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Research Article

Global Stability of Polytopic Linear Time-Varying Dynamic Systems under Time-Varying Point Delays and Impulsive Controls

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This paper investigates the stability properties of a class of dynamic linear systems possessing several linear time-invariant parameterizations (or configurations) which conform a linear time-varying polytopic dynamic system with a finite number of time-varying time-differentiable point delays. The parameterizations may be timevarying and with bounded discontinuities and they can be subject to mixed regular plus impulsive controls within a sequence of time instants of zero measure. The polytopic parameterization for the dynamics associated with each delay is specific, so that $(q + 1)$ polytopic parameterizations are considered for a system with q delays being also subject to delay-free dynamics. The considered general dynamic system includes, as particular cases, a wide class of switched linear systems whose individual parameterizations are timeinvariant which are governed by a switching rule. However, the dynamic system under consideration is viewed as much more general since it is time-varying with timevarying delays and the bounded discontinuous changes of active parameterizations are generated by impulsive controls in the dynamics and, at the same time, there is not a prescribed set of candidate potential parameterizations.

1. Introduction

The stabilization of dynamic systems is a very important question since it is the first requirement for most of applications. Powerful techniques for studying the stability of dynamic systems are Lyapunov stability theory and fixed point theory which can be easily extended from the linear time-invariant case to the time-varying one as well as to functional differential equations, as those arising, for instance, from the presence of internal delays, and to certain classes of nonlinear systems, [1, 2]. Dynamic systems which are of increasing interest are the so-called switched systems which consist of a set of individual parameterizations and a switching law which selects along time, which parameterization is active. Switched systems are essentially timevarying by nature even if all the individual parameterizations are timeinvariant. The interest of such systems arises from the fact that some existing systems in the real world modify their parameterizations to better adapt to their environments. Another important interest of some of such systems relies on the fact that changes of parameterizations through time can lead to benefits in certain applications, [3–13]. The natural way of modelling these situations lies in the definition of appropriate switched dynamic systems. For instance, the asymptotic stability of Liénard-type equations with Markovian switching is investigated in [4, 5]. Also, time-delay dynamic systems are very important in the real life for appropriate modelling of certain biological and ecology systems and they are present in physical processes implying diffusion, transmission, tele-operation, population dynamics, war and peace models, and so forth. (see, e.g., [1, 2, 12–18]). Linear switched dynamic systems are a very particular case of the dynamic system proposed in this paper. Switched systems are very important in practical applications since their parameterizations are not constant. A switched system can result, for instance, from the use of a multimodel scheme, a multicontroller scheme, a buffer system or a multiestimation scheme. For instance, a (nonexhaustive) list of papers deal with some of these questions related to switched systems follow

(1) In [15], the problem of delay-dependent stabilization for singular systems with multiple internal and external incommensurate delays is focused on. Multiple memoryless state-feedback controls are designed so that the resulting closed-loop system is regular, independent of delays, impulsefree and asymptotically stable. A relevant related problem for obtaining sufficiency-type conditions of asymptotic stability of a time-delay system is the asymptotic comparison of its solution trajectory with its delayfree counterpart provided that this last one is asymptotically stable, [19].

(2) In [20], the problem of the N-buffer switched flow networks is discussed based on a theorem on positive topological entropy.

(3) In [21], a multi-model scheme is used for the regulation of the transient regime occurring between stable operation points of a tunnel diode-based triggering circuit.

(4) In [22, 23], a parallel multi-estimation scheme is derived to achieve close-loop stabilization in robotic manipulators whose parameters are not perfectly known. The multi-estimation scheme allows the improvement of the transient regime compared to the use of a single estimation scheme while achieving at the same time closed-loop stability.

(5) In [24], a parallel multi-estimation scheme allows the achievement of an order reduction of the system prior to the controller synthesis so that this one is of reducedorder (then less complex) while maintaining closed-loop stability.

(6) In [25], the stabilization of switched dynamic systems is discussed through topologic considerations via graph theory.

(7) The stability of different kinds of switched systems subject to delays has been investigated in [11–13, 17, 26–28].

(8) The stability switch and Hopf bifurcation for a diffusive prey-predator system is discussed in [6] in the presence of delay.

(9) A general theory with discussed examples concerning dynamic switched systems is provided in [3].

(10) Some concerns of time-delay impulsive models are of increasing interest in the areas of stabilization, neural networks, and Biological models with particular interest in positive dynamic systems. See, for instance, [29–40] and references therein.

The dynamic system under investigation is a linear polytopic system subject to internal point delays and feedback state-dependent impulsive controls. Both parameters and delays are assumed to be timevarying in the most general case. The control impulses can occur as separate events from possible continuous-time or bounded-jump type parametrical variations. Furthermore, each delayed dynamics is potentially parameterized in its own polytope. Those are the main novelties of this paper since *it combines a time-varying parametrical polytopic nature with individual polytopes for the delay-free dynamics with time-varying parameters which are unnecessarily smooth for all time with a potential presence of delayed dynamics with point time-varying delays*. The case of switching between parameterizations at certain time instants, what is commonly known as a switched system, [3, 17, 20–28], is also included in the developed formalism as a particular case as being equivalent to define the whole systems as a particular parameterization of the polytopic system at one of its vertices. The delays are assumed to be time differentiable of bounded time-derivative for some of the presented stability results but just bounded for the rest of results. An important key point is that *if the system is stabilizable, then it can be stabilized via impulsive controls without requiring the delay-free dynamics of the system* as it is then shown in some of the given examples. Usually, for a given interimpulse time interval, the impulsive amplitudes are larger as the instability degree becomes larger, and the signs of the control components also should be appropriate, in order to compensate it by the stabilization procedure. Such a property also will hold for nonpolytopic parameterizations. The design philosophy adopted in the paper is that stabilization might be achieved through appropriate impulsive controls at certain impulsive time instants without requiring the design of a standard regular controller. The paper is organized as follows. Section 2 discusses the various evolution operators valid to build the state-trajectory solutions in the presence of impulsive feedback state-dependent controls. Analytic expressions are given to define such operators. In particular, an important operator defined and discussed in this paper is the so-called impulsive evolution operator. Such an evolution operator is sufficiently smooth within open time intervals between each two consecutive impulsive times, but it also depends on impulses at time instants with hose ones happen. Section 3 discusses new global stability and global asymptotic stability issues based on Krasovsky-Lyapunov functionals taking account of the feedback state-dependent control impulses. The relevance of the impulsive controls towards stabilization is investigated in the sense that the most general results do not require stability properties of the impulse-free system (i.e., that resulting as a particular case of the general one in the absence of impulsive controls). Some included very conservative stability results follow directly from the structures of the state-trajectory solution and the evolution operators of Section 2 without invoking Lyapunov stability theory. It is proven that stabilization is achievable if impulses occur at certain intervals and with the appropriate amplitudes. Finally, two application examples are given in Section 4.

Notation 1.1. **Z, R, and C** are the sets of integer, real, and complex numbers, respectively.

$\mathbf{Z}_+$ and $\mathbf{R}_+$ denote the positive subsets of $\mathbf{Z}$, respectively, and $\mathbf{C}_+$ denotes the subset of $\mathbf{C}$ of complex numbers with positive real part, and $\overline{n} := \{1, 2, \ldots, n\} \subset \mathbf{Z}_+$, for all $n \in \mathbf{Z}_+$.

$\mathbf{Z}_-$ and $\mathbf{R}_-$ denote the negative subsets of $\mathbf{Z}$, respectively, and $\mathbf{C}_-$ denotes the subset of $\mathbf{C}$ of complex numbers with negative real part.

$$\begin{aligned} &\mathbf{Z}_{0+} := \mathbf{Z}_+ \cup \{0\}, \quad \mathbf{R}_{0+} := \mathbf{R}_+ \cup \{0\}, \quad \mathbf{C}_{0+} := \mathbf{C}_+ \cup \{0\} \\ &\mathbf{Z}_{0-} := \mathbf{Z}_- \cup \{0\}, \quad \mathbf{R}_{0-} := \mathbf{R}_- \cup \{0\}, \quad \mathbf{C}_{0-} := \mathbf{C}_- \cup \{0\} \end{aligned} \tag{1.1}$$

Given some linear space X (usually $\mathbf{R}$ or $\mathbf{C}$), then $C^{(i)}(\mathbf{R}_{0+}, X)$ denotes the set of functions of class $C^{(i)}$. Also, $\mathrm{BPC}^{(i)}(\mathbf{R}_{0+}, X)$ and $\mathrm{PC}^{(i)}(\mathbf{R}_{0+}, X)$ denote the set of functions in $C^{(i-1)}(\mathbf{R}_{0+}, X)$ which, furthermore, possess bounded piecewise continuous or, respectively, piecewise continuous ith derivative on X.

$L(X)$ denotes the set of linear operators from X to X. In particular, the linear space denoted by X denotes the state space of the dynamic system with controls in the linear space U.

I_n denotes the nth identity matrix.

The symbols $M \succ 0$, $M \prec 0$, $M \succcurlyeq 0$, and $M \preceq 0$ stand for positive definite, negative definite, positive semidefinite, and negative semidefinite square real matrices M, respectively. The notations $M \succ D$, $M \prec D$, $M \succcurlyeq D$, and $M \preceq D$ stand correspondingly for $(M-D) \succ 0, (M-D) \prec 0, (M-D) \succcurlyeq 0,$ and $(M-D) \preceq 0$, and Superscript "T" stands for transposition of matrices and vectors.

$\lambda_{\max}(M)$ and $\lambda_{\min}(M)$stand for the maximum and minimum eigenvalues of a definite square real matrix $M = (m_{ij})$.

A finite or infinite strictly ordered sequence of impulsive time instants is defined by $\mathrm{Imp} := \{t_i \in \mathbf{R}_{0+} : t_{i+1} > t_i\}$, where an impulsive control $u(t_i)\delta(t-t_i)$ occurs with $\delta\,(\cdot)$ being the Dirac delta of the Dirac distribution.

2. The Dynamic System Subject to Time Delays and Impulsive Controls

Consider the following polytopic linear time-differential system of state vector and control of respective dimensions n and m and being subject to q time-varying point delays:

$$\dot{x}(t) = \sum_{i=0}^{q}\sum_{j=1}^{N} \lambda_{ij}(t)\left(A_{ij}(t)x(t-h_i(t)) + B_{ij}(t)u_{ij}(t)\right) = \lambda^T(t)(\overline{x}(t) + \overline{u}(t)), \tag{2.1}$$

where the incommensurate time-varying delays are $h_0(t) = 0$ for all $t \in \mathbf{R}_{0+}$, $h_i \in \mathrm{PC}^{(1)}(\mathbf{R}_{0+}, \mathbf{R}_{0+})$, for all $i \in \overline{q} := \{1, 2, \ldots, q\}$ (i.e., the delays are continuous time differentiable of bounded time derivative), and

$$\begin{aligned} \lambda^T(t) &= \left(\lambda_{01}(t)\lambda_{02}(t)\cdots\lambda_{0N}(t)\cdots\lambda_{q1}(t)\lambda_{q2}(t)\cdots\lambda_{qN}(t)\right), \\ \overline{x}^T(t) &= \left(x^T(t)\left(A_{01}^T(t)\cdots A_{0N}^T(t)\right)\cdots x^T(t-h_q(t))\left(A_{q1}^T(t)\cdots A_{qN}^T(t)\right)\right), \\ \overline{u}^T(t) &= \left(u_{01}^T(t)B_{01}^T(t)\cdots u_{0N}^T(t)B_{0N}^T(t)\cdots u_{q1}^T(t)B_{q1}^T(t)\cdots u_{qN}^T(t)B_{qN}^T(t)\right) \end{aligned} \tag{2.2}$$

are vector functions from $\mathbf{R}_{0+}$ to $\mathbf{R}^{(q+1)N}$, $\mathbf{R}^{(q+1)Nn}$ and $\mathbf{R}^{(q+1)Nm}$, respectively, and

(i) $x : \mathbf{R}_{0+} \cup [-h, 0) \to X \subset \mathbf{R}^n$ is the state vector, which is almost everywhere time differentiable on $\mathbf{R}_{0+}$ satisfying (2.1), subject to bounded piecewise continuous initial conditions on $[-\overline{h}, 0)$,that is, $x = \varphi \in \mathrm{BPC}^{(0)}([-\overline{h}, 0], \mathbf{R}^n)$, where $h = \overline{h}(0) = \max_{i \in \overline{q}} \sup(h_i(0)) \le \overline{h} := \max_{i \in \overline{q}} \sup_{t \in \mathbf{R}_{0+}}(h(t))$, and $u_{ij} : \mathbf{R}_{0+} \to U \subset \mathbf{R}^m$ are the control vectors for all $i \in \overline{q} \cup \{0\}$ for all $j \in \overline{N}$ and $A_{ij} \in \mathrm{BPC}^{(0)}(\mathbf{R}_{0+}, \mathbf{R}^{n \times n})$ and $B_{ij} \in \mathrm{BPC}^{(0)}(\mathbf{R}_{0+}, \mathbf{R}^{n \times m})$ parameterize the dynamic system.

(ii) $\lambda_{ij} \in \mathrm{BPC}^{(0)}(\mathbf{R}_{0+}, \mathbf{R}_{0+})$, subject to the constraint $\sum_{i=0}^{q} \sum_{j=0}^{N} \lambda_{ij}(t) \in [c_1, c_2] \subset \mathbf{R}_+$, for all $t \in \mathbf{R}_{0+}$ with $\infty > \varepsilon_2 \ge c_2 \ge c_1 \ge \varepsilon_1 \ge 0$ are the weighting scalar functions defining the polytopic system in the various delayed dynamics and parameterizations which are not all simultaneously zero at any time for some given lower-bound and upper-bound scalars ε_1 and ε_2. Note that there exist two summations in (2.1) related to these scalar functions, one them referring to the contribution of delayed dynamics for the various delays and the second one related to the system parameterization within the polytopic structure. It will be not assumed through the paper that the delay-free auxiliary system is stable. Note that the dynamic system can be seen as a convex polytopic dynamic system

$$\dot{x}(t) = \sum_{i=0}^{q} \sum_{j=1}^{N} \lambda_{ij}(t) \dot{x}_{ij}(t) \tag{2.3}$$

formed with subsystems of the form $\dot{x}_{ij}(t) = A_{ij}(t)x(t - h_i(t)) + B_{ij}(t)u_{ij}(t)$. The controls $u_{ij} : \mathbf{R}_{0+} \to U \subset \mathbf{R}^m$ are generated from the state-feedback impulsive controller as follow:

$$\begin{gathered} u_{ij}(t) = K_{ij}(t)x(t - h_i(t)) \quad \forall i \in \overline{q} = \{1, 2, \ldots, q\};\ \forall t \in \mathbf{R}_{0+}, \text{for } i = 0,\ t \notin \mathrm{Imp}, \\ u_{0j}(t^+) = \left(K_{0j}(t^+) + K'_{0j}(t)\right)x(t^+) \quad \text{for } i = 0,\ t \in \mathrm{Imp}, \end{gathered} \tag{2.4}$$

where the strictly ordered Imp: $= \{t_i \in \mathbf{R}_{0+} : t_{i+1} > t_i, i \in \mathbf{Z}_+\}$ is the so-called sequence of impulsive time instants where the control impulses occur whose elements form a monotonically increasing sequence; that is, for any well posed test function $f : \mathbf{R} \to \mathbf{R}$,

