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Research Article

Existence, Uniqueness and Ergodicity of Positive Solution of Mutualism System with Stochastic Perturbation

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We discuss a two-species Lotka-Volterra mutualism system with stochastic perturbation. We show that there is a unique nonnegative solution of this system. Furthermore, we investigate that there exists a stationary distribution for this system, and it has ergodic property.

1. Introduction

It is well known that the differential equation

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)[r_1 - a_{11}x_1(t) + a_{12}x_2(t)], \\ \dot{x}_2(t) &= x_2(t)[r_2 + a_{21}x_1(t) - a_{22}x_2(t)] \end{aligned} \tag{1.1}$$

denotes the population growth of mutualism system for the two species. $x_1(t)$ and $x_2(t)$ represent the densities of the two species at time t, respectively, and the parameters $r_i, a_{ij}, i, j = 1, 2$ are all positive. Goh [1] showed that the asymptotic stability equilibrium state of (1.1) in local must be asymptotic stability in global. That is, if $r_i > 0$, $a_{ij} > 0$, $i, j = 1, 2$, and $a_{11}a_{22} - a_{12}a_{21} > 0$, then

$$x_1(t) \longrightarrow x_1^*, \quad x_2(t) \longrightarrow x_2^*, \quad \text{as } t \longrightarrow \infty, \tag{1.2}$$

where $x^* = (x_1^*, x_2^*)$ is the unique positive equilibrium of system (1.1) and

$$x_1^* = \frac{r_1 a_{22} + r_2 a_{12}}{a_{11}a_{22} - a_{12}a_{21}} > 0, \qquad x_2^* = \frac{r_2 a_{11} + r_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}} > 0. \tag{1.3}$$

While if $a_{11}a_{22} - a_{12}a_{21} < 0$, then the population of both species increase to infinite. There are extensive literature concerned with mutualism system; see [2–7].

The papers mentioned above are all deterministic models, which do not incorporate the effect of fluctuating environment. In fact, environmental fluctuations are important components in the population system. Most of natural phenomena do not follow strictly deterministic laws, but rather oscillate randomly around some average values, hence the deterministic equilibrium is no longer an absolutely fixed state [8]. Therefore stochastic differential equation models play a significant role in various branches of applied sciences including the population system, as they provide some additional degree of realism compared to their deterministic counterpart [9–13]. Recently, many authors have paid attention to how population systems are affected by random fluctuations from environment (see, e.g., [14–18]). However, as far as we known, there is few paper consider how environmental noises affect the dynamical behaviors of the mutualism system, Zeng et al. [19] discussed the effects of noise and time delay on $C(s)$ (the normalized correlation function) and T_c (the associated relaxation time) of a mutualism system, in which they considered the intraspecies interaction parameters were stochastically perturbed. Motivated by this, the main aim of this paper is to study the dynamical behaviors of the mutualism system with stochastic perturbation.

In this paper, considering the effect of randomly fluctuating environment, we incorporate white noise in each equation of system (1.1). Here we assume that fluctuations in the environment will manifest themselves mainly as fluctuations in the natural growth rates r_i, $i = 1, 2$. Suppose $r_i \to r_i + \sigma_i \dot{B}_i(t)$, where $B_i(t)$, $i = 1, 2$ are mutually independent one dimensional standard Brownian motions with $B_i(0) = 0$, and $\sigma_i > 0$, $i = 1, 2$ are the intensities of white noises. The stochastic version corresponding to the deterministic system (1.1) takes the following form:

$$\begin{aligned} dx_1(t) &= x_1(t)\left[(r_1 - a_{11}x_1(t) + a_{12}x_2(t))dt + \sigma_1 dB_1(t)\right], \\ dx_2(t) &= x_2(t)\left[(r_2 + a_{21}x_1(t) - a_{22}x_2(t))dt + \sigma_2 dB_2(t)\right]. \end{aligned} \tag{1.4}$$

This paper is organized as follows. In Section 2, we show there is a unique positive solution of (1.4) if $a_{11}a_{22} - a_{12}a_{21} > 0$, and give out the estimation of the solution. The stability of system (1.4) is investigated in Section 3. Since (1.4) does not have interior equilibrium, we cannot discuss the stability as the deterministic system. First, we show there is a stationary distribution of (1.4) and it has ergodic property. Next, by estimating the p moment, we explore some properties of the solution.

Throughout this paper, unless otherwise specified, let $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let R_+^2 denote the positive cone of R^2, namely $R_+^2 = \{x \in R^2 : x_i > 0,\ i = 1, 2\}$. If A is a vector, its transpose is denoted by $A^\top$. For $x \in R^2$, $|x| = |x_1| + |x_2|$.

2. The Existence and Estimation of the Solution

To investigate the dynamical behavior, the first concern thing is whether the solution is global existence. Moreover, for a population model, whether the value is nonnegative is also considered. Hence in this section we first show the solution of (1.4) is global and nonnegative. As we have known, for a stochastic differential equation to have a unique global solution for any given initial value, the coefficients of the equation are generally required to satisfy the linear growth condition and local Lipschitz condition (cf. Arnold [20], Mao [21]). However, the coefficients of (1.4) do not satisfy the linear growth condition, though they are locally Lipschitz continuous, so the solution of (1.4) may explode at a finite time. In this section, following the way developed by Mao et al. [17], we show there is a unique positive solution of (1.4).

Theorem 2.1. *There is a unique positive solution $x(t)$ of system* (1.4) *for any given initial value $x(0) = x_0 \in R_+^2$ provided $a_{11}a_{22} > a_{12}a_{21}$.*

Proof. The proof is similar to Theorem 2.1 in [17]. Here we define a C^2-function $V : R_+^2 \to R_+$:

$$V(x_1, x_2) = a_{21}\left[x_1 - x_1^* - x_1^* \log \frac{x_1}{x_1^*}\right] + a_{12}\left[x_2 - x_2^* - x_2^* \log \frac{x_2}{x_2^*}\right], \tag{2.1}$$

where $x^* = (x_1^*, x_2^*)^\top$ satisfies

$$\begin{aligned} r_1 - a_{11}x_1^* + a_{12}x_2^* &= 0, \\ r_2 + a_{21}x_1^* - a_{22}x_2^* &= 0. \end{aligned} \tag{2.2}$$

□

Remark 2.2. Theorem 2.1 shows stochastic equation (1.4) also has a global positive solution under the same condition of the corresponding deterministic system (1.1). That is to say, the white noise does not affect the existence of the unique global positive solution.

