(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * W(x) \tag{3.28}$$

as a solution of (3.25) *with the boundary condition*

$$u(x) = (R^H_{2(k-2)}(\upsilon))^{(m)} * G(x) * (-1)^k S_{2k}(x) * R_{2k}(x) \tag{3.29}$$

for all $x \in \partial\Omega, m = (n-4)/2$, *and* $W(x)$ *is a continuous function for* $x \in \Omega \cup \partial\Omega$, *while* $R^H_{2(k-2)(\upsilon)}, S_{2k}(x)$, *and* $R_{2k}(x)$ *are given by* (2.2), (2.16), *and* (2.17) *with* $\alpha = 2(k-2), \alpha = 2k$, *and* $\gamma = 2k$, *respectively. Moreover, for* $k = 1$ *one obtains*

$$M(x) = \left(R^H_{-4}(x) * (-1)^2 R^e_{-4}(x)\right) * \left(O^{*1}(x)\right) * (-1)^k S_{-2}(x) * u(x) \tag{3.30}$$

as a solution of the inhomogeneous equation

$$\Box\Box_B M(x) = W(x), \tag{3.31}$$

where $\square$ *and* $\square_B$ *are defined by* (1.3) *and* (1.9) *with* $k = 1$, *respectively, and* $u(x)$ *is obtained from* (3.28). *Furthermore, If one puts* $p = k = 1$, *then the operators* $\square^k$ *and* $\square_B^k$ *reduce to*

$$\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \cdots - \frac{\partial^2}{\partial x_n^2}, \qquad B_{x_1} - B_{x_2} - B_{x_3} - \cdots - B_{x_n}, \tag{3.32}$$

respectively, and the solution $M(x) = I_2^H(x) * I_2(x) * W(x)$ *is the inhomogeneous wave equation*

$$\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \cdots - \frac{\partial^2}{\partial x_n^2}\right) \cdot \left(B_{x_1} - B_{x_2} - B_{x_3} - \cdots - B_{x_n}\right) M(x) = W(x), \tag{3.33}$$

where $I_2^H(x)$ *is defined by* (2.6) *with* $\alpha = 2$ *and* $I_2(x)$ *is defined by* (2.20) *with* $\gamma = 2$.

Proof. Since

$$\otimes^k \lozenge_B^k u(x) = \square\square^{k-1} L^k \Delta_B^k \square_B^k u(x) = f\left(x, \square^{k-1} L^k \Delta_B^k \square_B^k u(x)\right), \tag{3.34}$$

$u(x)$ has continuous derivative up to order $6k$ for $k = 1, 2, 3, \ldots$, and $\square^{k-1} L^k \Delta_B^k \square_B^k u(x)$ exists as the generalized function. Thus we can assume that

$$\square^{k-1} L^k \Delta_B^k \square_B^k u(x) = W(x), \quad \forall x \in \Omega. \tag{3.35}$$

Then (3.34) can be written in the form

$$\otimes^k \lozenge_B^k u(x) = \square W(x) = f(x, W(x)). \tag{3.36}$$

By(3.26)

$$\left| f(x, W(x)) \right| \leq N, \quad x \in \Omega, \tag{3.37}$$

and by(3.27) $W(x) = 0, x \in \partial\Omega$, or

$$\square^{k-1} L^k \Delta_B^k \square_B^k u(x) = 0, \quad \forall x \in \partial\Omega. \tag{3.38}$$

We obtain a unique solution of (3.28) which satisfies (3.27) by Lemma 2.8.

Since $R_{2(k-1)}^H(x), (-1)^k S_{2k}(x)$, and $R_{2k}(x)$ are the elementary solution of the operators $\square^{k-1}$, Δ_B^k, and $\square_B^k$, respectively, and by Lemma 2.10, we have that $K(x)$ is an elementary of the operator L^k where $L^k = ((3/4)\Delta^2 + (1/4)\square^2)^k$, that is,

$$\begin{aligned} \square^{k-1} R_{2(k-1)}^H(x) = \delta, &\qquad \Delta_B^k (-1)^k S_{2k}(x) = \delta, \\ {}_B^k R_{2k}(x) = \delta, &\qquad L^k K(x) = \delta. \end{aligned} \tag{3.39}$$