$$f(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t - \tau)d\tau = \int_{t}^{t^+} f(\tau)\delta(t - \tau)d\tau = \lim_{\varepsilon \to 0^+} \int_{t-\varepsilon}^{t+\varepsilon} f(\tau)\delta(t - \tau)d\tau, \tag{2.5}$$

where $\delta(t)$ is the Dirac distribution at time $t = 0$ with the following notational convention being used: $g(t^+) = \lim_{\varepsilon \to 0^+} g(t + \varepsilon) \ne g(t) = \lim_{\varepsilon \to 0^+} g(t - \varepsilon)$ either if $t \in \mathrm{Imp}$ or if g is bounded having left and right limits at a discontinuity point $t \in \mathbf{R}_{0+}$, and $g(t^+) = g(t)$ if $\mathbf{R}_{0+} \ni t \notin \mathrm{Imp}$ since the functions used are all left-continuous functions. Partial sequences of impulsive time instants are denoted by specifying the time intervals they refer to, as for instance, $\mathrm{Imp}[T_1, T_2] = \{t \in \mathrm{Imp} : t \in [T_1, T_2]\}$ and $\mathrm{Imp}[T_1, T_2) = \{t \in \mathrm{Imp} : t \in [T_1, T_2)\}$. Note that $\mathrm{Imp} = \mathrm{Imp}[0, \infty)$. The regular and impulsive controller gain matrices are, respectively, $K_{ij} \in \mathrm{BPC}^{(0)}(\mathbf{R}_{0+}, \mathbf{R}^{m \times n})$ and $K'_{ij} : \mathrm{Imp} \to \mathbf{R}^{m \times n}$ being a discrete sequence of bounded matrices. Note that, if $K_{0j}(t)$ is discontinuous at the time instant t, then $K_{0j}(t^+) \ne K_{0j}(t)$ even if $t \notin \mathrm{Imp}$. The extensions to vector and matrix test functions are obvious by using respective appropriate zero components or entries if impulses do not occur at time t, a particular component or matrix entry. The substitution of the control law (2.4) into the open-loop system equation (2.1) leads to the closed-loop functional dynamic system as follows:

$$\begin{aligned}\dot{x}(t) &= \sum_{j=1}^{N} \lambda_{0j}(t)\Big(A_{0j}^{*}(t) + B_{0j}(t)K_{0j}'(t)\delta(0)\Big)x(t) + \sum_{i=1}^{q}\sum_{j=1}^{N} \lambda_{ij}(t)A_{ij}^{*}(t)x(t-h_i(t)),\\ x(t^{+}) &= \Big(I_n + \sum_{j=1}^{N}\lambda_{0j}(t)B_{0j}(t)K_{0j}'(t)\Big)x(t);\end{aligned} \tag{2.6}$$

for all $t \in \mathbf{R}_{0+}$ with $K_{0j}'(t) = 0$; for all $t \notin \mathrm{Imp}$, where

$$A_{ij}^{*}(t) = A_{ij}(t) + B_{ij}(t)K_{ij}(t), \ \forall i \in \overline{q} \cup \{0\}\ \forall j \in \overline{N}, \tag{2.7}$$

Equation (2.6) becomes

$$\dot{x}(t) = \sum_{i=0}^{q}\sum_{j=1}^{N} \lambda_{ij}(t)A_{ij}^{*}(t)x(t-h_i(t)), \tag{2.8}$$

for all $t \notin \mathrm{Imp}$ and also at the left limits for all $t \in \mathrm{Imp}$, and $x(t^{+}) - x(t) = \sum_{j=1}^{N}\lambda_{0j}(t)B_{0j}(t)K_{0j}'(t)x(t)$, which is zero if $t \notin \mathrm{Imp}$, with

$$\dot{x}(t^{+}) = \sum_{j=1}^{N}\lambda_{0j}(t)A_{0j}^{*}(t^{+})x(t^{+}) + \sum_{i=1}^{q}\sum_{j=1}^{N}\lambda_{ij}(t)A_{ij}^{*}(t^{+})x(t^{+}-h_i(t)) \tag{2.9}$$

for the right limits of all $t \in \mathrm{Imp}$. Define $D := \mathrm{Imp} \cup D_p$, where

$$D_p := \left(\bigcup_{i\in\overline{q}\cup\{0\},\, j\in\overline{N}} D_{A_{ij}}\right) \cup \left(\bigcup_{i\in\overline{q}\cup\{0\},\, j\in\overline{N}} D_{B_{ij}}\right) \cup \left(\bigcup_{i\in\overline{q}\cup\{0\},\, j\in\overline{N}} D_{\lambda_{ij}}\right) \cup \left(\bigcup_{i\in\overline{q}\cup\{0\},\, j\in\overline{N}} D_{K_{ij}}\right) \tag{2.10}$$

is the total set of discontinuities on $\mathbf{R}_{0+}$ of $A_{ij} \in \mathrm{BPC}^{(0)}(\mathbf{R}_{0+}, \mathbf{R}^{n\times n})$, $B_{ij} \in \mathrm{BPC}^{(0)}(\mathbf{R}_{0+}, \mathbf{R}^{n\times m})$, $\lambda_{ij} \in \mathrm{BPC}^{(0)}(\mathbf{R}_{0+}, \mathbf{R}_{0+})$, and $K_{ij} \in \mathrm{BPC}^{(0)}(\mathbf{R}_{0+}, \mathbf{R}^{m\times n})$ for all $i \in \overline{q} \cup \{0\}$, for all $j \in \overline{N}$ which are in the respective sets $D_{A_{ij}}$, $D_{B_{ij}}$, $D_{\lambda_{ij}}$, and $D_{K_{ij}}$. The following technical assumptions are made.

Assumption 2.1. there exist $\upsilon \in \mathbf{R}_{+}$ such that $t_{k+1} - t_k \geq \upsilon$, for all $t_k, t_{k+1}(> t_k) \in \mathrm{Imp}$.

Assumption 2.2. $((\bigcup_{j\in\overline{N}} D_{B_{0j}}) \cup (\bigcup_{j\in\overline{N}} D_{\lambda_{ij}})) \cap \mathrm{Imp} = \emptyset$.

Assumption 2.1 implies that the sequence of impulsive time instants is a real sequence with no accumulation points. It is a technical assumption to guarantee the existence and uniqueness of an almost everywhere time-differentiable state-trajectory solution. Assumption 2.2 is needed for all the functions $\lambda_{0j}(B_{0j})_{k\ell} \in \mathrm{BPC}^{(0)}(\mathbf{R}_{0+}, \mathbf{R}_{0+})$ for all $j \in \overline{N}$, for all $k \in \overline{n}$ and for all $\ell \in \overline{m}$, build with the entries $B_{0j} \in \mathrm{BPC}^{(0)}(\mathbf{R}_{0+}, \mathbf{R}^{n\times m})$. This follows since they are piecewise continuous on $\mathbf{R}_{0+}$ and, furthermore, continuous at any small neighborhood around any point of the sequence of impulsive time instants where control impulses occur. From Picard-Lindeloff theorem, there is a unique solution for any vector function of initial conditions $\varphi \in \mathrm{BPC}^{(0)}([-\overline{h},0],\mathbf{R}^n)$ and $x \in \mathrm{BPC}^{(1)}(\mathbf{R}_+,\mathbf{R}^n)$. The state-trajectory solution of the closed-loop system (2.8)-(2.9) for initial conditions $\varphi \in \mathrm{BPC}^{(0)}([-\overline{h},0],\mathbf{R}^n)$ is given by

$$
\begin{aligned}
x(t) &= \Psi(t)\Bigg[\Psi^{-1}(0)x(0) + \sum_{i=1}^{q}\sum_{j=1}^{N}\int_0^t \Psi^{-1}(\tau)\lambda_{ij}(\tau)A_{ij}^{*}(\tau)x(\tau - h_i(\tau))d\tau \\
&\qquad + \sum_{t_k\in \mathrm{Imp}[0,t)}\sum_{j=1}^{N}\lambda_{0j}(t_k)\Psi^{-1}(t_k)B_{0j}(t_k)K'_{0j}(t_k)x(t_k)\Bigg] \\
&= \Bigg[\Psi_s(t,t_0)x(t_0) + \sum_{i=1}^{q}\sum_{j=1}^{N}\int_0^t \lambda_{ij}(\tau)\Psi_s(t,\tau)A_{ij}^{*}(\tau)x(\tau - h_i(\tau))d\tau \\
&\qquad + \sum_{t_k\in \mathrm{Imp}[t_0,t)}\sum_{j=1}^{N}\lambda_{0j}(t_k)\Psi_s(t,t_k)B_{0j}(t_k)K'_{0j}(t_k)x(t_k)\Bigg],
\end{aligned}
\tag{2.11}
$$

subject to $x(t) = \varphi(t)$, for all $t \in [-\overline{h},0]$, where

(1) $\Psi(t) \in C^{(0)}(\mathbf{R}_{0+},\mathbf{R}^{n\times n})$ is an almost everywhere differentiable matrix function on $\mathbf{R}_+$ (being time differentiable on the non connected real set $\bigcup_{t_i\in \mathrm{Imp}}(t_{i+1} - t_i)$) with unnecessarily continuous time derivatives which satisfies $\dot{\Psi}(t) = \sum_{j=1}^{N}\lambda_{0j}(t)A_{0j}^{*}(t)\Psi(t)$ on $\mathbf{R}_+$ with $\Psi(0) = I_n$. If A_{ij}, B_{ij}, λ_{ij}, and K_{ij} for all $i \in \overline{q} \cup \{0\}$, for all $j \in N$ are everywhere continuous on $\mathbf{R}_+$, then $\Psi(t) \in C^{(1)}(\mathbf{R}_{0+},\mathbf{R}^{n\times n})$, $\Psi_s(\cdot,\cdot) : \mathbf{R}_{0+}^2 \to \mathbf{R}^{n\times n}$ as $\Psi_s(t,\tau) = \Psi(t)\Psi^{-1}(\tau)$ for all $t \geq \tau$, and

(2) $\mathrm{Imp}[t_0,t)\colon = \{t_k \in \mathbf{R}_{0+} : t_0 \leq t_k (\in \mathrm{Imp}) < t\} \subset \mathrm{Imp}$ is the strictly ordered sequence of impulsive time instants with input impulses occurred on $[t_0,t)$ for any $t_0 \in \mathbf{R}_+$. Also, $\mathrm{Imp}(t_0,t)\colon = \{t_k \in \mathrm{Imp} : t_0 < t_k < t\} \subset \mathrm{Imp}$; $\mathrm{Imp}(t_0,t]\colon = \{t_k \in \mathrm{Imp} : t_0 < t_k \leq t\} \subset \mathrm{Imp}$ are defined in a closed way.

The solution (2.11) is identically defined by

$$
\begin{aligned}
x(t) = Z(t)\Bigg[&Z^{-1}(0)x(0) + \int_{-h}^{0} Z^{-1}(\tau)\varphi(\tau)d\tau \\
&+ \sum_{t_k\in \mathrm{Imp}(0,t)}\sum_{j=1}^{N}\lambda_{0j}(t_k)Z^{-1}(t_k)B_{0j}(t_k)K'_{0j}(t_k)x(t_k)\Bigg],
\end{aligned}
\tag{2.12}
$$

where $Z(t) \in C^{(0)}(\mathbf{R}_{0+},\mathbf{R}^{n\times n})$ is an almost everywhere differentiable matrix function on $\mathbf{R}_+$, with unnecessarily continuous time derivatives, which satisfies (2.8) on $\mathbf{R}_+$ with $Z(0) = I_n$, $Z(t) = 0$ for all $t \in \mathbf{R}_-$. Defining the matrix function $Z_s(\cdot,\cdot) : \mathbf{R}_{0+}^2 \to \mathbf{R}^{n\times n}$ as

$Z_s(t,\tau) = Z(t)Z^{-1}(\tau)$ for all $t \geq \tau$, one has from (2.12) for $t \in [t_k, t_{k+1}]$ for any two consecutive given $t_k, t_{k+1} \in \mathrm{Imp}$ as follow:

$$x(t) = Z_s(t,t_k)x(t_k^+) + \sum_{i=1}^{q} \int_{-h_i}^{0} Z_s(t, t_k+\tau)\varphi(t_k+\tau)d\tau + \left(\sum_{j=1}^{N} Z_s(t,t_k)\lambda_{0j}(t_k \ )B_{0j}(t_{k+1})x(t_{k+1}^+)K'_{0j}(t_{k+1}) \right), \tag{2.13}$$

which becomes for $t = t_{k+1}^+$ as follow:

$$\begin{aligned} x(t_{k+1}^+) &= \left(I_n + \sum_{j=1}^{N} Z_s(t_{k+1}, t_k)B_{0j}(t_{k+1})K'_{0j}(t_{k+1}) \right) x(t_{k+1}) \\ &= Z_s(t_{k+1}, t_k)x(t_k^+) + \sum_{i=1}^{q} \int_{-h_i}^{0} Z_s(t, t_k+\tau)\varphi(t_k+\tau)d\tau \\ &\quad + \sum_{j=1}^{N} Z_s(t_{k+1}, t_k)\lambda_{0j}(t_k)B_{0j}(t_{k+1})x(t_{k+1})K'_{0j}(t_{k+1})\delta(t, t_{k+1}), \end{aligned} \tag{2.14}$$

where $\delta(t, t_{k+1}) = 1$ if $t = t_{k+1}$ and zero otherwise is the Kronecker delta. In view of (2.12), the state-trajectory solution can be defined by the impulsive evolution operator $\{T(t,t_k) : t \in [t_k, t_{k+1}],$ for all $t_k \in \mathrm{Imp}\}$, associated with $\{Z(t) : t \in \mathbf{R}_{0+}\}$ where $T(\cdot,\cdot) : \{([t_k, t_{k+1}] : t_k \in \mathrm{Imp} \cup \{0\})\} \to L(X)$, which is represented by $x(t) = T(t,t_k)x_{t_k^+}$; for all $t \in [t_k, t_{k+1}]$, for all $t_k \in \mathrm{Imp}$ so that:

$$\begin{aligned} x(t) &= T(t,t_k)x_{t_k^+}, \\ x(t_{k+1}^+) &= T(t_{k+1}^+, t_k)x_{t_k^+} = \left(I_n + \sum_{j=1}^{N} \lambda_{0j}(t_{k+1})B_{0j}(t_{k+1})K'_{0j}(t_{k+1}) \right) T(t_{k+1}, t_k)x_{t_k^+}, \end{aligned} \tag{2.15}$$

for all $t \in [t_k, t_{k+1}]$, for all $t_k \in \mathrm{Imp}$, where x_t and x_{t^+} denote the strings of state solution trajectory and $\{x(\tau) : \tau \in [t-\overline{h}, t)\}$ and $\{x(\tau) : \tau \in [t-\overline{h}, t]\}$, respectively. The subsequent result follows directly for the state-trajectory solution from (2.11) for any initial conditions $\varphi \in \mathrm{BPC}^{(0)}([-\overline{h}, 0], \mathbf{R}^n)$.