In the remaining of this section, we give the estimation of the solution of system (1.4). Jiang and Shi [22] discussed a randomized nonautonomous logistic equation:

$$dN(t) = N(t)[(a(t) - b(t)N(t))dt + \alpha(t)dB(t)], \tag{2.3}$$

where $B(t)$ is 1-dimensional standard Brownian motion, $N(0) = N_0$ and N_0 is independent of $B(t)$. They showed the following.

Lemma 2.3 (see [22]). *Assume that $a(t)$, $b(t)$ and $\alpha(t)$ are bounded continuous functions defined on $[0, \infty)$, $a(t) > 0$ and $b(t) > 0$. Then there exists a unique continuous positive solution of* (2.3) *for any initial value $N(0) = N_0 > 0$, which is global and represented by*

$$N(t) = \frac{e^{\int_0^t [a(s) - \alpha^2(s)/2]ds + \alpha(s)dB(s)}}{1/N_0 + \int_0^t b(s) e^{\int_0^s [a(\tau) - \alpha^2(\tau)/2]d\tau + \alpha(\tau)dB(\tau)} ds}, \quad t \geq 0. \tag{2.4}$$

Theorem 2.4. *Assume that* $a_{11}a_{22} > a_{12}a_{21}$ *and* $x(t) \in R_+^2$ *is the solution of system* (1.4) *with initial value* $x_0 \in R_+^2$. *Then* $x(t)$ *has the property that*

$$x_1(t) \geq \phi_1(t), \qquad x_2(t) \geq \phi_2(t), \tag{2.5}$$

where $\phi_1(t)$ *and* $\phi_2(t)$ *are the solutions of equations:*

$$d\phi_1(t) = \phi_1(t)\left[(r_1 - a_{11}\phi_1(t))dt + \sigma_1 dB_1(t)\right], \quad \phi_1(0) = x_1(0), \tag{2.6}$$

$$d\phi_2(t) = \phi_2(t)\left[(r_2 - a_{22}\phi_2(t))dt + \sigma_2 dB_2(t)\right], \quad \phi_2(0) = x_2(0). \tag{2.7}$$

The result of Theorem 2.4 follows directly from the classical comparison theorem of stochastic differential equations (see [23]).

Remark 2.5. From Lemma 2.3, we see

$$\phi_1(t) = \frac{e^{(r_1-\sigma_1^2/2)t+\sigma_1 B_1(t)}}{1/x_1(0) + a_{11}\int_0^t e^{(r_1-\sigma_1^2/2)s+\sigma_1 B_1(s)}ds}, \qquad \phi_2(t) = \frac{e^{(r_2-\sigma_2^2/2)t+\sigma_2 B_2(t)}}{1/x_2(0) + a_{22}\int_0^t e^{(r_2-\sigma_2^2/2)s+\sigma_2 B_2(s)}ds}. \tag{2.8}$$

This together with Theorem 2.4 shows that if $r_i > (\sigma_i^2/2)(i = 1,2)$, then both species will not extinct.

3. Stationary Distribution and Ergodicity for System (1.4)

In the introduction, we have mentioned that if $r_i > 0$, $a_{ij} > 0$, $i,j = 1,2$, and $a_{11}a_{22} - a_{12}a_{21} > 0$, then the unique positive equilibrium (x_1^*, x_2^*) of (1.1) is globally stable. But there is none positive equilibrium for (1.4). We investigate there is a stationary distribution for system (1.4) instead of asymptotically stable equilibria [24]. Before giving the main theorem, we first give a lemma (see [25]).

Assumption B. There exists a bounded domain $U \subset E_l$ with regular boundary Γ, having the following properties.

(B.1) In the domain U and some neighbourhood thereof, the smallest eigenvalue of the diffusion matrix $A(x)$ is bounded away from zero.

(B.2) If $x \in E_l \setminus U$, the mean time τ at which a path issuing from x reaches the set U is finite, and $\sup_{x\in K} E_x\tau < \infty$ for every compact subset $K \subset E_l$.

Lemma 3.1 (see [25]). *If (B) holds, then the Markov process* $X(t)$ *has a stationary distribution* $\mu(A)$. *Let* $f(\cdot)$ *be a function integrable with respect to the measure* μ. *Then*

$$P_x\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T f(X(t))dt = \int_{E_l} f(x)\mu(dx)\right\} = 1 \tag{3.1}$$

for all $x \in E_l$.

Theorem 3.2. *Assume $a_{11}a_{22} > a_{12}a_{21}$, $\sigma_1 > 0$, $\sigma_2 > 0$ and $\delta < \min\{m_1(x_1^*)^2, m_2(x_2^*)^2\}$. Then there is a stationary distribution $\mu(A)$ for system* (1.4) *and it has ergodic property. Here (x_1^*, x_2^*) is the solution of* (2.2), *$\delta = (a_{21}x_1^*\sigma_1^2 + a_{12}x_2^*\sigma_2^2)/2$, $m_1 = a_{11}a_{21} - a_{12}a_{21}/\epsilon_0 > 0$ and $m_2 = a_{12}a_{22} - \epsilon_0 a_{12}a_{21} > 0$, where $\epsilon_0 > 0$ satisfies $a_{12}/a_{11} < \epsilon_0 < a_{22}/a_{21}$.*

Proof. Define $V : E_l = R_+^2 \to R_+$:

$$V(x_1, x_2) = a_{21}\left(x_1 - x_1^* - x_1^* \log \frac{x_1}{x_1^*}\right) + a_{12}\left(x_2 - x_2^* - x_2^* \log \frac{x_2}{x_2^*}\right). \tag{3.2}$$