From (3.35), we have

$$\square^{k-1} L^k \Delta_B^k \square_B^k u(x) = W(x). \tag{3.40}$$

Convolving the above equation by

$$R_{2(k-1)}^H(x) * K(x) * (-1)^k S_{2k}(x) * R_{2k}(x), \tag{3.41}$$

we obtain

$$\begin{aligned} &\left(R_{2(k-1)}^H(x) * K(x) * (-1)^k S_{2k}(x) * R_{2k}(x)\right) * \left(\square^{k-1} L^k \Delta_B^k \square_B^k u(x)\right) \\ &\quad = \left(R_{2(k-1)}^H(x) * K(x) * (-1)^k S_{2k}(x) * R_{2k}(x)\right) * W(x). \end{aligned} \tag{3.42}$$

By properties of convolution, we obtain

$$\begin{aligned} &\left(\square^{k-1} R_{2(k-1)}^H(x)\right) * \left(Ł^k K(x)\right) * \left(\Delta_B^k * (-1)^k S_{2k}\right) * \left(\square_B^k R_{2k}\right) * u(x) \\ &\quad = \left(R_{2(k-1)}^H(x) * K(x) * (-1)^k S_{2k}(x) * R_{2k}(x)\right) * W(x). \end{aligned} \tag{3.43}$$

By (3.39) we obtain

$$\delta * \delta * \delta * \delta * u(x) = \left(R_{2(k-1)}^H(x) * K(x) * (-1)^k S_{2k}(x) * R_{2k}(x)\right) * W(x). \tag{3.44}$$

Thus

$$u(x) = \left(R_{2(k-1)}^H(x) * K(x) * (-1)^k S_{2k}(x) * R_{2k}(x)\right) * W(x), \tag{3.45}$$

as a solution of (3.25).

Next, consider the boundary condition (3.38). From

$$\square^{k-1} L^k \Delta_B^k \square_B^k u(x) = 0, \tag{3.46}$$

by Lemma 2.8, we have

$$L^k \Delta_B^k \square_B^k u(x) = \left(R_{2(k-2)}^H(\upsilon)\right)^{(m)}, \tag{3.47}$$

where $m = (n-4)/2, n \geq 4$ and n is even. Convolving both sides of (3.47) by

$$K(x) * (-1)^k S_{2k}(x) * R_{2k}(x), \tag{3.48}$$

we obtain

$$\begin{aligned} &\left(K(x)*(-1)^k S_{2k}(x) *R_{2k}(x)\right)*\left(L^k \Delta_B^k \Box_B^k\right) * u(x) \\ &\quad = K(x) *(-1)^k S_{2k}(x) * R_{2k}(x) * \left(R^H_{2(k-2)}(\upsilon)\right)^{(m)}. \end{aligned} \tag{3.49}$$

By the properties of convolution, we obtain

$$\begin{aligned} &\left(L^k K(x)\right) * \left(\Delta_B^k (-1)^k S_{2k}\right) * \left(\Box_B^k R_{2k}\right) *u(x) \\ &\quad = K(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * (R^H_{2(k-2)}(\upsilon))^{(m)}. \end{aligned} \tag{3.50}$$

By (3.39), we obtain

$$\delta * \delta * \delta * u(x) = \left(K(x) * (-1)^k S_{2k}(x) * R_{2k}(x)\right) * (R^H_{2(k-2)}(\upsilon))^{(m)}. \tag{3.51}$$

Thus, for $x \in \partial\Omega$ and $k = 2, 3, 4, 5, \ldots,$

$$u(x) = K(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * \left(R^H_{2(k-2)}(\upsilon)\right)^{(m)}, \tag{3.52}$$

as required.

Now, for $k = 1$ in (3.28), we have

$$u(x) = \delta(x) * G(x) * (-1) S_2(x) * R_2(x) * W(x). \tag{3.53}$$

By (2.47), we have

$$G(x) = \left(R^H_6(x) * (-1)^2 R^e_4(x)\right) * \left(O^{*1}(x)\right)^{*-1}. \tag{3.54}$$

Taking into account (3.53), we obtain

$$u(x) = \left(R^H_6(x) * (-1)^2 R^e_4(x)\right) * \left(O^{*1}(x)\right)^{*-1} * (-1)^1 S_2(x) * R_2(x) * W(x) \tag{3.55}$$

as a solution of (3.25) for $k = 1$.