Theorem 2.3. *The following properties hold.*

(*i*) *The state-trajectory solution satisfies the following equations on any interval* $[\zeta, t) \subset \mathbf{R}_{0+}$ *for any* $\varphi \in BPC^{(0)}([-\overline{h}, 0], \mathbf{R}^n)$*:*

$$x(t_{k+1}^{+}) = \left(I_n + \sum_{j=1}^{N} \Psi_s(t_{k+1}, t_{k+1})\lambda_{0j}(t_{k+1})B_{0j}(t_{k+1})K'_{0j}(t_{k+1})\right)x(t_{k+1}) \tag{2.16}$$

$$\begin{aligned} &= \Psi_s(t_{k+1}, \zeta)x(\zeta^+) + \int_{\zeta^+}^{t_{k+1}} \Psi_s(t_{k+1}, \tau)\left(\sum_{i=1}^{q}\sum_{j=1}^{N} \lambda_{ij}(\tau)A_{ij}^{*}(\tau)x(\tau - h_i(\tau))\right)d\tau \\ &\quad + \sum_{t_i \in \mathrm{Imp}[\zeta, t_{k+1}]} \sum_{j=1}^{N} \lambda_{0j}(t_i)\Psi_s(t_{k+1}, t_i)B_{0j}(t_i)K'_{0j}(t_i)x(t_i) \end{aligned} \tag{2.17}$$

$$= \left(I_n + \sum_{j=1}^{N} Z_s(t_{k+1}, t_{k+1})B_{0j}(t_{k+1})K'_{0j}(t_{k+1})\right)x(t_{k+1}) \tag{2.18}$$

$$\begin{aligned} &= Z_s(t_{k+1}, \zeta)x(\zeta^+) + \sum_{i=1}^{q}\int_{-h_i}^{0} Z_s(t_{k+1}, \zeta + \tau)x(\zeta + \tau)d\tau \\ &\quad + \sum_{t_i \in \mathrm{Imp}[\zeta, t_{k+1}]} \sum_{j=1}^{N} \lambda_{0j}(t_i)Z_s(t_{k+1}, t_i)B_{0j}(t_i)x(t_i)K'_{0j}(t_i) \end{aligned} \tag{2.19}$$

$$= T(t_{k+1}^{+}, \zeta)x_{\zeta^+} \tag{2.20}$$

$$= \prod_{t_i, t_{i+1} \in \mathrm{Imp}[\zeta, t_{k+1}]} \left[\left(I_n + \sum_{j=1}^{N} \lambda_{0j}(t_{i+1})B_{0j}(t_{i+1})K'_{0j}(t_{i+1})\right)T(t_{i+1}, t_i)\right]x_{\zeta^+}, \tag{2.21}$$

for all $t_{k+1}(> \zeta) \in \mathrm{Imp}$*, for all* $\zeta \in \mathbf{R}_{0+}$ *with* $T(t_{k+1}, t_{k+1}) = Z_s(t_{k+1}, t_{k+1}) = \Psi_s(t_{k+1}, t_{k+1}) = I_n$*. Equations* (2.17) *and* (2.19) *are also valid by replacing* $t_{k+1} \to t$*, for all* $t \in (t_{k+1}, t_{k+2})$ *if* $t_{k+2} \in \mathrm{Imp}$ *and for all* $t \in (t_{k+1}, \infty)$ *if* $(t_{k+1}, \infty) \cap \mathrm{Imp} = \emptyset$*, that is, if the sequence of impulsive time instants is finite with the last impulsive time instant being* t_{k+1}*. Equation* (2.21) *has to be modified by replacing* $t_{k+1} \to t$ *and then by premultiplying it by* $T(t, t_{k+1})$*.*

(*ii*) *Assume that*

$$\left\|\prod_{t_i, t_{i+1} \in \mathrm{Imp}[\zeta, t_{k+1}]} \left[\left(I_n + \sum_{j=1}^{N} \lambda_{0j}(t_{i+1})B_{0j}(t_{i+1})K'_{0j}(t_{i+1})\right)T(t_{i+1}, t_i)\right]\right\| \le M_T \le 1 \tag{2.22}$$

$$\left\|\left(I_n + \sum_{j=1}^{N} \lambda_{0j}(t)B_{0j}(t)K'_{0j}(t)\right)T\left(t, t_{c_{\mathrm{imp}}}\right)\right\| \le M_T \le 1, \tag{2.23}$$

for all $t_{k+1}(> \zeta) \in \mathrm{Imp}$*, for all* $\zeta \in \mathbf{R}_{0+}$*, and for all* $t \ge t_{c_{\mathrm{imp}}}$ *provided that* $c_{\mathrm{imp}} := \mathrm{card}\,\mathrm{Imp}[0, \infty) < \infty$*, then* $\|\Gamma x\|_{L^p(\mathbf{R}_+, X)} \le C_\Gamma$*, where* $\Gamma : \mathrm{Dom}(\Gamma) \equiv X \to L^p(\mathbf{R}_+, X)$ *is defined by* $(\Gamma x)(t) = T(t, \theta)x$*. for all* $x \in X$*.*

Proof. (i) It follows directly for the state-trajectory solution from (2.11), (2.14), and (2.15) for any time interval $[\zeta - h, \zeta)$ of initial conditions $\varphi \in \mathrm{BPC}^{(0)}([-\overline{h}, 0], \mathbf{R}^n)$.

(ii) The first part follows from the definition of the impulsive evolution operator. If, in addition, $M_T < 1$, then it follows from the following given constraints:

$$\exists \lim_{t\to\infty} T(t,\theta)\xi = 0,\ \forall t(>\theta) \in \mathbf{R}_+,\ \theta \in \mathbf{R}_{0+},\ \forall \xi \in X \Longrightarrow T(t,\theta)\xi \tag{2.24}$$

is bounded, for all $\xi \in X$, for all $t(>\theta) \in \mathbf{R}_+, \theta \in \mathbf{R}_{0+}$

$$\Longrightarrow \|T(t,\theta)\| \leq C_T\ \text{, some } \mathbf{R} \ni C_T \geq 1, \forall\ t(>\theta) \in \mathbf{R}_+, \theta \in \mathbf{R}_{0+} \tag{2.25}$$

(from the uniform boundedness principle). Now, note that the operator $\Gamma : \mathrm{Dom}(\Gamma) \equiv X \to L^p(\mathbf{R}_+, X)$ is closed and then bounded from the closed graph theorem, so that the proof of Property (ii) is complete. □

Remark 2.4. Stabilization by impulsive controls may be combined with the design of regular stabilization controllers or used as the sole stabilization tool. Some advantages related to the use of impulsive control for stabilization of stabilizable systems arise in the cases when the classical regular controller are of high design and maintenance costs.

3. Stability

The global asymptotic stability of the controlled system is now investigated. Firstly, a conservative stability result follows from Theorem 2.3 (2.16)–(2.21), which does not take into account possible compensations of the impulsive controls for stabilization purposes.

Theorem 3.1. *Assume that the sequence* Imp *is infinite,* $\|\Psi_s(t,\tau)\| \leq k_\Psi e^{-\rho_\psi(t-\tau)}$*, for all* $t \geq \tau + t_0$*, some finite* $t_0 > 0$*, some* $\mathbf{R}_+ \ni k_\Psi > 0$*, and some* $\rho_\psi \in \mathbf{R}_+$ *as follow:*

$$k_\Psi \left(1 + \frac{\sup_{t_{k+ip} \leq \tau \leq t_{k+(i+1)p}} \left\| \sum_{i=1}^{q} \sum_{j=1}^{N} \lambda_{ij}(\tau) A_{ij}^{*}(\tau) \right\|_2}{\rho_\Psi} + \left\| \sum_{t_j \in \, \mathrm{Imp}\left[t_{k+ip}, t_{k+(i+1)p}\right]} \sum_{j=1}^{N} \lambda_{0j}(t_j) B_{0j}(t_j) K_{0j}'(t_j) e^{-\rho_\Psi (t_{k+(j+1)p} - t_j)} \right\|_2 \right) \leq 1, \tag{3.1}$$

for some $p \in \mathbf{Z}_+$*, some finite* $k \in \mathbf{Z}_{0+}$*, some subsequence* $\{t_{k+ip}\} \in$ Imp*, for all* $i \in \mathbf{Z}_{0+}$*. Thus, the closed-loop system* (2.8)-(2.9) *is globally stable. If the above inequality is strict, then the system is globally asymptotically stable. Also, if the sequence* Imp *is finite, then the results are valid* $k_\Psi (1 + \sup_{t_k \leq \tau < \infty} \| \sum_{i=1}^{q} \sum_{j=1}^{N} \lambda_{ij}(\tau) A_{ij}^{*}(\tau) \|_2 / \rho_\Psi) \leq 1 (< 1)$ *with* t_k *being the last element of the finite sequence Imp*

Now, a general stability result follows, which proves that (in general, nonasymptotic) global stability is achievable by some sequence of impulsive controls generated from appropriate impulsive controller gains.

Theorem 3.2. *There is a sequence of impulsive time instants* Imp $:= \{t_i \in \mathbf{R}_{0+}\}$ *such that the closed-loop system* (2.6)–(2.7) *is globally stable for any function of initial conditions* $\varphi \in BPC^{(0)}([-\overline{h}, 0], \mathbf{R}^n)$ *for some sequence of impulsive controller gains* $K'_{0j} : \mathbf{R}_{0+} \to \mathbf{R}^{n\times m}$, *for all* $j \in \overline{N}$, *for all* $i \in \overline{q} \cup \{0\}$.

Proof. The basic equation to build the stability proof is $x(t^+) - x(t) = \sum_{j=1}^{N} \lambda_{0j}(t) B_{0j}(t) K'_{0j}(t) x(t)$, for all $t \in$ Imp and any sequence of impulsive time instants Imp. Consider prefixed real constants $K_i \in \mathbf{R}_+$ $(i \in \overline{4})$ fulfilling $K_1 \leq K_3 - \varepsilon_1$ and $K_4 \leq K_2 - \varepsilon_2$ with $\varepsilon_1 \in (0, K_3) \cap \mathbf{R}_{0+}$ and $\varepsilon_2 \in (0, K_2) \cap \mathbf{R}_{0+}$ such that $x_k(0) \in [K_3, K_4] \subset [K_1 + \varepsilon_1, K_2 - \varepsilon_2] \subset [K_1, K_2]$, for all $k \in \overline{n}$. The proof of global stability is now made by complete induction. Assume that some finite or infinite $t \in \mathbf{R}_+$ exists such that $x_k(\tau) \in [K_1, K_2]$; for all $\tau \in [0, t)$, but $x_k(t) \in ((-\infty, K_3) \cup (K_4, \infty)) \cap [K_1, K_2]$ for some $k \in \overline{n}$, some $K_3 \in \mathbf{R}$ with an existing (perhaps empty) partial sequence of impulsive time instants $\mathrm{Imp}[0, t)$ until time t. Such a time t always exists from the boundedness and almost everywhere continuity of the state-trajectory solution. Then, $t \in$ Imp so that $\mathrm{Imp}[0, t] = \mathrm{Imp}[0, t) \cup \{t\}$ is fixed as the first impulsive time instant and

$$-\infty < K_3 \leq x_k(t^+) = \left(\delta(k, \ell) + \sum_{j=1}^{N} \sum_{i=1}^{m} \sum_{\ell=1}^{n} \lambda_{0j}(t) B_{0jki}(t) K'_{0ji\ell}(t) \right) x_\ell(t) \leq K_4 < \infty, \tag{3.2}$$

where the entry notation $M = (M_{ij})$ for a matrix M is used, provided that the impulsive controller gain $K'_{0jik}(t)$ is chosen so that the following constraint holds:

$$\begin{aligned} &\frac{K_3 - \left(\sum_{j=1}^{N} \sum_{i(\neq k)=1}^{m} \sum_{\ell(\neq k)=1}^{n} \lambda_{0j}(t) B_{0jki}(t) K'_{0ji\ell}(t) \right) x_\ell(t) - x_k(t)}{\left(1 + \sum_{j=1}^{N} \lambda_{0j}(t) B_{0jki}(t) K'_{0jik}(t) \right) x_k(t)} \\ &\quad \leq K'_{0jkk}(t) \leq \frac{K_4 - \left(\sum_{j=1}^{N} \sum_{i(\neq k)=1}^{m} \sum_{\ell(\neq k)=1}^{n} \lambda_{0j}(t) B_{0jki}(t) K'_{0ji\ell}(t) \right) x_\ell(t) - x_k(t)}{\left(1 + \sum_{j=1}^{N} \lambda_{0j}(t) B_{0jki}(t) K'_{0jkk}(t) \right) x_k(t)}. \end{aligned} \tag{3.3}$$

Note by direct inspection of (3.3) that such a controller gain always exists. As a result, $x_k(t^+) \in [K_3, K_4] \subset [K_1, K_2]$, for all $k \in \overline{n}$. By continuity of the state-trajectory solution, there exists a finite $T(t, K') \in \mathbf{R}_+$ such that $x_k(\tau) \in [K_3 - K', K_4 + K'] \subset [K_1, K_2]$ for any prefixed $K' \in \mathbf{R}_+$, for all $\tau \in [t, t + T(t, K'))$, for all $k \in \overline{n}$ provided that $K_3 - K_1 \leq K' \leq K_2 - K_4$. Since $x_k(t + T(t, K')) \in ((-\infty, K_3) \cup (K_4, \infty)) \cap [K_1, K_2]$ then $x_k(\tau) \subset [K_1, K_2]$, for all $\tau \in [0, t + T(t))$, for all $k \in \overline{n}$. Also, $x_k(\tau) \subset [K_1, K_2]$ for al $\tau \in [0, t + T(t)]$, for all $k \in \overline{n}$ if an impulsive controller gain is chosen at time $t + T(t)$ by replacing $t \to t + T(t)$ in (3.3) and $\mathrm{Imp}[0, t + T(t)] = \mathrm{Imp}[0, t + T(t)) \cup \{t + T(t)\}$ with $\mathrm{Imp}[0, t + T(t)) = \mathrm{Imp}[0, t]$. It has been proven that $x_k(\tau) \in [K_1, K_2]$, for all $\tau \in [0, t)$ for any given $t \in \mathbf{R}_{0+}$, for all $k \in \overline{n}$ then $x_k(\tau) \in [K_1, K_2]$, for all $\tau \in [0, t + T(t)]$, for some $T(t) \in \mathbf{R}_+$, and for all $k \in \overline{n}$ so that the result holds by complete induction for for all $t \in \mathbf{R}_{0+}$ with a bounded sequence of impulsive controller gains at some appropriate sequence of impulsive time instants Imp: $= \{t_i \in \mathbf{R}_{0+}\}$. □

Remark 3.3. Note that Theorem 3.2 holds irrespective of the values of the regular controller gain functions $K_{ij} : \mathbf{R}_{0+} \to \mathbf{R}^{m\times n}$ for some appropriate sequence of impulsive controller gains $K'_{0j} : \mathbf{R}_{0+} \to \mathbf{R}^{n\times m}$, for all $j \in \overline{N}$, for all $i \in \overline{q} \cup \{0\}$. The reason is that the stabilization mechanism consists of decreasing the absolute values of the state components as much as necessary at its right limits at the impulsive time instants for any values of their respective left-hand-side limits and values at previous values at the intervals between consecutive impulsive time instants.

The subsequent result establishes that the stabilization is achievable with the stabilizing impulsive controller gains being chosen arbitrarily except at some subsequence of the impulsive time instants.