Then

$$\begin{aligned} dV &= a_{21}\left(1 - \frac{x_1^*}{x_1}\right)dx_1 + \frac{a_{21}}{2}\frac{x_1^*}{x_1^2}(dx_1)^2 + a_{12}\left(1 - \frac{x_2^*}{x_2}\right)dx_2 + \frac{a_{12}}{2}\frac{x_2^*}{x_2^2}(dx_2)^2 \\ &= a_{21}(x_1 - x_1^*)[(r_1 - a_{11}x_1 + a_{12}x_2)dt + \sigma_1 dB_1(t)] + \frac{1}{2}a_{21}x_1^*\sigma_1^2 dt \\ &\quad + a_{12}(x_2 - x_2^*)[(r_2 + a_{21}x_1 - a_{22}x_2)dt + \sigma_2 dB_2(t)] + \frac{1}{2}a_{12}x_2^*\sigma_2^2 dt \\ &:= LV dt + a_{21}\sigma_1(x_1 - x_1^*)dB_1(t) + a_{12}\sigma_2(x_2 - x_2^*)dB_2(t), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} LV &= a_{21}(x_1 - x_1^*)(r_1 - a_{11}x_1 + a_{12}x_2) + \frac{1}{2}a_{21}x_1^*\sigma_1^2 \\ &\quad + a_{12}(x_2 - x_2^*)(r_2 + a_{21}x_1 - a_{22}x_2) + \frac{1}{2}a_{12}x_2^*\sigma_2^2 \\ &= a_{21}(x_1 - x_1^*)[-a_{11}(x_1 - x_1^*) + a_{12}(x_2 - x_2^*)] \\ &\quad + a_{12}(x_2 - x_2^*)[a_{21}(x_1 - x_1^*) - a_{22}(x_2 - x_2^*)] + \delta \\ &= -a_{11}a_{21}(x_1 - x_1^*)^2 + 2a_{12}a_{21}(x_1 - x_1^*)(x_2 - x_2^*) \\ &\quad - a_{12}a_{22}(x_2 - x_2^*)^2 + \delta, \end{aligned} \tag{3.4}$$

according to the equality (2.2). By Young inequality, we have

$$2a_{12}a_{21}|x_1 - x_1^*||x_2 - x_2^*| \le a_{12}a_{21}\left[\frac{(x_1 - x_1^*)^2}{\epsilon_0} + \epsilon_0(x_2 - x_2^*)^2\right]. \tag{3.5}$$

Then

$$\begin{aligned} LV &\le -\left(a_{11}a_{21} - \frac{a_{12}a_{21}}{\epsilon_0}\right)(x_1 - x_1^*)^2 - (a_{12}a_{22} - \epsilon_0 a_{12}a_{21})(x_2 - x_2^*)^2 + \delta \\ &= -m_1(x_1 - x_1^*)^2 - m_2(x_2 - x_2^*)^2 + \delta. \end{aligned} \tag{3.6}$$

Note that $\delta < \min\{m_1(x_1^*)^2, m_2(x_2^*)^2\}$; then the ellipsoid

$$-m_1(x_1 - x_1^*)^2 - m_2(x_2 - x_2^*)^2 + \delta = 0 \tag{3.7}$$

lies entirely in R_+^2. We can take U to be any neighborhood of the ellipsoid with $\overline{U} \subseteq E_l = R_+^2$, so for $x \in U \setminus E_l$, $LV \le 0$, which implies that condition (B.2) in Lemma 3.1 is satisfied. Besides, there is $M > 0$ such that

$$\sum_{i,j=1}^{n}\left(\sum_{k=1}^{n} g_{ik}(x)g_{jk}(x)\right)\xi_i\xi_j = \sigma_1^2 x_1^2 \xi_1^2 + \sigma_2^2 x_2^2 \xi_2^2 \ge M\left|\xi^2\right| \quad x \in \overline{U},\ \xi \in R^2, \tag{3.8}$$

which implies condition (B.1) is also satisfied. Therefore, the stochastic system (1.4) has a stable a stationary distribution $\mu(A)$ and it is ergodic. □

Remark 3.3. If $a_{11}a_{22} > a_{12}a_{21}$ and $a_{22} > a_{21}$, we can choose $\epsilon_0 = (1/2)(a_{12}/a_{11} + a_{22}/a_{21})$, then $m_1 = a_{11}a_{12}(a_{22} - a_{21})/(a_{11}a_{22} + a_{12}a_{21})$ and $m_2 = a_{12}(a_{11}a_{22} - a_{12}a_{21})/2a_{11}$.

Since system (1.4) is ergodic, next we explore some properties of the solution.

Consider the equation

$$\dot{N}(t) = N(t)[a - bN(t)] \tag{3.9}$$

with initial value $N_0 > 0$. It is well known, when $a, b > 0$, (3.9) has a unique positive solution

$$N(t) = \frac{e^{at}}{1/\widetilde{N}_0 + (b/a)(e^{at} - 1)}, \quad t \ge 0, \tag{3.10}$$

$$\lim_{t\to\infty} N(t) = \frac{a}{b}, \qquad \lim_{t\to\infty}\frac{\log N(t)}{t} = 0. \tag{3.11}$$

Lemma 3.4. *Suppose that $r_1 > \sigma_1^2/2$ and $\phi_1(t)$ is the solution of* (2.6)*, then one has*

$$\psi_1(t)e^{-\sigma_1(\max_{0\le s\le t} B_1(s) - B_1(t))} \le \phi_1(t) \le \psi_1(t)e^{-\sigma_1(\min_{0\le s\le t} B_1(s) - B_1(t))}, \tag{3.12}$$

where $\psi_1(t)$ is the solution of

$$\begin{gathered}\dot{\psi}_1(t) = \psi_1(t)\left[r_1 - \frac{\sigma_1^2}{2} - a_{11}\psi_1(t)\right], \\ \psi_1(0) = x_1(0).\end{gathered} \tag{3.13}$$