Convolving both sides of (3.55) by

$$\left(R^H_{-4}(x) * (-1)^2 R^e_{-4}(x)\right) * \left(O^{*1}(x)\right) * (-1) S_{-2}(x), \tag{3.56}$$

by Lemma 2.12, we obtain

$$\left(R^H_{-4}(x) * (-1)^2 R^e_{-4}(x)\right) * \left(O^{*1}(x)\right) * (-1) S_{-2}(x) * u(x) = R^H_2(x) * R_2(x) * W(x). \tag{3.57}$$

By Lemma 2.6, we obtain

$$M(x) = \left(R^H_{-4}(x) * (-1)^2 R^e_{-4}(x)\right) * \left(O^{*1}(x)\right) * (-1)S_{-2}(x) * u(x) \tag{3.58}$$

as a solution of the inhomogeneous equation

$$\Box\Box_B M(x) = W(x). \tag{3.59}$$

Now, consider the boundary condition for $k = 1$ in (3.27); we have

$$L\Delta_B \Box_B u(x) = 0, \quad \text{or} \quad \Box_B L\Delta_B u(x) = 0 \tag{3.60}$$

for $x \in \partial\Omega$. Thus by Lemma 2.8, for $k = 1$, we have

$$L\Delta_B u(x) = \delta^{(m)}(\upsilon) \quad \text{for } x \in \partial\Omega, \tag{3.61}$$

where $\delta^{(m)}(x) = R^H_0(x)$. Convolving the above equation by $K(x) * (-1)S_2(x)$ where $K(x)$ is defined by (2.47) with $k = 1$ and $S_2(x)$ is defined by (2.16) with $\alpha = 2$, we obtain

$$K(x) * (-1)S_2(x) * (L\Delta_B u(x)) = \delta^{(m)}(\upsilon) * K(x) * (-1)S_2(x). \tag{3.62}$$

By properties of convolution,

$$LK(x) * \Delta_B(-1)S_2(x) * u(x) = \delta^{(m)}(\upsilon) * K(x) * (-1)S_2(x). \tag{3.63}$$

By Lemmas 2.10 and 2.5, we obtain

$$\delta(x) * \delta(x) * u(x) = \delta^{(m)}(\upsilon) * K(x) * (-1)S_2(x). \tag{3.64}$$

It follows that

$$u(x) = \delta^{(m)}(\upsilon) * K(x) * (-1)S_2(x). \tag{3.65}$$

By (2.47) with $k = 1$, we have

$$K(x) = \left(R^H_4(x) * (-1)^2 R^e_4(x)\right) * \left(O^{*1}(x)\right)^{*-1}. \tag{3.66}$$

Taking into account (3.65), we obtain

$$u(x) = \delta^{(m)}(\upsilon) * \left(R^H_4(x) * (-1)^2 R^e_4(x)\right) * \left(O^{*1}(x)\right)^{*-1} * (-1)S_2(x) \quad \text{for } x \in \partial\Omega. \tag{3.67}$$

Now consider the case $k = 1, p = 1$, and $q = n - 1$, that is, from (3.59), $R_2^H(x)$ reduced to $I_2^H(x)$ where $I_2^H(x)$ is defined by (2.2) with $\alpha = 2$ and $R_2(x)$ reduced to $I_2(x)$ where $I_2(x)$ is defined by (2.17) with $\gamma = 2$, and then the operator $\Box$ defined by (1.3) reduces to the wave operator

$$\Box^* = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \cdots - \frac{\partial^2}{\partial x_n^2}, \tag{3.68}$$

$\Box_B$ defined by (1.9) reduces to the Bessel wave operator

$$\Box_B^* = B_{x_1} - B_{x_2} - B_{x_3} - \cdots - B_{x_n}, \tag{3.69}$$

and then the solution $M(x)$ reduced to

$$M(x) = I_2^H(x) * I_2(x) * W(x), \tag{3.70}$$

which is the solution of inhomogeneous wave equation

$$\Box^* \Box_B^* M(x) = W(x), \tag{3.71}$$

or

$$\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \cdots - \frac{\partial^2}{\partial x_n^2}\right) \cdot \left(B_{x_1} - B_{x_2} - B_{x_3} - \cdots - B_{x_n}\right) M(x) = W(x). \tag{3.72}$$

With the boundary condition for $x \in \partial\Omega$,

$$L^* \Box_B^* \Delta_B u(x) = 0, \tag{3.73}$$

where $L^* = (3/4)\Delta^2 + (1/4)(\Box^*)^2$ and $\Box^*$ is defined by (3.68), or for $x \in \partial\Omega$ and by (3.65), we obtain

$$u(x) = \delta^{(m)}(s) * \left(I_4^H(x) * (-1)^2 R_4^e(x)\right) * \left(D^{*1}(x)\right)^{*-1} * (-1) S_2(x), \tag{3.74}$$

where $I_4(x)$ is defined by (2.20) with $\gamma = 4$, $s = x_1^2 - x_2^2 - x_3^2 - \cdots - x_n^2$, and $D(x)$ reduced from $O(x)$ where it is defined by (2.34), that is, $D(x) = (3/4) I_4^H(x) + (1/2)(-1)^2 R_4^e(x)$. $\Box$

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