Theorem 3.4. *The closed-loop system* (2.6)–(2.7) *is globally stable for any* $\varphi \in BPC^{(0)}([-\overline{h}, 0], \mathbf{R}^n)$ *and any given set of regular controller gain functions* $K_{ij} : \mathbf{R}_{0+} \rightarrow \mathbf{R}^{n\times m}$ *if the sequence of impulsive time instants* $Imp := \{t_i \in \mathbf{R}_{0+}\}$ *is chosen so that*

(1) *the sequence of impulsive controller gains* $K'_{0j} : \mathbf{R}_{0+} \rightarrow \mathbf{R}^{n\times m}$, *for all* $j \in \overline{N}$; *for all* $i \in \overline{q} \cup \{0\}$ *is chosen appropriately for some subsequence of impulsive time instants* $Imp^* := \{t_k^*\} \subset Imp$ *satisfying* $t_{k+1}^* - t_k^* \leq T^*(t_k^*) < \infty$, *for each two consecutive* $t_k^*, t_{k+1}^* \in Imp^*$

(2) *such a sequence of impulsive controller gains is chosen arbitrarily for the sequence* $Imp \setminus Imp^*$.

Proof. Consider the following Lyapunov functional candidate $V : \mathbf{R}_{0+} \times \mathbf{R}^n \rightarrow \mathbf{R}_{0+}$, [17]:

$$V(t, x_t) := x^T(t)Px(t) + \sum_{i=1}^{q} \int_{t-h_i(t)}^{t} x^T(\tau)S_i(\tau)x(\tau)d\tau, \tag{3.4}$$

where $\mathbf{R}^{n\times n} \ni P = P^T \succ 0$ and $S_i \in \mathrm{BPC}^{(0)}(\mathbf{R}_{0+}, \mathbf{R}^{n\times n})$ fulfils $S_i(t) \succ 0$, for all $t \in \mathbf{R}_{0+}$, for all $i \in \overline{q}$. One gets by taking time-derivatives in (3.4) using (2.6) as follow:

$$\begin{aligned}\dot{V}(t, x_t) := {}& 2x^T(t)P\left[\sum_{j=1}^{N} \lambda_{0j}(t)B_{0j}(t)K'_{0j}(t)\delta(0) + \sum_{i=0}^{q}\sum_{j=1}^{N} \lambda_{ij}(t)A^*_{ij}(t)x(t-h_i(t))\right]x(t) \\ &+ \sum_{i=1}^{q}\left(x^T(t)S_i(t)x(t) - \left(1-\dot{h}_i(t)\right)x^T(t-h_i(t))S_i(t-h_i(t))x^T(t-h_i(t))\right)\end{aligned} \tag{3.5}$$

$$= \hat{x}^T(t)Q(t)\hat{x}(t) = -\hat{x}^T(t)(Q_d(t) + Q_{od}(t))\hat{x}(t), \tag{3.6}$$

where

$$\begin{aligned}\hat{x}(t) &= \left(x^T(t)x^T(t-h_1(t))\cdots x^T(t-h_q(t))\right)^T, \\ Q(t) &:= \text{Block matrix}\left(Q_{ij}(t) : i, j \in \overline{q+1}\right),\end{aligned} \tag{3.7}$$

with

$$Q_{11}(t) = \left(\sum_{j=1}^{N} \lambda_{0j}(t)\left(A_{0j}^{*}(t) + B_{0j}(t)K_{0j}'(t)\delta(0)\right)\right)^{T} P + P\left(\sum_{j=1}^{N} \lambda_{0j}(t)\left(A_{0j}^{*}(t) + B_{0j}(t)K_{0j}'(t)\delta(0)\right)\right) + \sum_{i=1}^{q} S_i(t)$$

$$Q_{1,i+1}(t) = Q_{i+1,1}^{T}(t) := \sum_{j=1}^{N} \lambda_{ij}(t) P A_{ij}^{*}(t), \quad \forall i \in \overline{q},$$

$$Q_{ii}(t) := -\left(1 - \dot{h}_i(t)\right) S_i(t - h_i(t)), \quad Q_{ij}(t) = 0, \quad \forall i, j(\neq i) \in \overline{q+1} \setminus \{1\}, \tag{3.8}$$

$$Q_d(t) = \text{Block diag}\left(-Q_{11}(t) - Q_{22}(t) - Q_{q+1,q+1}(t)\right),$$

$$Q_{od}(t) = -(Q(t) + Q_d(t)) = \begin{bmatrix} 0 & -Q_{12}(t) & \cdots & -Q_{1,q+1}(t) \\ -Q_{12}^{T}(t) & 0 & -Q_{23}(t) \cdots & -Q_{2,q+1}(t) \\ \vdots & \vdots & \ddots & \vdots \\ -Q_{q+1,1}^{T}(t) & -Q_{q+1,2}^{T}(t)(t) & -Q_{q+1,q}^{T}(t)(t) \cdots & 0 \end{bmatrix},$$

so that the following cases arise:

(1) if $t \notin D$, then

$$Q_{11}(t) = \left(\sum_{j=1}^{N} \lambda_{0j}(t) A_{0j}^{*T}(t)\right) P + P\left(\sum_{j=1}^{N} \lambda_{0j}(t) A_{0j}^{*}(t)\right) + \sum_{i=1}^{q} S_i(t),$$

$$Q_{1,i+1}(t) = Q_{i+1,1}^{T}(t) := \sum_{j=1}^{N} \lambda_{ij}(t) P A_{ij}^{*}(t), \quad \forall i \in \overline{q}, \tag{3.9}$$

$$Q_{ii}(t) : = -\left(1 - \dot{h}_i(t)\right) S_i(t - h_i(t)), \quad Q_{ij}(t) = 0, \quad \forall i, j(\neq i) \in \overline{q+1} \setminus \{1\},$$

(2) if $t \in D \setminus \text{Imp}$, then (3.8) still holds to the left of any $t \in \mathbf{R}_{0+}$. Similar equations as (3.9) stand for t^{+} by replacing $t \rightarrow t^{+}$ in all the matrix functions entries which become modified only if the time instant t is a discontinuity point of the corresponding matrix function entry,

(3) if $t \in \text{Imp}$, then the left-hand-side limit of $Q(t)$ is defined with block matrices as follow:

$$Q_{11}(t) = \left(\sum_{j=1}^{N} \lambda_{0j}(t)\left(A_{0j}^{*}(t) + B_{0j}(t)K_{0j}'(t)\delta(0)\right)\right)^{T} P + P\left(\sum_{j=1}^{N} \lambda_{0j}(t)\left(A_{0j}^{*}(t) + B_{0j}(t)K_{0j}'(t)\delta(0)\right)\right) + \sum_{i=1}^{q} S_i(t),$$

$$Q_{1,i+1}(t) = Q_{i+1,1}^T(t) := \sum_{j=1}^{N} \lambda_{ij}(t) P A_{ij}^*(t), \quad \forall i \in \overline{q},$$
$$Q_{ii}(t) : = -\left(1 - \dot{h}_i(t)\right) S_i(t - h_i(t)), \quad Q_{ij}(t) = 0, \quad \forall i, j(\neq i) \in \overline{q+1} \setminus \{1\}, \tag{3.10}$$

and the right-hand-side limits are defined with block matrices as follow:

$$Q_{11}(t^+) = \left(\sum_{j=1}^{N} \lambda_{0j}(t) \left(A_{0j}^*(t^+) + B_{0j}(t) K'_{0j}(t) \right) \right)^T P$$
$$+ P \left(\sum_{j=1}^{N} \lambda_{0j}(t) \left(A_{0j}^*(t^+) + \sum_{j=1}^{N} B_{0j}(t) K'_{0j}(t) \right) \right) + \sum_{i=1}^{q} S_i(t^+) \tag{3.11}$$
$$Q_{1,i+1}(t^+) = Q_{i+1,1}^T(t^+) := \sum_{j=1}^{N} \lambda_{ij}(t) P A_{ij}^*(t^+), \quad \forall i \in \overline{q},$$
$$Q_{ii}(t^+) := -\left(1 - \dot{h}_i(t)\right) S_i(t - h_i(t))^+, \quad Q_{ij}(t) = 0, \quad \forall i, j(\neq i) \in \overline{q+1} \setminus \{1\},$$

since from Assumption 2.1, the scalar functions $\lambda_{ij}(t)$ and the matrix functions $B_{0j}(\ t)$, for all $i \in \overline{q} \cup \{0\}$, for all $j \in \overline{N}$ cannot be discontinuous at the sequence Imp. As in (3.11), a matrix function entry at t^+ is more distinct than its left-hand-side limit at t only if it has a discontinuity at the time instant t. Thus,

$$\dot{V}(t^+, x_{t^+}) - \dot{V}(t, x_t) = \widehat{x}^T(t)(Q(t^+) - Q(t))\widehat{x}(t), \quad V(t^+, x_t) - V(t, x_t) = 0, \quad \forall t \notin \text{Imp},$$
$$\dot{V}(t^+, x_{t^+}) - \dot{V}(t, x_t) = 0, \quad \forall t \notin D \text{ since } Q(t^+) = Q(t). \tag{3.12}$$

Furthermore, in view of (3.5),

$$\dot{V}(t^+, x_{t^+}) - \dot{V}(t, x_t) = \widehat{x}^T(t^+) Q(t^+) \widehat{x}(t^+) - \widehat{x}^T(t) Q(t) \widehat{x}(t), \quad \forall t \in \text{Imp}. \tag{3.13}$$

If, in addition, $t \notin D_p$, that is, if $t \in \text{Imp} \cap \overline{D_p}$, and since $Q(t^+) = Q(t)$, (3.13) becomes

$$\dot{V}(t^+, x_{t^+}) - \dot{V}(t, x_t) = \widehat{x}^T(t^+) Q(t) \widehat{x}(t^+) - \widehat{x}^T(t) Q(t) \widehat{x}(t), \tag{3.14}$$

Furthermore,

$$V(t^+, x_{t^+}) - V(t, x_t) = \int_t^{t^+} \dot{V}(\tau, x_\tau) d\tau$$
$$\times x^T(t)\left(\left(\sum_{j=1}^{N} \lambda_{0j}(t) B_{0j}(t) K'_{0j}(t)\right)^T P\left(\sum_{j=1}^{N} \lambda_{0j}(t) B_{0j}(t) K'_{0j}(t)\right)\right.$$
$$\left. + 2P\left(\sum_{j=1}^{N} \lambda_{0j}(t) B_{0j}(t) K'_{0j}(t)\right)\right) x(t), \quad \forall t \in \mathrm{Imp}. \tag{3.15}$$

since $x(t^+) - x(t) = \sum_{j=1}^{N} \lambda_{0j}(t) B_{0j}(t) K'_{0j}(t) x(t)$ from (2.4) (which results to be zero for $t \notin \mathrm{Imp}$). Now, for any $k \in \mathbf{Z}_{0+}$ and some $p_k \in \mathbf{Z}_+$, consider a sequence of consecutive impulsive time instants $\mathrm{Imp}(t_k, t_{k+p_k}) := \{t_k, t_{k+1}, \ldots, t_{k+p_k}\} \subset \mathrm{Imp}$, so that,

$$V\left(t^+_{k+p_k}, x_{t^+_{k+p_k}}\right) - V\left(t^+_k, x_{t^+_k}\right)$$
$$= \int_{t^+_k}^{t^+_{k+p_k}} \dot{V}(\tau, x_\tau) d\tau$$
$$= \sum_{i=1}^{p_k} \left[\int_{t^+_{k+i-1}}^{t^-_{k+i}} \dot{V}(\tau, x_\tau) d\tau + 2x^T(t_{k+i}) P\left(\sum_{j=1}^{N} \lambda_{0j}(t_{k+i}) B_{0j}(t_{k+i}) K'_{0j}(t_{k+i})\right) x(t_{k+i})\right.$$
$$+ x^T(t_{k+i})\left(\sum_{j=1}^{N} \lambda_{0j}(t_{k+i}) K'^T_{0j}(t_{k+i}) B^T_{0j}(t_{k+i})\right)$$
$$\left. \times P\left(\sum_{j=1}^{N} \lambda_{0j}(t_{k+i}) B_{0j}(t_{k+i}) K'_{0j}(t_{k+i})\right) x(t_{k+i})\right)\Bigg] \tag{3.16}$$

$$\leq -\sum_{i=1}^{p_k} \left[\int_{t^+_{k+i-1}}^{t_{k+i}} \alpha(\tau)\left|\overline{x}^T(\tau) Q_d(\tau) \overline{x}(\tau)\right| d\tau + \int_{t^+_{k+i-1}}^{t_{k+1}} \beta(\tau)\left|\overline{x}^T(\tau) Q_{od}(\tau) \overline{x}(\tau)\right| d\tau\right.$$
$$+ x^T(t_{k+i})\left(\sum_{j=1}^{N} \lambda_{0j}(t_{k+i}) K'^T_{0j}(t_{k+i}) B^T_{0j}(t_{k+i})\right)$$
$$\times P\left(\sum_{j=1}^{N} \lambda_{0j}(t_{k+i}) B_{0j}(t_{k+i}) K'_{0j}(t_{k+i})\right) x(t_{k+i})\Bigg)$$
$$\left. -2\mu(t_{k+i})\left|x^T(t_{k+i}) P\left(\sum_{j=1}^{N} \lambda_{0j}(t_{k+i}) B_{0j}(t_{k+i}) K'_{0j}(t_{k+i})\right) x(t_{k+i})\right|\right] \tag{3.17}$$

by using the binary indicator functions as follow:

(a) $\alpha : \mathbf{R}_{0+} \to \{1, -1\}$ defined by $\alpha(t) = 1$ if $\overline{x}^T(t)Q_d(t)\overline{x}(t) > 0$ and $\alpha(t) = -1$ otherwise, for all $t \in \mathbf{R}_{0+}$,

(b) $\beta : \mathbf{R}_{0+} \to \{1, -1\}$ defined by $\beta(t) = 1$ if $\overline{x}^T(t)Q_{od}(t)\overline{x}(t) > 0$ and $\beta(t) = -1$ otherwise, for all $t \in \mathbf{R}_{0+}$,