Proof. From the representation of the solution $\phi_1(t)$, we have

$$\begin{aligned}\frac{1}{\phi_1(t)} &= \frac{1}{x_1(0)}e^{-(r_1-\sigma_1^2/2)t-\sigma_1 B_1(t)} + a_{11}\int_0^t e^{-(r_1-\sigma_1^2/2)(t-s)-\sigma(B_1(t)-B_1(s))}ds \\ &= e^{-\sigma_1 B_1(t)}\left[\frac{1}{x_1(0)}e^{-(r_1-\sigma_1^2/2)t} + a_{11}\int_0^t e^{-(r_1-\sigma_1^2/2)(t-s)}e^{\sigma_1 B_1(s)}ds\right] \\ &\le e^{-\sigma_1 B_1(t)}\left[\frac{1}{x_1(0)}e^{-(r_1-\sigma^2/2)t} + a_{11}e^{\sigma_1 \max_{0\le s\le t}B_1(s)}\int_0^t e^{-(r_1-\sigma_1^2/2)(t-s)}ds\right] \\ &\le e^{\sigma_1[\max_{0\le s\le t}B_1(s)-B_1(t)]}\left[\frac{1}{x_1(0)}e^{-(r_1-\sigma_1^2/2)t} + a_{11}\int_0^t e^{-(r_1-\sigma_1^2/2)(t-s)}ds\right],\end{aligned} \tag{3.14}$$

where the last inequality is based on the property of Brownian motion that $B(0) = 0$. Similarly, we have

$$\frac{1}{\phi_1(t)} \ge e^{\sigma_1(\min_{0\le s\le t}B_1(s)-B_1(t))}\left[\frac{1}{x_1(0)}e^{-(r_1-\sigma_1^2/2)t} + a_{11}\int_0^t e^{-(r_1-\sigma_1^2/2)(t-s)}ds\right]. \tag{3.15}$$

Therefore

$$e^{\sigma_1(\min_{0\le s\le t}B_1(s)-B_1(t))}\frac{1}{\psi_1(t)} \le \frac{1}{\phi_1(t)} \le e^{\sigma_1(\max_{0\le s\le t}B_1(s)-B_1(t))}\frac{1}{\psi_1(t)}, \tag{3.16}$$

which is as required. □

Lemma 3.5. *Suppose that $r_1 > \sigma_1^2/2$ and $\phi_1(t)$ is the solution of* (2.6), *then one has*

$$\lim_{t\to\infty}\frac{\log\phi_1(t)}{t} = 0. \tag{3.17}$$

Proof. It is easy to drive from Lemma 3.4 that

$$\sigma_1\left(B_1(t) - \max_{0\le s\le t}B_1(s)\right) \le \log\phi_1(t) - \log\psi_1(t) \le \sigma_1\left(B_1(t) - \min_{0\le s\le t}B_1(s)\right). \tag{3.18}$$

Note that the distribution of $\max_{0\le s\le t}B_1(s)$ is the same as $|B_1(t)|$, and that $\min_{0\le s\le t}B_1(s)$ has the same distribution as—$\max_{0\le s\le t}B_1(s)$, then by (3.11) and the strong law of large numbers, we get

$$\lim_{t\to\infty}\frac{\log\phi_1(t)}{t} = \lim_{t\to\infty}\frac{\log\psi_1(t)}{t} = 0. \tag{3.19}$$

This completes the proof of Lemma 3.5. □

Now consider the solution of (2.7), by the same reasons as Lemmas 3.4 and 3.5, we have the following.

Lemma 3.6. *Suppose that $r_2 > \sigma_2^2/2$ and $\phi_2(t)$ is the solution of* (2.7), *then one has*

$$\lim_{t\to\infty} \frac{\log \phi_2(t)}{t} = 0. \tag{3.20}$$

Lemma 3.7. *Let $M(t) = \int_0^t e^s dB(s)$, where $B(t)$ is 1-dimensional standard Brownian motion, then*

$$\limsup_{t\to\infty} \frac{\left|e^{-t} M(t)\right|}{\sqrt{\log t}} = 1 \quad a.s. \tag{3.21}$$

Proof. The proof can be found on [21, page 70]. □

Based on these lemmas, now we show the main result in this section.

Let $y_1(t) = \log x_1(t)$, $y_2(t) = \log x_2(t)$; then by Itô's formula we obtain

$$\begin{aligned} dy_1(t) &= \left(r_1 - \frac{\sigma_1^2}{2} - a_{11}x_1(t) + a_{12}x_2(t) \right) dt + \sigma_1 dB_1(t), \\ dy_2(t) &= \left(r_2 - \frac{\sigma_2^2}{2} + a_{21}x_1(t) - a_{22}x_2(t) \right) dt + \sigma_2 dB_2(t). \end{aligned} \tag{3.22}$$

If $a_{11}a_{22} > a_{12}a_{21}$ and $2r_1 > \sigma_1^2$, $2r_2 > \sigma_2^2$, then the equation

$$\begin{aligned} r_1 - \frac{\sigma_1^2}{2} - a_{11}x_1 + a_{12}x_2 &= 0, \\ r_2 - \frac{\sigma_2^2}{2} + a_{21}x_1 - a_{22}x_2 &= 0 \end{aligned} \tag{3.23}$$

has a unique positive solution:

$$\tilde{x}_1^* = \frac{a_{22}\left(r_1 - \sigma_1^2/2\right) + a_{12}\left(r_2 - \sigma_2^2/2\right)}{a_{11}a_{22} - a_{12}a_{21}}, \qquad \tilde{x}_2^* = \frac{a_{21}\left(r_1 - \sigma_1^2/2\right) + a_{11}\left(r_2 - \sigma_2^2/2\right)}{a_{11}a_{22} - a_{12}a_{21}}. \tag{3.24}$$

Lemma 3.8. *Assume $a_{11}a_{22} > a_{12}a_{21}$ and $2r_1 > \sigma_1^2, 2r_2 > \sigma_2^2$. Then for any initial value $x_0 \in R_+^2$, the solution $x(t)$ of system* (1.4) *has the following property:*

$$\lim_{t\to\infty} \overline{x}_1(t) = \widetilde{x}_1^*, \qquad \lim_{t\to\infty} \overline{x}_2(t) = \widetilde{x}_2^*, \tag{3.25}$$

where $\overline{x}_i(t) = (1/t)\int_0^t x_i(s)ds,\ i = 1,2.$

Proof. It follows from (3.22) that

$$\begin{aligned}\frac{y_1(t)}{t} &= \frac{y_1(0)}{t} + r_1 - \frac{\sigma_1^2}{2} - a_{11}\overline{x}_1(t) + a_{12}\overline{x}_2(t) + \sigma_1\frac{B_1(t)}{t},\\ \frac{y_2(t)}{t} &= \frac{y_2(0)}{t} + r_2 - \frac{\sigma_2^2}{2} + a_{21}\overline{x}_1(t) - a_{22}\overline{x}_2(t) + \sigma_2\frac{B_2(t)}{t}.\end{aligned} \tag{3.26}$$

Obviously, to prove the result, it is an easy consequence of

$$\lim_{t\to\infty}\frac{y_i(t)}{t} = \lim_{t\to\infty}\frac{\log x_i(t)}{t} = 0, \quad i = 1,2 \text{ a.s.} \tag{3.27}$$

We first show that

$$\liminf_{t\to\infty}\frac{\log x_i(t)}{t} \geq 0, \quad i = 1,2 \text{ a.s.} \tag{3.28}$$

In fact, the results of Theorem 2.4 and Lemmas 3.5 and 3.6 imply that (3.28) is true.

Next, we will prove

$$\limsup_{t\to\infty}\frac{\log x_i(t)}{t} \leq 0, \quad i = 1,2 \text{ a.s.} \tag{3.29}$$

If $a_{11}a_{22} > a_{12}a_{21}$, then there exist positive constants c_1, c_2, m_1, m_2 such that

$$\begin{aligned}-a_{11}c_1 + a_{21}c_2 &= -m_1,\\ a_{12}c_1 - a_{22}c_2 &= -m_2.\end{aligned} \tag{3.30}$$

From (3.22) we get

$$
\begin{aligned}
&d\big(c_1y_1(t)+c_2y_2(t)\big)\\
&\quad= c_1 d\log x_1(t) + c_2 d\log x_2(t)\\
&\quad= c_1\left[\left(r_1-\frac{\sigma_1^2}{2}-a_{11}x_1(t)+a_{12}x_2(t)\right)dt+\sigma_1 dB_1(t)\right]\\
&\qquad+c_2\left[\left(r_2-\frac{\sigma_2^2}{2}+a_{21}x_1(t)-a_{22}x_2(t)\right)dt+\sigma_2 dB_2(t)\right]\\
&\quad=\left[\left(r_1-\frac{\sigma_1^2}{2}\right)c_1+\left(r_2-\frac{\sigma_2^2}{2}\right)c_2+(a_{21}c_2-a_{11}c_1)x_1(t)+(a_{12}c_1-a_{22}c_2)x_2(t)\right]dt\\
&\qquad+c_1\sigma_1 dB_1(t)+c_2\sigma_2 dB_2(t)\\
&\quad=\left[\left(r_1-\frac{\sigma_1^2}{2}\right)c_1+\left(r_2-\frac{\sigma_2^2}{2}\right)c_2-m_1x_1(t)-m_2x_2(t)\right]dt\\
&\qquad+c_1\sigma_1 dB_1(t)+c_2\sigma_2 dB_2(t),\\
&d\big[e^t\big(c_1y_1(t)+c_2y_2(t)\big)\big]\\
&\quad=e^t d\big(c_1y_1(t)+c_2y_2(t)\big)+e^t\big(c_1y_1(t)+c_2y_2(t)\big)dt\\
&\quad=e^t\left[\left(r_1-\frac{\sigma_1^2}{2}\right)c_1+\left(r_2-\frac{\sigma_2^2}{2}\right)c_2+c_1y_1(t)-m_1e^{y_1(t)}+c_2y_2(t)-m_2e^{y_2(t)}\right]dt\\
&\qquad+e^t\left[c_1\sigma_1 dB_1(t)+c_2\sigma_2 dB_2(t)\right].
\end{aligned}
\tag{3.31}
$$

Note that the function $c_1y_1 - m_1e^{y_1}$ has its maximum value $c_1^* = c_1\log(c_1/m_1) - c_1$ at $y_1 = \log(c_1/m_1)$, and the function $c_2y_2 - m_2e^{y_2}$ has its maximum value $c_2^* = c_2\log(c_2/m_2) - c_2$ at $y_2 = \log(c_2/m_2)$; then

$$
\begin{aligned}
d\big[e^t\big(c_1y_1(t)+c_2y_2(t)\big)\big] &\le e^t\left[\left(r_1-\frac{\sigma_1^2}{2}\right)c_1+\left(r_2-\frac{\sigma_2^2}{2}\right)c_2+c_1^*+c_2^*\right]dt\\
&\quad+e^t\left[c_1\sigma_1 dB_1(t)+c_2\sigma_2 dB_2(t)\right]\\
&:= c^*e^t dt+e^t\left[c_1\sigma_1 dB_1(t)+c_2\sigma_2 dB_2(t)\right],
\end{aligned}
\tag{3.32}
$$

where $c^* = (r_1 - \sigma_1^2/2)c_1 + (r_2 - \sigma_2^2/2)c_2 + c_1^* + c_2^*$. Integrating both sides of it from 0 to t, yields

$$\begin{aligned}
e^t\left(c_1y_1(t) + c_2y_2(t)\right) &\le c_1y_1(0) + c_2y_2(0) + c^*\left(e^t - 1\right) + \int_0^t e^s[c_1\sigma_1 dB_1(s) + c_2\sigma_2 dB_2(s)], \\
c_1y_1(t) + c_2y_2(t) &\le c^* + \left(c_1y_1(0) + c_2y_2(0) - c^*\right)e^{-t} + e^{-t}\int_0^t e^s[c_1\sigma_1 dB_1(s) + c_2\sigma_2 dB_2(s)] \\
&\le c^* + c_1y_1(0) + c_2y_2(0) + e^{-t}\int_0^t e^s[c_1\sigma_1 dB_1(s) + c_2\sigma_2 dB_2(s)] \\
&:= c + e^{-t}\int_0^t e^s[c_1\sigma_1 dB_1(s) + c_2\sigma_2 dB_2(s)],
\end{aligned} \tag{3.33}$$

where $c = c^* + c_1y_1(0) + c_2y_2(0)$. It is easy to drive from Lemma 3.7 that

$$c_1y_1(t) + c_2y_2(t) \le c + O_{\text{a.s.}}\left(\sqrt{\log t}\right), \tag{3.34}$$

which implies

$$c_1\limsup_{t\to\infty}\frac{y_1(t)}{t} + c_2\limsup_{t\to\infty}\frac{y_2(t)}{t} \le 0, \quad \text{a.s.} \tag{3.35}$$