(c) $\mu : \mathrm{Imp} \to \{1, -1\}$ defined by,

$\mu(t_k) = 1$ if $x^T(t_j^+)P(\sum_{j=1}^{N} \lambda_{0j}(t_j)B_{0j}(t_j)K'_{0j}(t_j))x(t_j^+) > 0$ and $\mu(t_k) = -1$ otherwise; for all $t_k \in \mathrm{Imp}$. Equation (3.17) is less than or equal to zero, which implies that $V(t_{k+p_k}^+, x_{t_{k+p_k}^+}) \le V(t_k^+, x_{t_k^+})$ if

$$\begin{aligned}
&\sum_{i=1}^{p_k}\Bigg[\int_{t_{k+i-1}^+}^{t_{k+i}} \alpha(\tau)\left|\overline{x}^T(\tau)Q_d(\tau)\overline{x}(\tau)\right|d\tau + \int_{t_{k+i-1}^+}^{t_{k+i}} \beta(\tau)\left|\overline{x}^T(\tau)Q_{od}(\tau)\overline{x}(\tau)\right|d\tau \\
&\quad + x^T(t_{k+i})\left(\left(\sum_{j=1}^{N} \lambda_{0j}(t_{k+i})K_{0j}'^{T}(t_{k+i})B_{0j}^T(t_{k+i})\right)\right. \\
&\quad \times P\left(\sum_{j=1}^{N} \lambda_{0j}(t_{k+i})B_{0j}(t_{k+i})K'_{0j}(t_{k+i})\right)x(t_{k+i})\Bigg) \\
&\quad - 2\mu(t_{k+i})\left|x^T(t_{k+1})P\left(\sum_{j=1}^{N} \lambda_{0j}(t_{k+i})B_{0j}(t_{k+i})K'_{0j}(t_{k+i})\right)x(t_{k+i})\right|\Bigg] \ge 0,
\end{aligned} \tag{3.18}$$

which holds with an existing $\mathrm{Imp} \ni t_{k^*} = t_{k+p_k} \in [t_k, t_{k+p_k}]$ for each $t_{k+i} \in \mathrm{Imp}$ (for all $i \in \overline{p_k} \cup \{0\}$) with impulsive control gains $K'_{0j}(t_{k^*}) = \Lambda_{0j}(t_{k^*})B_{0j}^T(t_{k^*})P$ of the jth parameterization of the polytopic system, where $\mathbf{R}^{n\times n} \ni \Lambda_{0j}(t_{k^*}) = \Lambda_{0j}^T(t_{k^*})$, for all $j \in \overline{N}$ if

$$\begin{aligned}
&\sum_{j=1}^{N}\Bigg(\lambda_{0j}(t_{k^*})\lambda_{\max}(\Lambda_{0j}(t_{k^*}))\left\|PB_{0j}(t_{k^*})\right\|_2 \\
&\quad \times\left(1 - \lambda_{0j}(t_{k^*})\frac{\lambda_{\min}^2(\Lambda_{0j}(t_{k^*}))\lambda_{\min}\left(B_{0j}^T(t_{k^*})P^2B_{0j}(t_{k^*})\right)}{\lambda_{\max}(\Lambda_{0j}(t_{k^*}))}\right)\Bigg)
\end{aligned} \tag{3.19}$$

$$\leq \frac{1}{\|x(t_{k^*})\|_2^2}\left(\sum_{i=1}^{k^*}\left[\int_{t_{k+i-1}^+}^{t_{k+i}}\alpha(\tau)\left|\overline{x}^T(\tau)Q_d(\tau)\overline{x}(\tau)\right|d\tau+\int_{t_{k+i-1}^+}^{t_{k+i}^-}\beta(\tau)\left|\overline{x}^T(\tau)Q_{od}(\tau)\overline{x}(\tau)\right|d\tau\right]\right.$$
$$+\sum_{i=1}^{k^*-1}\left[x^T(t_{k+i})\left(\sum_{j=1}^{N}\lambda_{0j}(t_{k+i})K_{0j}^{\prime T}(t_{k+i})B_{0j}^T(t_{k+i})\right)\right.$$
$$\times P\left(\sum_{j=1}^{N}\lambda_{0j}(t_{k+i})B_{0j}(t_{k+i})K'_{0j}(t_{k+i})\right)x(t_{k+i})\Bigg)$$
$$\left.\left.-2\left|x^T(t_{k+i})\left(\sum_{j=1}^{N}\lambda_{0j}(t_{k+i})PB_{0j}(t_{k+i})K'_{0j}(t_{k+i})\right)x(t_{k+i})\right|\right]\right), \tag{3.20}$$

where $t_{k^*}:=\max_{i\in\overline{p}_k}(t_{k+i}\in \text{Imp}: x^T(t_{k+i})(\sum_{j=1}^N\lambda_{0j}(t_{k+i})PB_{0j(t_{k+i})}K'_{0j}(t_{k+i}))x(t_{k+i})\neq 0)$

The existence of $t_{k^*}\in[t_k,t_{k+p_k}]$ has been proven for time instants $t_{k+i}\in\text{Imp}$ (for all $i\in\overline{p_k}\cup\{0\}$), and some $p_k\in\mathbf{Z}_{0+}$ such that $x^T(t_k^*)(\sum_{j=1}^N\lambda_{0j}(t_k^*)PB_{0j}(t_k^*)K'_{0j}(t_k^*))x(t_k^*)\neq 0$ if $x(t_k^*)\neq 0$ for appropriate impulsive controller gains $K'_{0j}(t_k)$, for all $j\in\overleftarrow{N}$. In particular, if $\Lambda_{0j}(t_{k^*})=\nu(t_{k^*})I_m\neq 0$ with $\nu(t_{k^*})\in\mathbf{R}\setminus\{0\}$ being a scalar common for the impulses injected at all the parameterizations of the polytopic system, then the condition in (3.19) becomes in particular,

$$\nu(t_{k^*})\leq\frac{1}{\rho(t_{k^*})\|x(t_{k^*})\|_2^2}$$
$$\times\left(\sum_{i=1}^{k^*}\left[\int_{t_{k+i-1}^+}^{t_{k+i}}\alpha(\tau)\left|\overline{x}^T(\tau)Q_d(\tau)\overline{x}(\tau)\right|d\tau+\int_{t_{k+i-1}^+}^{t_{k+i}}\beta(\tau)\left|\overline{x}^T(\tau)Q_{od}(\tau)\overline{x}(\tau)\right|d\tau\right]\right.$$
$$+\sum_{i=1}^{k^*-1}\left[x^T(t_{k+i})\left(\sum_{j=1}^{N}\lambda_{0j}(t_{k+i})K_{0j}^{\prime T}(t_{k+i})B_{0j}^T(t_{k+i})\right)\right.$$
$$\times P\left(\sum_{j=1}^{N}\lambda_{0j}(t_{k+i})B_{0j}(t_{k+i})K'_{0j}(t_{k+i})\right)x(t_{k+i})\Bigg) \tag{3.21}$$
$$\left.\left.-2\left|x^T(t_{k+i})\left(\sum_{j=1}^{N}\lambda_{0j}(t_{k+i})PB_{0j}(t_{k+i})K'_{0j}(t_{k+i})\right)x(t_{k+i})\right|\right]\right),$$
$$\rho(t_{k^*}):=\sum_{j=1}^{N}\left(\lambda_{0j}(t_{k^*})\lambda_{\max}(\Lambda_{0j}(t_{k^*}))\left\|PB_{0j}(t_{k^*})\right\|_2\right.$$
$$\left.\times\left(1-\lambda_{0j}(t_{k^*})\frac{\lambda_{\min}^2(\Lambda_{0j}(t_{k^*}))\lambda_{\min}\left(B_{0j}^T(t_{k^*})P^2B_{0j}(t_{k^*})\right)}{\lambda_{\max}(\Lambda_{0j}(t_{k^*}))}\right)\right).$$

It follows by simple inspection that $\nu(t_{k^*})$ may be chosen to satisfy (3.21) and, furthermore, $-\infty \leq \nu(t_{k^*}) \leq \infty$ for some $t_{k^*} \in \text{Imp} \cap [t_k, t_{k+p_k}]$. It is now proven by contradiction that the sequences $\{\|x(t_k)\|_{t_k \in \text{Imp}}\}$and $\{\|x(t_k^+)\|_{t_k \in \text{Imp}}\}$ are both bounded. Assume that $S_\nu := \{\nu(t_{k^*})\}_{t_{k^*} \in \text{Imp} \cap [t_k, t_{k+p_k}]}$ is an unbounded sequence. Then, there is an infinite subsequence $S'_\nu \subset S_\nu$ such that $S'_\nu \ni \nu(t_{k^*}) \to \pm\infty$ as $t_{k^*} \to \infty$, for all $t_k \in \text{Imp}$. From the definition of the Lyapunov function candidate (3.4) and the guaranteed property $V(t_{k^*}^+, x_{t_{k^*}^+}) \leq V(t_k^+, x_{t_k^+}) \leq V(t_1^+, x_{t_1^+}) < \infty$, (from (3.17), if (3.21) holds), it follows that $\|x(t_{k^*}^+)\| \leq M_{\varphi k^*}^+ < \infty$ for any positive finite constant M_φ depending on the bounded function of initial conditions of the system (3.4). Also, $\|x(t_{k^*})\| \leq M_{\varphi k^*} < \infty$ since the discontinuities at the state trajectory solution caused by impulses are second-class finite jump-type discontinuities. Then, the sequences $\{\|x(t_k^*)\|\}_{t_k^* \in \text{Imp}^*}$ and $\{\|x(t_k^{*+})\|\}_{t_k^* \in \text{Imp}^*}$ are bounded by positive real constants, $M_\varphi = \max_{t_k \in \text{ Imp}}(\max_{t_k^* \in [t_k, t_{k+p_k}]} M_{\varphi k^*})$ and $M_\varphi^+ = \max_{t_k \in \text{ Imp}}(\max_{t_k^* \in [t_k, t_{k+p_k}]} M_{\varphi k^*}^+)$, respectively. This implies that (3.21) may be fulfiled with $-\infty < \nu(t_{k^*}) < \infty$, also, since

(1) the state-trajectory solution of the closed-loop system is continuous and almost everywhere time differentiable except at second-class discontinuity points on a set of zero measure, and

(2) the state-trajectory solution of the closed-loop system is bounded on the subsequence Imp^*. Thus, it cannot beunbounded on $\text{Imp} \setminus \text{Imp}^*$ since, otherwise, it could not be an almost everywhere smooth state-trajectory solution.

As a result, it exist $C = C(T^*, \text{Imp}) \in \mathbf{R}_+$ and $C^+ = C^+(T^*, \text{Imp}) \in \mathbf{R}_+$ such that $\|x(t_k)\| \leq C M_\varphi$ and $\|x(t_k^+)\| \leq C^+ M_\varphi^+$, for all $t_k \in \text{Imp}$ and the candidate (3.4) is a Lyapunov functional. The result has been proven. □

The proof of the global asymptotic stability of the system requires to extend Theorem 3.4 by guaranteeing that the state-trajectory solution converges asymptotically to zero as time tends to infinity. This requires also stabilizability-type conditions on the nonimpulsive part of the closed-loop solution. The following result holds.

Theorem 3.5. *The closed-loop system* (2.6)–(2.7) *is globally asymptotically stable for any* $\varphi \in BPC^{(0)}([-\overline{h}, 0], \mathbf{R}^n)$ *and a given sequence of impulsive time instants if the regular controller gain functions* $K_{ij} : \mathbf{R}_{0+} \to \mathbf{R}^{n \times m}$ *and the sequence of impulsive controller gains* $K'_{0j} : \mathbf{R}_{0+} \to \mathbf{R}^{n \times m}$, *for all* $j \in \overline{N}$, *for all* $i \in \overline{q} \cup \{0\}$ *are chosen so that the following matrix inequalities hold for some* $\mathbf{R}^{n \times n} \ni P = P^T \succ 0$ *and* $S_i \in BPC^{(0)}(\mathbf{R}_{0+}, \mathbf{R}^{n \times n})$ *which fulfils* $S_i(t) \succ 0$, *for all* $t \in \mathbf{R}_{0+}$, *for all* $i \in \overline{q}$ *as follow:*

$$Q_{11}(t) \prec 0, \quad \overline{Q}_{22}(t) - \overline{Q}_{12}^T(t) Q_{11}^{-1}(t) \overline{Q}_{12}^T(t) \prec 0, \quad \forall t \in \mathbf{R}_{0+} \setminus D, \tag{3.22}$$

$$\begin{aligned} &Q_{11}(t^+) \prec 0, \quad \overline{Q}_{22}(t^+) - \overline{Q}_{12}^T(t^+) Q_{11}^{-1}(t^+) \overline{Q}_{12}^T(t^+) \prec 0, \quad \forall t \in D \setminus \text{Imp}, \\ &Q_{11}(t^+) \prec 0, \quad \overline{Q}_{22}(t^+) - \overline{Q}_{12}^T(t^+) Q_{11}^{-1}(t^+) \overline{Q}_{12}^T(t^+) \prec 0, \quad \forall t \in D \setminus \text{Imp}, \end{aligned} \tag{3.23}$$

$$Q(t^+) - Q(t) \leq 0, \quad \forall t \in \text{Imp}, \tag{3.24}$$

where

$$\overline{Q}_{12}(t) = \overline{Q}_{21}^{T}(t) = \left(\sum_{j=1}^{N} \lambda_{1j}(t) P A_{1j}^{*}(t) \sum_{j=1}^{N} \lambda_{2j}(t) P A_{1j}^{*}(t) \cdots \sum_{j=1}^{N} \lambda_{qj}(t) P A_{1j}^{*}(t) \right), \tag{3.25}$$

$$\begin{aligned}\overline{Q}_{22}(t) :&= -\text{Block diag}\big((1 - \dot{h}_1(t)) S_{11}(t - h_1(t)) (1 - \dot{h}_2(t)) \\ &\quad \times S_{22}(t - h_2(t)) \cdots (1 - \dot{h}_q(t)) S_{qq}(t - h_q(t))\big).\end{aligned} \tag{3.26}$$

Proof. From (3.6) and (3.12), the system is globally asymptotically stable if the Lyapunov functional candidate (3.4) is in fact a Lyapunov functional which is guaranteed if

(a) $\dot{V}(t, x_t) < 0$, for all $t \in \mathbf{R}_{0+}$ such that $x_t \neq 0$ what holds if and only if $Q(t) \preceq 0$, for all $t \in \mathbf{R}_{0+}$

(b) $V(t^+, x_{t^+}) \leq V(t, x_t)$, for all $t \in \text{Imp}$ what holds if and only if $Q(t^+) \preceq Q(t)$, for all $t \in \mathbf{R}_{0+}$.

The first condition holds from (3.9) via Schur's complement if (3.22)-(3.23) hold. The second condition holds if (3.24) holds. □

Remark 3.6. Theorem 3.5 can be tested directly from (3.9)–(3.11) with direct algebraic tests. However, it is very restrictive since it does not provide with conditions guaranteeing a cooperative achievement of global asymptotic stability among the non-impulsive and impulsive parts. Note that necessary conditions for the fulfilment of Theorem 3.5 are from (3.9)–(3.11): $Q_{11}(t) \prec 0$, $Q_{11}(t^+) \prec 0$ (i.e., the Lyapunov matrix inequality holds for for all $t \in \mathbf{R}_{0+} \setminus D$, and for the left and right limits of all $t \in D$), and $Q_{11}(t^+) \preceq Q_{11}(t) \prec 0$, for all $t \in \text{Imp}$.

Remark 3.7. Note that (3.22)-(3.23) imply that

$$\overline{Q}_{22}(t) \prec \overline{Q}_{12}^{T}(t) Q_{11}^{-1}(t) \overline{Q}_{12}^{T}(t) \preceq 0, \quad \forall t \in \mathbf{R}_{0+} \setminus D, \tag{3.27}$$

$$\begin{aligned}\overline{Q}_{22}(t) &\prec \overline{Q}_{12}^{T}(t) Q_{11}^{-1}(t) \overline{Q}_{12}^{T}(t) \preceq 0, \\ \overline{Q}_{22}(t^+) &\prec \overline{Q}_{12}^{T}(t^+) Q_{11}^{-1}(t^+) \overline{Q}_{12}^{T}(t^+) \preceq 0,\end{aligned} \tag{3.28}$$

for all $t \in D \setminus \text{Imp}$ since $\overline{Q}_{12}^{T}(t) Q_{11}^{-1}(t) \overline{Q}_{12}^{T}(t)$ is symmetric and $Q_{11}^{-1}(t) \prec 0$ and

$$\begin{aligned}\overline{Q}_{22}(t) &\prec \overline{Q}_{12}^{T}(t) Q_{11}^{-1}(t) \overline{Q}_{12}^{T}(t) \preceq 0, \\ \overline{Q}_{22}(t^+) &\prec \overline{Q}_{12}^{T}(t^+) Q_{11}^{-1}(t^+) \overline{Q}_{12}^{T}(t^+) \preceq 0, \quad \forall t \in D \setminus \text{Imp},\end{aligned} \tag{3.29}$$

As a result, $\sup_{t \in \mathbf{R}_0^+} [\max_{i \in \overline{q}}(\max(\dot{h}_i(t), \dot{h}_i(t^+)))] < 1$ is a necessary condition for Theorem 3.5 to hold.