Moreover (3.35) together with (3.28) shows that

$$c_1\limsup_{t\to\infty}\frac{y_1(t)}{t} \le -c_2\limsup_{t\to\infty}\frac{y_2(t)}{t} \le -c_2\liminf_{t\to\infty}\frac{y_2(t)}{t} \le 0, \quad \text{a.s.} \tag{3.36}$$

that is,

$$\limsup_{t\to\infty}\frac{y_1(t)}{t} \le 0, \quad \text{a.s.} \tag{3.37}$$

Similarly, we have

$$\limsup_{t\to\infty}\frac{y_2(t)}{t} \le 0, \quad \text{a.s.} \tag{3.38}$$

which is as required. □

Lemma 3.9. *Assume $a_{11} > a_{12}$ and $a_{22} > a_{21}$. Then for any initial value $x_0 \in R_+^2$, there exists a positive constant $K(p)$ such that the solution $x(t)$ of system* (1.4) *has the following property:*

$$E\left[c_1x_1^p(t) + c_2x_2^p(t)\right] \le K(p), \quad \forall t \in [0,\infty],\ p > 1. \tag{3.39}$$

Proof. By Itô's formula and Young inequality, we compute

$$
\begin{aligned}
d\left(\frac{1}{p}x_1^p\right) &= x_1^{p-1}dx_1 + \frac{p-1}{2}x_1^{p-2}(dx_1)^2 \\
&= \left[\left(r_1 + \frac{p-1}{2}\sigma_1^2\right)x_1^p - a_{11}x_1^{p+1} + a_{12}x_1^p x_2\right]dt + \sigma_1 x_1^p dB_1(t) \\
&\le \left[\left(r_1 + \frac{p-1}{2}\sigma_1^2\right)x_1^p - a_{11}x_1^{p+1} + \frac{a_{12}p}{p+1}x_1^{p+1} + \frac{a_{12}}{p+1}x_2^{p+1}\right]dt + \sigma_1 x_1^p dB_1(t), \\
d\left(\frac{1}{p}x_2^p\right) &= x_2^{p-1}dx_2 + \frac{p-1}{2}x_2^{p-2}(dx_2)^2 \\
&= \left[\left(r_2 + \frac{p-1}{2}\sigma_2^2\right)x_2^p + a_{21}x_1x_2^p - a_{22}x_2^{p+1}\right]dt + \sigma_2 x_2^p dB_2(t) \\
&\le \left[\left(r_2 + \frac{p-1}{2}\sigma_2^2\right)x_2^p + \frac{a_{21}}{p+1}x_1^{p+1} + \frac{a_{21}p}{p+1}x_2^{p+1} - a_{22}x_2^{p+1}\right]dt + \sigma_2 x_2^p dB_2(t).
\end{aligned}
\tag{3.40}
$$

Hence, for positive constants c_1 and c_2, we have

$$
\begin{aligned}
c_1 d\left(\frac{1}{p}x_1^p\right) + c_2 d\left(\frac{1}{p}x_2^p\right) \le \Bigg[&c_1\left(r_1 + \frac{p-1}{2}\sigma_1^2\right)x_1^p - \left(c_1a_{11} - \frac{c_1a_{12}p}{p+1} - \frac{c_2a_{21}}{p+1}\right)x_1^{p+1} \\
&+ c_2\left(r_2 + \frac{p-1}{2}\sigma_2^2\right)x_2^p - \left(c_2a_{22} - \frac{c_1a_{12}}{p+1} - \frac{c_2a_{21}p}{p+1}\right)x_2^{p+1}\Bigg]dt \\
&+ c_1\sigma_1 x_1^p dB_1(t) + c_2\sigma_2 x_2^p dB_2(t).
\end{aligned}
\tag{3.41}
$$

Next, we claim that there are $c_1 > 0,\ c_2 > 0$ such that

$$
c_1a_{11} - \frac{c_1a_{12}p}{p+1} - \frac{c_2a_{21}}{p+1} > 0, \qquad c_2a_{22} - \frac{c_1a_{12}}{p+1} - \frac{c_2a_{21}p}{p+1} > 0, \tag{3.42}
$$

if $a_{11} > a_{12}$ and $a_{22} > a_{21}$. In fact, we only need $a_{21}/(a_{11}(p+1) - a_{12}p) < c_1/c_2 < (a_{22}(p+1) - a_{21}p)/a_{12}$, which can be simplified to $a_{12}a_{21} < [a_{11}(p+1) - a_{12}p][\mathrm{a}_{22}(p+1) - a_{21}p]$. It is obviously true, if $a_{11} > a_{12}$ and $a_{22} > a_{21}$.

Let $\alpha_1 := p(r_1 + ((p-1)/2)\sigma_1^2)$, $\alpha_2 := p(r_2 + ((p-1)/2)\sigma_2^2)$, $\beta_1 := c_1^{-(p+1)/p} p(c_1 a_{11} - c_1 a_{12} p/(p+1) - c_2 a_{21}/(p+1))$, $\beta_2 := c_2^{-(p+1)/p} p(c_2 a_{22} - c_1 a_{12}/(p+1) - c_2 a_{21} p/(p+1))$. Then $\alpha_1 > 0,\ \alpha_2 > 0,\ \beta_1 > 0,\ \beta_2 > 0$ and

$$\begin{aligned} d\left(c_1 x_1^p + c_2 x_2^p\right) &\le \left[c_1\alpha_1 x_1^p + c_2\alpha_2 x_2^p - c_1^{(p+1)/p}\beta_1 x_1^{p+1} - c_2^{(p+1)/p}\beta_2 x_2^{p+1}\right]dt \\ &\quad + \frac{c_1}{p}\sigma_1 x_1^p dB_1(t) + \frac{c_2}{p}\sigma_2 x_2^p dB_2(t) \\ &\le \left[\max\{\alpha_1,\alpha_2\}\left(c_1 x_1^p + c_2 x_2^p\right) - \min\{\beta_1,\beta_2\}\left(c_1^{(p+1)/p} x_1^{p+1} + c_2^{(p+1)/p} x_2^{p+1}\right)\right]dt \\ &\quad + \frac{c_1}{p}\sigma_1 x_1^p dB_1(t) + \frac{c_2}{p}\sigma_2 x_2^p dB_2(t). \end{aligned} \tag{3.43}$$

Hence,

$$\begin{aligned} dE\left[c_1 x_1^p + c_2 x_2^p\right] &\le \left\{\max\{\alpha_1,\alpha_2\}E\left[c_1 x_1^p + c_2 x_2^p\right] - \min\{\beta_1,\beta_2\}E\left[c_1^{(p+1)/p} x_1^{p+1} + c_2^{(p+1)/p} x_2^{p+1}\right]\right\}dt \\ &\le \left\{\max\{\alpha_1,\alpha_2\}E\left[c_1 x_1^p + c_2 x_2^p\right] - 2^{-p}\min\{\beta_1,\beta_2\}E\left[c_1 x_1^p + c_2 x_2^p\right]^{(p+1)/p}\right\}dt. \end{aligned} \tag{3.44}$$

By comparison theorem, we can get

$$\limsup_{t\to\infty} E\left[c_1 x_1^p + c_2 x_2^p\right] \le \left[\frac{2^p \max\{\alpha_1,\alpha_2\}}{\min\{\beta_1,\beta_2\}}\right]^p := C(p), \tag{3.45}$$

which implies that there is a $T > 0$, such that

$$E\left[c_1 x_1^p + c_2 x_2^p\right] \le 2C(p), \quad \forall t > T. \tag{3.46}$$

Besides, note that $E[c_1 x_1^p + c_2 x_2^p]$ is continuous, then there is a $\widetilde{C}(p) > 0$ such that

$$E\left[c_1 x_1^p + c_2 x_2^p\right] \le \widetilde{C}(p), \quad \forall t \in [0,T]. \tag{3.47}$$

Let $K(p) = \max\{2C(p), \widetilde{C}(p)\}$, then

$$E\left[c_1 x_1^p + c_2 x_2^p\right] \le K(p), \quad \forall t \in [0,\infty]. \tag{3.48}$$

□

By the ergodic property, for $m > 0$, we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\left(x_i^p(s)\wedge m\right)ds = \int_{R_+^2}\left(z_i^p\wedge m\right)\mu(dz_1,dz_2), \quad \text{a.s.} \tag{3.49}$$

On the other hand, by dominated convergence theorem, we can get

$$E\left[\lim_{t\to\infty}\frac{1}{t}\int_0^t\left(x_i^p(s)\wedge m\right)ds\right] = \lim_{t\to\infty}\frac{1}{t}\int_0^t E\left[x_i^p(s)\wedge m\right]ds \le K(p), \quad i = 1,2, \tag{3.50}$$

which together with (3.49) implies

$$\int_{R_+^2}\left(z_i^p\wedge m\right)\mu(dz_1,dz_2) \le K(p), \quad i = 1,2. \tag{3.51}$$

Letting $m \to \infty$, we get

$$\int_{R_+^2} z_i^p\mu(dz_1,dz_2) \le K(p), \quad i = 1,2. \tag{3.52}$$

That is to say, functions $f_1(x) = x_1^p$ and $f_2(x) = x_2^p$ are integrable with respect to the measure μ. Therefore one has the following.

Theorem 3.10. *Assume $a_{11} > a_{12}$, $a_{22} > a_{21}$, $2r_1 > \sigma_1^2 > 0$, $2r_2 > \sigma_2^2 > 0$ and $\delta < \min\{m_1(x_1^*)^2, m_2(x_2^*)^2\}$, where $\delta = (a_{21}x_1^*\sigma_1^2 + a_{12}x_2^*\sigma_2^2)/2$, $m_1 = a_{11}a_{12}(a_{22}-a_{21})/a_{11}a_{22}+a_{12}a_{21}$ and $m_2 = a_{12}(a_{11}a_{22} - a_{12}a_{21})/2a_{11}$. Then for any initial value $x_0 \in R_+^2$, the solution $x(t)$ of system* (1.4) *has the following property:*

$$P\left\{\lim_{t\to\infty}\overline{x}_1(t) = \int_{R_+^2} z_1\mu(dz_1,dz_2) = \tilde{x}_1^*\right\} = 1, \qquad P\left\{\lim_{t\to\infty}\overline{x}_2(t) = \int_{R_+^2} z_2\mu(dz_1,dz_2) = \tilde{x}_2^*\right\} = 1. \tag{3.53}$$

Moreover, we can get the following.

Theorem 3.11. *Assume $a_{11} > a_{12}$, $a_{22} > a_{21}$ and $2r_1 > \sigma_1^2, 2r_2 > \sigma_2^2$. Then for any initial value $x_0 \in R_+^2$, the solution $x(t)$ has following property*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\left(a_{11}a_{21}x_1^2(s) - a_{12}a_{22}x_2^2(s)\right)ds = a_{21}r_1\tilde{x}_1^* - a_{12}r_2\tilde{x}_2^*, \tag{3.54}$$

where $(\tilde{x}_1^*, \tilde{x}_2^*)$ is defined as in (3.24).