Concerning Theorem 3.5, (3.22)-(3.23), note that isolated bounded discontinuities in $\dot{V}(t, x_t)$ do not affect to maintain $V(t, x_t)$ as a positive strictly monotonically decreasing functional on $\mathbf{R}_{0+}$. Therefore, Theorem 3.5 can be relaxed by removing a set of zero measure of $\mathbf{R}_{0+} \setminus D$ (and then of $\mathbf{R}_{0+}$) to evaluate (3.22)-(3.23) and also bounded discontinuities at the sequence Imp from (3.24). The resulting modified stability result follows.

Corollary 3.8. *The closed-loop system* (2.6)–(2.7) *is globally asymptotically stable for any* $\varphi \in BPC^{(0)}([-\overline{h},0],\mathbf{R}^n)$ *and a given sequence of impulsive time instants if the regular controller gain functions* $K_{ij} : \mathbf{R}_{0+} \to \mathbf{R}^{n\times m}$ *and the sequence of impulsive controller gains* $K'_{0j} : \mathbf{R}_{0+} \to \mathbf{R}^{n\times m}$*, for all* $j \in \overline{N}$*, for all* $i \in \overline{q} \cup \{0\}$ *are chosen so that the following matrix inequalities hold for some* $\mathbf{R}^{n\times n} \ni P = P^T \succ 0$ *and* $S_i \in BPC^{(0)}(\mathbf{R}_{0+},\mathbf{R}^{n\times n})$ *which fulfils* $S_i(t) \succ 0$*, for all* $t \in \mathbf{R}_{0+}$*, for all* $i \in \overline{q}$ *as follow:*

$$\left(\sum_{j=1}^{N}\lambda_{0j}(t)A_{0j}^{*^T}(t)\right)P + P\left(\sum_{j=1}^{N}\lambda_{0j}(t)A_{0j}^{*}(t)\right) + \sum_{i=1}^{q}S_i(t) \prec 0, \tag{3.30}$$

almost everywhere in $\mathbf{R}_{0+}$*,*

$$\begin{aligned}
&\text{Block diag}\big((1-\dot{h}_1(t))S_{11}(t-h_1(t))(1-\dot{h}_2(t)) \\
&\qquad\qquad \times S_{22}(t-h_2(t))\cdots(1-\dot{h}_q(t))S_{qq}(t-h_q(t))\big) \\
&\succ \left(\sum_{j=1}^{N}\lambda_{1j}(t)PA_{1j}^{*}(t)\sum_{j=1}^{N}\lambda_{2j}(t)PA_{1j}^{*}(t)\cdots\sum_{j=1}^{N}\lambda_{qj}(t)PA_{1j}^{*}(t)\right)^T \\
&\quad\times\left[-\left(\sum_{j=1}^{N}\lambda_{0j}(t)A_{0j}^{*^T}(t)\right)P - P\left(\sum_{j=1}^{N}\lambda_{0j}(t)A_{0j}^{*}(t)\right) - \sum_{i=1}^{q}S_i(t)\right]^{-1} \\
&\quad\times\left(\sum_{j=1}^{N}\lambda_{1j}(t)PA_{1j}^{*}(t)\sum_{j=1}^{N}\lambda_{2j}(t)PA_{1j}^{*}(t)\cdots\sum_{j=1}^{N}\lambda_{qj}(t)PA_{1j}^{*}(t)\right),
\end{aligned} \tag{3.31}$$

almost everywhere in $\mathbf{R}_{0+}$*, and*

$$\begin{aligned}
&\overline{Q}(t^+) - Q(t) \\
&= \begin{bmatrix}\left(\sum_{j=1}^{N}\lambda_{0j}(t)K_{0j}^{\prime T}(t)B_{0j}^{T}(t)\right)P + P\left(\sum_{j=1}^{N}\lambda_{0j}(t)B_{0j}(t)K'_{0j}(t)\lambda_{0j}(t)B_{0j}(t)K'_{0j}(t)\right) & 0 \\ 0 & 0\end{bmatrix} \preceq 0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall t \in \text{Imp},
\end{aligned} \tag{3.32}$$

where $\overline{Q}_{ij}(t^+) = Q_{ij}(t)$ *if* $Q_{ij}(t)$ *is not impulsive and* $\overline{Q}_{ij}(t^+) = Q_{ij}(t^+)$*, otherwise.*

Proof. Equations (3.30)-(3.31) follow from Theorem 3.5 by expanding $Q_{11}(t) \prec 0, \overline{Q}_{22}(t) - \overline{Q}_{12}^T(t)Q_{11}^{-1}(t)\overline{Q}_{12}^T(t) \prec 0$ from (3.8)-(3.9) and (3.25)-(3.26) on $\mathbf{R}_{0+}$ excepting time instants of bounded isolated discontinuities. Equation (3.32) follow from (3.24), for all $t \in$ Imp also excluding bounded discontinuities at the time-derivative of the Lyapunov functional since they are irrelevant for analysis since they do not generate bounded jumps at the Lyapunov functional. □

Corollary 3.8 holds in terms of more restrictive but it is easier to test conditions given in the subsequent result.

Corollary 3.9. *Corollary 3.8 holds if*

$$\left(\sum_{j=1}^{N}\lambda_{0j}(t)A_{0j}^{*^T}(t)\right)P+P\left(\sum_{j=1}^{N}\lambda_{0j}(t)A_{0j}^{*}(t)\right)+\sum_{i=1}^{q}S_i(t)\prec -q(t)I_n, \tag{3.33}$$

almost everywhere in $\mathbf{R}_{0+}$ *for some* $q\in BPC^{(0)}(\mathbf{R}_{0+},\mathbf{R}_{+})$ *which satisfies*

$$q(t)>\min\left(q_0,\frac{\left\|\left(\sum_{j=1}^{N}\lambda_{1j}(t)PA_{1j}^{*}(t)\sum_{j=1}^{N}\lambda_{2j}(t)PA_{1j}^{*}(t)\cdots\sum_{j=1}^{N}\lambda_{qj}(t)PA_{1j}^{*}(t)\right)\right\|_2^2}{\min_{i\in\overline{q}}\lambda_{\min}(S_{ii}(t-h_i(t)))},\right) \tag{3.34}$$

provided that

$$\max_{i\in\overline{q}}\dot{h}_i(t)<\min\left(\gamma,1-\frac{\left\|\left(\sum_{j=1}^{N}\lambda_{1j}(t)PA_{1j}^{*}(t)\sum_{j=1}^{N}\lambda_{2j}(t)PA_{1j}^{*}(t)\cdots\sum_{j=1}^{N}\lambda_{qj}(t)PA_{1j}^{*}(t)\right)\right\|_2^2}{q(t)\min_{i\in\overline{q}}\lambda_{\min}(S_{ii}(t-h_i(t)))}\right), \tag{3.35}$$

almost everywhere in $\mathbf{R}_{0+}$, *and* (3.32) *holds for all* $t\in\mathrm{Imp}$.

The following result states that stabilization is achievable under impulsive control impulses which respect a maximum separation time interval and exceed an upper bound of the maximum delay provided that it is bounded.

Corollary 3.10. *Assume that*

(1) *all the delays are uniformly bounded for all time,*

(2) $\neg\exists t\in\mathbf{R}_{0+}:\lambda_{0j}(t)B_{0j}(t)\neq 0$ *for all* $j\in\overline{N}$ *and fix a real constant* $\infty>\overline{T}\geq T(\geq\sup_{t\in\mathbf{R}_{0+}}\overline{h}(t))$. *Fix a real constant* $\infty>\overline{T}\geq T(\geq\sup_{t\in\mathbf{R}_{0+}}\overline{h}(t))$. *Thus, there is always a globally stabilizing impulsive control law by appropriate design of one of the impulsive controller gains and choice of the interval sequences of impulsive instants as follow:*

$$K'_{0\ell}:\mathbf{R}_{0+}\longrightarrow\mathbf{R}^{m\times n},\quad \ell\in\overline{N},\quad Imp(jT,(j+1)T] \tag{3.36}$$

for each time interval $(jT,(j+1)T]$, *for all* $j(\geq j_0)\in\mathbf{Z}_{0+}$ *and some given arbitrary finite* $j_0\in\mathbf{Z}_{0+}$.

Proof. One has from (3.4) that

$$\Delta V(t, x_t) = V(t^+, x_{t^+}) - V(t, x_t) = x^T(t^+)Px(t^+) - x^T(t)Px(t), \quad \forall t \in \mathbf{R}_{0+}, \tag{3.37}$$

which equalizes zero at $t \notin \mathrm{Imp}$, since

$$\sum_{i=1}^{q} \int_{(t-h_i(t))^+}^{t^+} x^T(\tau)S_i(\tau)x(\tau)d\tau = \sum_{i=1}^{q} \int_{(t-h_i(t))}^{t} x^T(\tau)S_i(\tau)x(\tau)d\tau, \quad \forall t \in \mathbf{R}_{0+}, \tag{3.38}$$

since the discontinuities of the state vector at $t \in \mathrm{Imp}$ are bounded. Thus, one has for any arbitrary $T \in \mathbf{R}_+$ that

$$\begin{aligned} V(t^+ + T) - V(t^+) &\leq -\int_t^{t+T} \widehat{x}^T(\tau)Q(\tau)\widehat{x}(\tau)d\tau \\ &\quad + \sum_{t_i \in \mathrm{Imp}(t,t+T]} \left(x^T(t_i^+)Px(t_i^+) - x^T(t_i)Px(t_i)\right) \\ &\leq -\int_t^{t+T} \widehat{x}^T(\tau)Q(\tau)\widehat{x}(\tau)d\tau \\ &\quad + \sum_{t_i \in \mathrm{Imp}(t,t+T]} \left(x^T(t_i)\left[\left(I_n + \sum_{j=1}^{N} \lambda_{0j}(t_i)K_{0j}^{\prime T}(t_i)B_{0j}(t_i)\right)\right.\right. \\ &\qquad \left.\left. \times P\left(I_n + \sum_{j=1}^{N} \lambda_{0j}(t_i)B_{0j}(t_i)K_{0j}^{\prime}(t_i)\right) - P\right]x(t_i)\right), \\ &\qquad \forall t \in \mathbf{R}_{0+}. \end{aligned} \tag{3.39}$$

Define

$$\begin{aligned} t^* = t^*(t,T) = \{ & t_i \in \mathrm{Imp}(t, t+T] : x(t^*) \neq 0 \wedge \left(\mathrm{Imp}(t^*, t+T] \neq \emptyset \Longrightarrow x\left(t_i^*\right) = 0, \right. \\ & \left. \forall t_i \in \mathrm{Imp}(t^*, t+T]\right)\} \in (t, t+T] \cap \mathbf{R}_{0+} \end{aligned} \tag{3.40}$$

as the last impulsive sampling instant in $(t, t+T]$, where the state vector is nonzero. Thus, $V(t^+ + T) \leq V(t^+)$ if

$$\begin{aligned} &\left(x^T(t^*)\left[\left(I_n + \sum_{j=1}^{N} \lambda_{0j}(t^*)K_{0j}^{\prime T}(t^*)B_{0j}^T(t^*)\right)\right.\right. \\ &\qquad \left.\left. \times P\left(I_n + \sum_{j=1}^{N} \lambda_{0j}(t^*)B_{0j}(t^*)K_{0j}^{\prime}(t^*)\right) - P\right]x(t^*)\right) \\ &\qquad \leq \int_t^{i+T} \widehat{x}^T(\tau)Q(\tau)\widehat{x}(\tau)d\tau \end{aligned}$$

$$- \sum_{t_i \in \operatorname{Imp}(t, t^*(t,T))} \left(x^T(t_i) \left[\left(I_n + \sum_{j=1}^{N} \lambda_{0j}(t_i) B_{0j}^T(t_i) \right) \times P \left(I_n + \sum_{j=1}^{N} \lambda_{0j}(t_i) B_{0j}(t_i) K'_{0j}(t_i) \right) - P \right] x(t_i) \right) \tag{3.41}$$

from (3.6). Since the interval $(t, t+T)$ is finite, it follows that the Lyapunov functional candidate is bounded on the interval, provided that it is bounded at a single point. The result follows by applying the above upper-bounding constraint recursively for $t = jT$, for all $j \geq j_0$ and appropriate choice of the impulsive sequence $\operatorname{Imp}(jT, (j+1)T]$ since the state vector cannot be identically zero on $(jT, (j+1)T]$ for $\infty > \overline{T} \geq T (\geq \sup_{t \in \mathbf{R}_{0+}} \overline{h}(t))$ except for the trivial state-trajectory solution. □

Remark 3.11. Corollary 3.10 may be directly reformulated under weaker (but easier to deal with) conditions by using

$$\begin{aligned} \dot{V}(t, x_t) &\leq \widehat{x}^T(t) Q(t) \widehat{x}(t) = -\widehat{x}^T(t)(Q_d(t) + Q_{od}(t))\widehat{x}(t) \\ &\leq -\left(\lambda_{\min}(Q_d(t)) - \sqrt{\lambda_{\max}\left(Q_{0d}^T(t) Q_{0d}(t)\right)} \right) \|\widehat{x}(t)\|_2^2, \end{aligned} \tag{3.42}$$

$$\begin{aligned} \Delta V(t, x_t) &:= V(t^+, x_{t^+}) - V(t, x_t) \\ &= x^T(t) \left[\left(\sum_{j=1}^{N} \lambda_{0j}(t) K_{0j}^{\prime T}(t) B_{0j}^T(t) \right) P \left(\sum_{j=1}^{N} \lambda_{0j}(t) B_{0j}(t) K'_{0j}(t) \right) \right. \\ &\quad \left. + 2 \left(\sum_{j=1}^{N} \lambda_{0j}(t) K_{0j}^{\prime T}(t) B_{0j}^T(t) \right) P \right] x(t). \end{aligned} \tag{3.43}$$

4. Examples

4.1. Example for Scalar Systems

$\dot{x}(t) = a x(t) + a_0(t) x(t-h) + \sum_{t_k \in \operatorname{Imp}(0,t)} K(t_k) x(t_k) \delta(t - t_k)$ for some constant delay $h \geq 0$. Its solution satisfies for $T_k = t_{k+1} - t_k$, for all $\theta \in [0, T_k]$ with $U(t)$ being the unit step (Heaviside) function,

$$x(t_k^+ + \theta) = e^{a\theta} \left[x(t_k^+) + \int_0^\theta e^{-a\tau} a_0(t_k + \tau) x(t_k + \tau - h) d\tau + K(t_{k+1}) U(\theta - T_k) x(t_{k+1}) \right] \times \left(\sup_{\tau \in [t_k - h, \max(t_{k+1} - h, t_k)]} |x(\tau)| \right)$$

$$
\begin{aligned}
&\leq |1+K(t_{k+1})U(\theta-T_k)|\left|e^{a\theta}\left(x(t_k^+)+\int_0^{\theta}e^{-a\tau}a_0(t_k+\tau)x(t_k+\tau-h)d\tau\right)\right|,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\theta\in[0,T_k]\\
&\Longrightarrow \sup_{\theta\in[t_k,t_{k+1}]}|x(\tau)| \leq \max_{\theta\in[0,T_k]}|1+K(t_{k+1})U(\theta-T_k)|\\
&\max_{\theta\in[0,T_k]}\left|\left(e^{a\theta}\left|x(t_k^+)\right|+\left|\frac{e^{a\theta}-1}{a}\right|\right.\right.\\
&\qquad\left.\left.\times\max_{\tau\in[0,T_k]}|a_0(t_k+\tau)|\left(\sup_{\tau\in[t_k-h,\max(t_{k+1}-h,t_k)]}|x(\tau)|\right)\right)\right|,\quad \forall\theta\in[0,T_k]\\
&\Longrightarrow \sup_{\tau\in[t_k,t_{k+1}]}|x(\tau)| \leq \max\left(1,\max_{\theta\in[0,T_k]}|1+K(t_{k+1})U(\theta-T_k)|\right.\\
&\qquad\qquad\qquad\left.\times\max_{\theta\in[0,T_k]}\left(e^{a\theta}\left|x(t_k^+)\right|+\left|\frac{e^{a\theta}-1}{a}\right|\max_{\tau\in[0,T_k]}|a_0(t_k+\tau)|\right)\right)\\
&\times\max\left(\left(\sup_{\tau\in[t_k-h,t_{k+1}-h]}|x(\tau)|\right),\sup_{\tau\in[t_{k+1}-h,t_k]}|x(\tau)|U(t_k-t_{k+1}-h)\right).
\end{aligned}
\tag{4.1}
$$