Proof. By (3.39), for $\delta > 0$, we have

$$P\left\{\omega : \sup_{(n-1)\delta \le t \le n\delta} \frac{x_1(t)}{t} > \delta\right\} \le \frac{E\left[x_1^p\right]}{(n-1)^p\delta} \le \frac{K(p)}{(n-1)^p\delta}, \quad p > 1. \tag{3.55}$$

In view of the well-known Borel-Cantelli lemma, we see that for almost all $\omega \in \Omega$,

$$\sup_{(n-1)\delta \le t \le n\delta} \frac{x_1(t)}{t} \le \delta \tag{3.56}$$

holds for all but finitely many n. Hence there exists a $n_0(\omega)$, for all $\omega \in \Omega$ excluding a P-null set, for which (3.56) holds whenever $n \ge n_0$. Consequently, letting $\delta \to 0$, we have, for almost all ω

$$\lim_{t\to\infty} \frac{x_1(t)}{t} = 0. \tag{3.57}$$

Similarly, we can obtain

$$\lim_{t\to\infty} \frac{x_2(t)}{t} = 0. \tag{3.58}$$

Besides, by (3.39) and its ergodic property, we get

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t x_1(s)dB_1(s) = 0, \qquad \lim_{t\to\infty} \frac{1}{t}\int_0^t x_2(s)dB_2(s) = 0. \tag{3.59}$$

On the other hand, we have

$$d(a_{21}x_1 - a_{12}x_2) = [a_{21}x_1(r_1 - a_{11}x_1) - a_{12}x_2(r_2 - a_{22}x_2)]dt + a_{21}\sigma_1 x_1 dB_1(t) - a_{12}\sigma_2 x_2 dB_2(t). \tag{3.60}$$

Then

$$\begin{aligned} a_{21}\frac{x_1(t)}{t} - a_{12}\frac{x_2(t)}{t} = a_{21}\frac{x_1(0)}{t} - a_{12}\frac{x_2(0)}{t} + a_{21}r_1\frac{1}{t}\int_0^t x_1(s)ds - a_{12}r_2\frac{1}{t}\int_0^t x_2(s)ds \\ - a_{11}a_{21}\frac{1}{t}\int_0^t x_1^2(s)ds + a_{12}a_{22}\frac{1}{t}\int_0^t x_2^2(s)ds + a_{21}\sigma_1\frac{1}{t}\int_0^t x_1(s)dB_1(s) \\ - a_{12}\sigma_2\frac{1}{t}\int_0^t x_2(s)dB_2(s). \end{aligned} \tag{3.61}$$

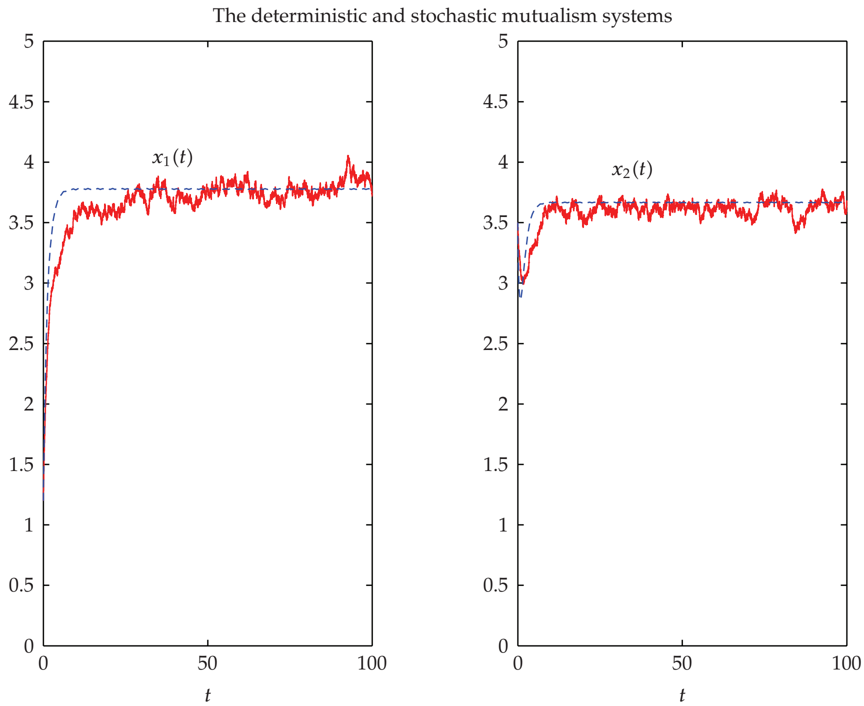

Figure 1: The solution of system (1.1) and system (1.4) with $(x_1(0), x_2(0)) = (1.2, 3.5)$, $a_{11} = 0.6$, $a_{12} = 0.4$, $a_{21} = 0.3$, $a_{22} = 0.5$, $\sigma_1 = 0.02$, $\sigma_2 = 0.02$, and $\Delta t = 0.001$ such that $a_{11}a_{22} > a_{12}a_{21}$. The imaginary lines represent the solution of system (1.1), and the real lines represent the solution of system (1.4).

Therefore, (3.25), (3.57), (3.58), and (3.59) imply

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\Big(a_{11}a_{21}x_1^2(s) - a_{12}a_{22}x_2^2(s)\Big)ds = a_{21}r_1\widetilde{x}_1^* - a_{12}r_2\widetilde{x}_2^*. \tag{3.62}$$

□

At the end of this section, to conform the results above, we numerically simulate the solution of (1.4). By the method mentioned in [26], we consider the discretized equation:

$$\begin{aligned} x_{1,k+1} &= x_{1,k} + x_{1,k}\left[(r_1 - a_{11}x_{1,k} + a_{12}x_{2,k})\Delta t + \sigma_1\epsilon_{1,k}\sqrt{\Delta}t + \frac{1}{2}\sigma_1^2\left(\epsilon_{1,k}^2\Delta t - \Delta t\right)\right], \\ x_{2,k+1} &= x_{2,k} + x_{2,k}\left[(r_2 + a_{21}x_{1,k} - a_{22}x_{2,k})\Delta t + \sigma_2\epsilon_{1,k}\sqrt{\Delta}t + \frac{1}{2}\sigma_1^2\left(\epsilon_{1,k}^2\Delta t - \Delta t\right)\right], \end{aligned} \tag{3.63}$$

given the values of $(x_{1,0}, x_{2,0})$ and parameters in the system, by Matlab software we get Figure 1.

Figure 1 gives the solutions of (1.1) and (1.4), and the real lines and the imaginary lines represent the deterministic and the stochastic, respectively. In this figure, we choose parameters such that the conditions said in theorems are satisfied. Hence, although there is no equilibrium of the stochastic system (1.4) as the deterministic system (1.1), but the solution of (1.4) is ergodicity. From the figure, we can see that the solution of system (1.4) is fluctuating around a constant.

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