Note that

$$
\begin{aligned}
|x(t_{k+1})| &\leq C(t_{k+1})\left(\sup_{\tau\in[t_k-h,\max(t_{k+1}-h,t_k)]}\right)|x(\tau)|,\\
\left|x(t_{k+1}^+)\right| &\leq C(t_{k+1}^+)\left(\sup_{\tau\in[t_k-h,\max(t_{k+1}-h,t_k)]}\right)|x(\tau)|,
\end{aligned}
\tag{4.2}
$$

with

$$
C(t_{k+1}^+)=|1+K(t_{k+1})|C(t_{k+1}),\qquad C(t_{k+1})=\left(e^{aT_k}+\left|\frac{e^{aT_k}-1}{a}\right|\max_{\tau\in[0,T_k]}|a_0(t_k+\tau)|\right),\tag{4.3}
$$

and also

$$
\begin{aligned}
|x(t_{k+1}+\theta)| &\leq C_\theta(t_{k+1})\left(\sup_{\tau\in[t_k-h,\max(t_{k+1}-h,t_k)]}|x(\tau)|\right),\\
\left|x(t_{k+1}^++\theta)\right| &\leq C_\theta(t_{k+1}^+)\left(\sup_{\tau\in[t_k-h,\max(t_{k+1}-h,t_k)]}|x(\tau)|\right),
\end{aligned}
\tag{4.4}
$$

for all $\theta \in [0, T_{k+1}]$, so that

$$|x(t_{k+1}+\theta)| \le \max_{i\in\overline{p}}(C_\theta(t_{k+i}))\left(\sup_{\tau\in\left[t_k-h,\max\left(t_k+\sum_{i=1}^{p}T_{k+i}-h,t_k\right)\right]}|x(\tau)|\right)$$
$$\left|x(t_{k+1}^{+}+\theta)\right| \le \max_{i\in\overline{p}}(C_\theta(t_{k+i}^{+}))\left(\sup_{\tau\in\left[t_k-h,\max\left(t_k+\sum_{i=1}^{p}T_{k+i}-h,t_k\right)\right]}|x(\tau)|\right) \tag{4.5}$$

for all $\theta \in [0, \sum_{i=1}^{p} T_{k+i}]$ and any finite $p \in \mathbf{Z}_+$. Thus, there exist $\Omega_\theta(t_{k+1}-h, t_k) \in [1,\infty) \cap \mathbf{R}_+$, which might be computed with direct simple calculations via (4.3), which equalizes $\max_{i\in\overline{p}_{k+1}}(C_\theta(t_{k+i}^{+}))$ with p_{k+1} being a positive integer accounting for a subsequence of consecutive impulsive time instants $\{t_{k+i} : i \in \overline{p}_{k+1}\}$. Thus, it follows from (4.5) that

$$\max\left(\left(\sup_{\tau\in[t_k-h,t_{k+1}-h]}|x(\tau)|\right), \sup_{\tau\in[t_{k+1}-h,t_k]}|x(\tau)|U(t_k-t_{k+1}-h)\right)$$
$$\le \Omega_\theta(t_{k+1}-h,t_k)\sup_{\tau\in[t_k-h,t_{k+1}-h]}|x(\tau)|. \tag{4.6}$$

It follows directly from (4.6) into (4.1) and complete induction that if

$$\max_{\theta\in[0,t_{k+1}-t_k]}|1+K(t_{k+1})U(\theta-T_k)|\max_{\theta\in[0,t_{k+1}-t_k]}e^{a\theta}$$
$$+\left|\frac{e^{a\theta}-1}{a}\right|\max_{\tau\in[0,t_{k+1}-t_k]}|a_0(t_k+\tau)|\Omega_\theta(t_{k+1}-h,t_k)\le 1, \tag{4.7}$$

for all $t_k \in \operatorname{Imp}(t_0,\infty)$ for some finite $t_0 \in \mathbf{R}_{0+}$ then the system is globally uniformly stable for any admissible function of initial conditions $\varphi \in \mathrm{BPC}^{(0)}([-h,0],\mathbf{R})$ with

$$\sup_{t\in\mathbf{R}_{0+}}|x(t)| \le \sup_{t\in[-h,t_0],}|x(t)| \le K_x < \infty, \tag{4.8}$$

with $x(t)=\varphi(t)$, for all $t \in [-h,0]$. If the inequality in (4.7) is strict, then the system is globally asymptotically stable for any $\varphi \in \mathrm{BPC}^{(0)}([-h,0],\mathbf{R})$.

Note that if, furthermore, $t_{k+1} > t_k + h$, for all $t_k, t_{k+1} \in \operatorname{Imp}$, then $\Omega_\theta(t_{k+1}-h,t_k) = 1$, for all $t_k, t_{k+1} \in \operatorname{Imp}$ so that (4.7) holds if the subsequent constraints hold for some real constant $\gamma \in (0,1] \cap \mathbf{R}_{0+}$ as follow:

$$\max_{\theta\in[0,t_{k+1}-t_k]}e^{a\theta}+\left|\frac{e^{a\theta}-1}{a}\right|\max_{\tau\in[0,t_{k+1}-t_k]}|a_0(t_k+\tau)|\le\gamma,$$
$$\max_{\theta\in[0,t_{k+1}-t_k]}|1+K(t_{k+1})U(\theta-T_k)|\le\gamma^{-1}, \tag{4.9}$$

which may be fulfiled without requiring neither $a \leq 0$ (global stability of the auxiliary system with no delayed dynamics) nor $a + |a_0(t)| \leq 0$ (global stability independent of the delay size) by using appropriate impulses of appropriate signs so that the above inequalities hold. A similar consideration applies for global asymptotic stability one of the inequalities in (4.9) being well posed and strict without requiring neither $a < 0$ (global asymptotic stability of the auxiliary system with no delayed dynamics) nor $a + |a_0(t)| < 0$ (global asymptotic stability independent of the delay size). Note also that these above results are particular results of Theorem 3.1 for a scalar system (2.1)-(2.4) with a single parameterization with the non-impulsive controller being identically zero and the control parameter b being unity. If the scalar dynamic system is of polytopic type 3d by

$$\begin{aligned}
\dot{x}(t) &= \sum_{j=1}^{N} \lambda_{0j}(t)\left(a_{0j}(t)x(t) + \sum_{t_k \in \mathrm{Imp}(0,t)} K_{0j}(t_k)x(t_k)\delta(t-t_k)\right) \\
&\quad + \sum_{i=1}^{q}\sum_{j=1}^{N} \lambda_{ij}(t)a_{ij}(t)x(t-h_i(t)) \\
&= ax(t) + \sum_{j=1}^{N}\sum_{t_k \in \mathrm{Imp}(0,t)} \lambda_{0j}(t_k)K_{0j}(t_k)x(t_k)\delta(t-t_k) \\
&\quad + \sum_{i=0}^{q}\sum_{j=1}^{N} \lambda_{ij}(t)\overline{a}_{ij}(t)x(t-h_i(t)),
\end{aligned} \tag{4.10}$$

provided that $\sum_{j=1}^{N} \lambda_{0j}(t) = 1$, $\lambda_{0j}(t) \in \mathbf{R}_{0+}$, $\overline{a}_{0j}(t) = a_{0j}(t) - a$, $\overline{a}_{ij}(t) = a_{ij}(t)$, for all $i \in \overline{q}$, for all $j \in \overline{N}$, for all $t \in \mathbf{R}_{0+}$ and any arbitrary constant $a \in \mathbf{R}$ so that,

$$\begin{aligned}
x\left(t_k^+ + \theta\right) = e^{a\theta}\Bigg[& x\left(t_k^+\right) + \int_0^{\theta} \sum_{i=0}^{q}\sum_{j=1}^{N} \lambda_{ij}(t_k+\tau)e^{-a\tau}\overline{a}_{ij}(t_k+\tau)x(t_k+\tau-h_i(\tau))d\tau \\
& + \sum_{j=1}^{N} \lambda_{0j}(t_{k+1})K_{0j}(t_{k+1})U(\theta - T_k)x(t_{k+1})\Bigg], \quad \forall \theta \in [0, T_k].
\end{aligned} \tag{4.11}$$

Thus, the first inequality of (4.1) becomes,

$$\begin{aligned}
\sup_{\theta \in [t_k, t_{k+1}]} |x(\tau)| \leq & \max_{\theta \in [0,T_k]} \left| 1 + \sum_{j=1}^{N} \lambda_{0j}(t_{k+1})K_{0j}(t_{k+1})U(\theta - T_k) \right| \\
& \times \max_{\theta \in [0,T_k]} \left| \left(e^{a\theta}\left| x\left(t_k^+\right)\right| + \left| \frac{e^{a\theta}-1}{a} \right| \max_{\tau \in [0,T_k]} \left| \sum_{i=0}^{q}\sum_{j=1}^{N} \lambda_{ij}(t_k+\tau)\overline{a}_{ij}(t_k+\tau) \right| \right.\right. \\
& \left.\left. \times \left(\sup_{\tau \in \left[t_k - \overline{h}(t), \max\left(t_{k+1} - \overline{h}(t), t_k\right)\right]} |x(\tau)| \right)\right)\right|,
\end{aligned} \tag{4.12}$$

for all $\theta \in [0, T_k]$, where $\overline{h}(t) = \max_{i\in\overline{q}}\ \sup(h_i(t))$. Thus, (4.7) is modified as follows:

$$\begin{aligned}
&\max_{\theta\in[0,t_{k+1}-t_k]}\left|1+\sum_{j=1}^{N}\lambda_{0j}(t_{k+1})K_{0j}(t_{k+1})U(\theta-T_k)\right| \\
&\times \max_{\theta\in[0,t_{k+1}-t_k]^{e^{a\theta}}} + \left|\frac{e^{a\theta}-1}{a}\right| \max_{\tau\in[0,t_{k+1}-t_k]}\left|\sum_{i=0}^{q}\sum_{j=1}^{N}\lambda_{ij}(t_k+\tau)\overline{a}_{ij}(t_k+\tau)\right| \\
&\times \Omega_\theta(t_{k+1}-h, t_k) \le 1,
\end{aligned} \tag{4.13}$$

which guarantees global stability from Theorem 3.1 and if the above inequality is strict, then global asymptotic stability is guaranteed.

Example 4.1. This example refers to the stability of the impulsive closed-loop system (2.8), subject to (2.7) and (2.9), by application of Corollary 3.10 to Theorems 3.4-3.5 and Remark 3.11. Assume that the non impulsive controller gains $K_{ij}(t)$are identically zero for all time so that $A^*_{ij}(t) = A_{ij}(t)$, for all $i \in \overline{q} \cup \{0\}$, for all $j \in \overline{N}$ and (3.7)–(3.12) are stated for this particular case. Then, the system is controlled by the impulsive controller gains which are nonzero only at set of zero measure defined by all the sequence of impulsive time instants. Note from Remark 3.11 that $\Delta V(t, x_t) = 0$ if $t \notin \mathrm{Imp}$. Note also that if there only one $\mathrm{Imp} \ni t_i \in (t, t+T]$ at which $x(t_i) \neq 0$ so that for the controller gain choice $K'_{0j}(t_i) = \nu_0(t_i) I_n$ then

$$\begin{aligned}
&V(t^+ + T, x_{t^+}) - V(t, x_t) \\
&\le -\int_{t^+}^{t+T}\left(\lambda_{\min}(Q_d(\tau)) - \sqrt{\lambda_{\max}\left(Q^T_{0d}(\tau)Q_{0d}(\tau)\right)}\right)\|\hat{x}(\tau)\|_2^2 d\tau \\
&\quad + \left[\lambda_{\min}\left(\left(\sum_{j=1}^{N}\lambda_{0j}(t_i)K'^T_{0j}(t_i)B^T_{0j}(t_i)\right)P\left(\sum_{j=1}^{N}\lambda_{0j}(t_i)B_{0j}(t_i)K'_{0j}(t_i)\right)\right)\right. \\
&\quad \left. -2\left\|\left(\sum_{j=1}^{N}\lambda_{0j}(t_i)K'^T_{0j}(t_i)B^T_{0j}(t_i)\right)P\right\|_2\right]\|x(t_i)\|_2^2 \\
&\le -\int_{t^+}^{t+T}\left(-\sqrt{\lambda_{\max}\left(Q^T_{0d}(\tau)Q_{0d}(\tau)\right)}\lambda_{\min}(Q_d(\tau))\right)\|\hat{x}(\tau)\|_2^2 d\tau \\
&\quad + \left[\lambda_{\min}\left(\left(\sum_{j=1}^{N}\lambda_{0j}(t_i)B^T_{0j}(t_i)\right)P\left(\sum_{j=1}^{N}\lambda_{0j}(t_i)B_{0j}(t_i)\right)\right)\nu_0(t_i)\right. \\
&\quad \left. -2\left\|\left(\sum_{j=1}^{N}\lambda_{0j}(t_i)B^T_{0j}(t_i)\right)P\right\|_2\right]\nu_0(t_i)\|x(t_i)\|_2^2 \le 0
\end{aligned} \tag{4.14}$$

if

$$\nu_0(t_i) \in \left[0, \frac{b(t_i) + \sqrt{b^2(t_i) + 4a(t_i)c(t_i)}}{2a(t_i)}\right], \quad \lambda_{\min}(Q_d(t)) \geq \sqrt{\lambda_{\max}(Q_{0d}^T(t)Q_{0d}(t))}, \quad \forall t \in \mathbf{R}_{0+}, \tag{4.15}$$

where

$$\begin{aligned} a(t_i) &:= \lambda_{\min}\left(\left(\sum_{j=1}^{N} \lambda_{0j}(t_i)B_{0j}^T(t_i)\right)P\left(\sum_{j=1}^{N} \lambda_{0j}(t_i)B_{0j}(t_i)\right)\right)\|x(t_i)\|_2^2, \\ b(t_i) &:= 2\left\|\left(\sum_{j=1}^{N} \lambda_{0j}(t_i)B_{0j}^T(t_i)\right)P\right\|_2 \|x(t_i)\|_2^2, \\ c(t_i) &:= \int_{t^+}^{t+T}\left(\lambda_{\min}(Q_d(\tau)) - \sqrt{\lambda_{\max}(Q_{0d}^T(\tau)Q_{0d}(\tau))}\right)\|\widehat{x}(\tau)\|_2^2 d\tau, \end{aligned} \tag{4.16}$$

which can always be fulfilled with $\nu_{0i}(t_i) = 0$ (i.e., zero impulsive controller of the given class of impulsive controllers) since the right-hand equation (4.16) holds, which is a condition of global stability of the impulse-free system. If (4.16) is replaced with

$$\nu_0(t_i) \in \left(0, \frac{b(t_i) + \sqrt{b^2(t_i) - 4a(t_i)c(t_i)}}{2a(t_i)}\right), \quad \lambda_{\min}(Q_d(t)) > \sqrt{\lambda_{\max}(Q_{0d}^T(t)Q_{0d}(t))}, \quad \forall t \in \mathbf{R}_{0+}, \tag{4.17}$$

then global asymptotic stability is guaranteed. However, assume that $\lambda_{\min}(Q_d(t)) < \sqrt{\lambda_{\max}(Q_{0d}^T(t)Q_{0d}(t))} \Rightarrow c(t_i) \leq 0$ (except possibly on a set of zero measure) implying $c(t_i) \leq 0$. Then, global stability is not guaranteed without impulsive controls since the candidate is not a Lyapunov functional. However, the choice $\nu_0(t_i) \in [0, (b(t_i) + \sqrt{(b^2(t_i) - 4a(t_i)|c(t_i)|}\ )/2a(t_i)]$ and a sufficiently small $T(t) \in \mathbf{R}_+$ containing each impulsive time instant ensuring that

$$|c(t_i)| \leq T(t) \max_{\tau\in[t,t+T(t)]}\left|\left(\lambda_{\min}(Q_d(\tau)) - \sqrt{\lambda_{\max}(Q_{0d}^T(\tau)Q_{0d}(\tau))}\|\widehat{x}(\tau)\|_2^2\right)\right| \leq \frac{b^2(t_i)}{4a(t_i)} \tag{4.18}$$

also guarantees global stability even although the impulsive-free system is not stable. If $\nu_0(t_i) \in (0, (b(t_i) + \sqrt{(b^2(t_i) - 4a(t_i)|c(t_i)|}) /2a(t_i))$ then global asymptotic stability is guaranteed provided that $\lambda_{\min}(Q_d(t)) > \sqrt{\lambda_{\max}(Q_{0d}^T(t)Q_{0d}(t))} \Rightarrow (c(t_i) > 0)$ on a connected subset of $\mathbf{R}_{0+}$ of infinite measure in order to guarantee the global asymptotic convergence to zero of the state-trajectory solution. That means that asymptotic stability is guaranteed under the last conditions for finite time intervals but, after some finite time, the conditions (4.17) are fulfilled. Note that it has not been assumed that the polytope of vertices $A_{ij}^*(t) = A_{ij}(t)$, for all $i \in \overline{q} \cup \{0\}$, for all $j \in \overline{N}$ is a stability matrix at any time. The example is very easily extendable to the case of simultaneous control under a standard control and an impulsive one so that $A_{ij}^*(t) = A_{ij}(t) + B_{ij}K_{ij}$, for all $i \in \overline{q} \cup \{0\}$, for all $j \in \overline{N}$.

Example 4.2. An automatic steering device was designed by Minorsky for the battleship New Mexico in 1962, [32]. There is a direction indicating instrument tracking the current direction of motion and there is also an instrument defining the suitable reference motion. Another problem solved by Minorsky for ships is that of the stabilization of the rolling by the activated tanks method in which ballast water is pumped from a position to another one by means of a propeller pump controlled by electronic instrumentation. The second-order delayed resulting dynamics for rolling control of the ship has the following standard form:

$$\ddot{y}(t) + \alpha\dot{y}(t) + \beta\dot{y}(t-h) + \omega_0^2 y(t) = u_0(t), \tag{4.19}$$

where the various parameters are positive, where the last left-hand side term is related to stiffness, α is the standard dumping coefficient excluding delay effects, and β is the dumping coefficient produced by pumping which has a delay when the dump becomes overworked (in not overworked normal operation points, the delay $h = 0$ and the dumping coefficient is $\alpha + \beta$). If the open-loop control action is modified using feedback to improve the original dynamics as follows:

$$u_0(t) \longrightarrow u(t) = u_0(t) + k_\alpha\dot{y}(t) + k_\beta\dot{y}(t-h) + k_\omega y(t) \tag{4.20}$$

then, the resulting closed-loop differential equation becomes,

$$\ddot{y}(t) + (\alpha - k_\alpha)\dot{y}(t) + (\beta - k_\beta)\dot{y}(t-h) + \left(\omega_0^2 - k_\omega\right)y(t) = u_0(t), \tag{4.21}$$

which can be also described in the state-space form (1) through two first-order differential equations by the state variables $x_1(t) = y(t)$, $x_2(t) = \dot{y}(t)$ as

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ k_\omega - \omega_0^2 & k_\alpha - \alpha\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & k_\beta - \beta\end{bmatrix}\begin{bmatrix}x_1(t-h)\\ x_2(t-h)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u_0(t). \tag{4.22}$$

The above system is positive if and only if $k_\omega \geq \omega_0^2$ and $k_\beta \geq \beta$ irrespective of the value of $(k_\alpha - \alpha)$ since $A_0 := \left[\begin{smallmatrix}0 & 1\\ k_\omega - \omega_0^2 & k_\alpha - \alpha\end{smallmatrix}\right]$ (the system delay-free matrix) is a Metzler matrix, the control vector $b > 0$ and the delayed matrix of dynamics $A_1 := \left[\begin{smallmatrix}0 & 0\\ 0 & k_\beta - \beta\end{smallmatrix}\right] > 0$. A complete discussion about positivity is found in [32]. The fundamental matrix of the above system is

$$\Psi(t,0) = e^{A_0 t}\left(I + \int_0^{t-h} e^{-A_0\tau}A_1\Psi(\tau - h, 0)U(t-h)d\tau\right), \tag{4.23}$$

where $U(t) = 1(t)$ is the unit step (Heaviside) function. In Minorsky's problem $u_0(t) \equiv a\sin\omega t$ which is not a positive control for all time. Now, consider the stability problem rather than the positivity one under a polytopic parameterization numbered by "1" and "2" one being stable while the other being unstable. Consider the case where switches occur between both vertices of the polytope. The polytope model is adopted to deal wit the uncertainty in the parameter $(k_\omega - \omega_0^2)$ which is known to be close zero, but its sign is unknown if, for instance, it is slightly time varying around zero.

(1) Assume that the uncontrolled parameterization 1 is stable independent of the delay under the following constraints:

$$k_{\omega 1} < \omega_{01}^2, \quad k_{\alpha 1} < \alpha_1, \quad \|A_{11}\|_2 < \frac{1}{2}\left|\lambda_{\max}\left(A_{01} + A_{01}^T\right)\right|, \tag{4.24}$$

where $A_{01} = \begin{bmatrix} 0 & 1 \\ k_{\omega 1}-\omega_{01}^2 & k_{\alpha 1}-\alpha_1 \end{bmatrix}$, $A_{11} = \begin{bmatrix} 0 & 0 \\ 0 & k_{\beta 1}-\beta_1 \end{bmatrix}$. The two first constraints ensure that A_{01} is a stability matrix while the third one ensures stability independent of the delay of the uncontrolled system or under any control guaranteeing that the modified closed-loop matrices $\overline{A}_{i1}$ $(i = 1, 2)$ satisfiy similar stability constraints.

(2) Assume that the uncontrolled parameterization 2 is unstable under the following constraints:

$$k_{\omega 2} > \omega_{02}^2, \quad k_{\alpha 2} < \alpha_2, \quad \|A_{12}\|_2 < \frac{1}{2}\left|\lambda_{\max}\left(A_{02} + A_{02}^T\right)\right|, \tag{4.25}$$

where $A_{02} = \begin{bmatrix} 0 & 1 \\ k_{\omega 2}-\omega_{02}^2 & k_{\alpha 2}-\alpha 2 \end{bmatrix}$, $A_{12} = \begin{bmatrix} 0 & 0 \\ 0 & k_{\beta 2}-\beta_2 \end{bmatrix}$. The two first constraints ensure that A_{01} is a stability matrix while the third one ensures stability independent of the delay of the uncontrolled system or under any control guaranteeing that the modified closed-loop matrices $\overline{A}_{i1}$ $(i = 1, 2)$ satisfy similar stability constraints. There are several possibilities to stabilize the system by choosing to generate impulsive controls at certain switching time instants in between parameterizations. Two of them are the following.

(1) Stabilizing Law 1 via Impulse-Free Switching between Parameterizations with Minimum Residence Time at the Stable Parameterization 1

Choose $u_0 \equiv 0$. Let Imp$\equiv \Xi := \{t_i \in \mathbf{R}_{0+}\}_{i \in \mathbf{Z}_{0+}}$ be the sequence of switching time instants in-between the parameterizations 1 and 2 and vice-versa. Prefix a designer's choice of indexing index integer $\hat{i} \in \mathbf{Z}_{0+}$ which might be sufficiently large but finite. Thus, for any $\mathbf{Z}_{0+} \ni i(\text{even}) \geq \hat{i}$ the active 2-parameterization is unstable on $[t_i, t_{i+1})$ with switching to parameterization 1 at $t = t_{i+1}$. Proceed as follows. Choose $t_{i+2} > t_{i+1}$ with sufficiently large residence time interval $T_{i+1} := t_{i+2} - t_i$ at the active stable parameterization 1 so that the subsequent stability constraint holds

$$\|\Psi(t_{i+2}, t_{i+1})\|_2 \|\Psi(t_{i+1}, t_i)\|_2 \leq \sigma(t_{i+2}, t_{i+1}) \leq 1, \tag{4.26}$$

with the prefixed real sequence $\Theta := \{\sigma(t_{i+2}, t_{i+1}) \leq 1\}_{i(\geq \hat{i}) \in \mathbf{Z}_{0+}}$ for any $t_{i+1} > t_i$. The above switching law between parameterizations generates a stable polytopic system with switches at the polytope vertices. This simple law has to direct immediate extensions. (a) The use of an impulsive-free stabilizing control law which makes the parameterization 1 stable with a greater stability degree that its associate open-loop counterpart. (b) To guarantee the stability constraint by considering strips including some finite number of consecutive switches in-between parameterizations 1-2 by guaranteeing a sufficiently large residence time at the current stable active parameterization 1. Note that if the sequence Θ has infinitely many members strictly less than one, the global exponential stability of the polytopic system with switches in between vertices is guaranteed.

(2) Stability Might Be Achieved with Impulsive Controls at Switching Time Instants for a Switching Sequence Ξ Indexed for $\mathbf{Z}_{0+} \ni i(Even) \geq \hat{i}$ as Above

Proceed as follows. (1) Choose $t_{i+2} \in \Xi$ at time instants such that $x_1(t_{i+2}) \neq 0$ for each triple of switching time instants which does not respect the stability constraint (i.e., if $\|\Psi(t_{i+2}, t_{i+1})\|_2 \|\Psi(t_{i+1}, t_i)\|_2 > 1$). (2) Define a sequence of real numbers $\{\varepsilon(t_{i+2})\}_{i(\geq \hat{i}) \in \mathbf{Z}_{0+}}$ defined by $\varepsilon(t_{i+2}) := |\varepsilon(t_{i+2})| \operatorname{sgn} x_1(t_{i+2})$ if $x_2(t_{i+2}) \neq 0$ and $\varepsilon(t_{i+2}) = 0$ if $x_2(t_{i+2}) = 0$ such that $\varepsilon(t_i)$ is zero if and only if $x_2(t_{i+2}) = 0$. (3) Generate an impulsive control $u_0(t_{i+2}) = K(t_{i+2})\delta(t - t_{i+2})$ with controller sequence $\{K(t_{i+2})\}_{i(\geq \hat{i}) \in \mathbf{Z}_{0+}}$ defined as

$$K(t_{i+2}) = \begin{cases} \dfrac{\varepsilon(t_{i+2}) \operatorname{sgn} x_1(t_{i+2}) - x_2(t_{i+2})}{x_2(t_{i+2})}, & \text{if } x_2(t_{i+2}) \neq 0, \\ -1 & \text{if } x_2(t_{i+2}) = 0, \end{cases}$$
$$x_1(t_{i+2}^+) = x_1(t_{i+2}), \quad x_2(t_{i+2}^+) = (1 + K(t_{i+2}))x_2(t_{i+2}) = \varepsilon(t_{i+2}). \tag{4.27}$$

Now, note by taking into account from the companion form of the state-space realization that $x_2(t) = \dot{x}_1(t)$, $k_{\omega 2} > \omega_{02}^2$, and $k_{\alpha 2} < \alpha_2$, it follows for $\delta > 0$ from the mean value theorem for integrals of continuous integrands that,

$$\dot{x}_1(t_{i+2}^+) = \begin{cases} x_2(t_{i+2}^+) = (1 + K(t_{i+2}))x_2(t_{i+2}) = \varepsilon(t_{i+2}) := |\varepsilon(t_{i+2})| \operatorname{sgn} x_1(t_{i+2}) & \text{if } x_2(t_{i+2}) \neq 0, \\ x_2(t_{i+2}^+) = 0 & \text{if } x_2(t_{i+2}) = 0 \end{cases} \tag{4.28}$$

Since $x_2(t)$ is Lipschitz-continuous, then for any given $\varepsilon_0 \in \mathbf{R}_+$, it exists $\delta^* = \delta^*(t_{i+2}, \varepsilon_0) \in \mathbf{R}_+$ being a monotone increasing function of the argument σ such that using the mean value theorem for integrals of continuous bounded integrands, one has for all $\sigma \in [0, \sigma^*)$; for all $\sigma^* \in \mathbf{R}_+$ as follow:

$$\begin{aligned} \dot{x}_1(t_{i+2}+\sigma) &= x_2(t_{i+2}^+ + \sigma) \in (\varepsilon(t_{i+2}) - \varepsilon_0, \varepsilon(t_{i+2}) + \varepsilon_0) \Longrightarrow |x_2(t_{i+2}+\sigma)| \leq |\varepsilon(t_{i+2})| + \varepsilon_0, \\ x_1(t_{i+2}+\sigma) &= x_1(t_{i+2}) + \int_{t_{i+2}}^{t_{i+2}+\sigma} \dot{x}_1(\tau) d\tau = x_1(t_{i+2}) + \sigma x_2(\zeta) \Longrightarrow |x_1(t_{i+2}+\sigma)| \\ &\leq (1-\sigma)(|\varepsilon(t_{i+2})| + \varepsilon_0), \end{aligned} \tag{4.29}$$

for some $\mathbf{R}_+ \ni \zeta \in (t_{i+2}, t_{i+2} + \delta)$. Thus, for sufficiently small $|\varepsilon(t_{i+2})|$ and ε_0, one has

$$\|x(t_{i+2} + \sigma)\|_2 \leq \sqrt{1 + (1 - \sigma)^2} (|\varepsilon(t_{i+2})| + \varepsilon_0), \tag{4.30}$$

so that there is a close time instant $t = t_{i+2} + \sigma$ to t_{i+2} (for a sufficiently small $\sigma \in \mathbf{R}_+$) such that $\|x(t_{i+2} + \sigma)\|_2$ is arbitrarily small by choosing a sufficiently small $|\varepsilon(t_{i+2})|$ in the sequence and a sufficiently small ε_0. Thus stabilization is achievable via impulsive controls proceeding in this way at the unstable parameterization when necessary through the above technique.

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