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Research Article

Limiting Phase Trajectories and Resonance Energy Transfer in a System of Two Coupled Oscillators

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We study a problem of energy exchange in a system of two coupled oscillators subject to 1 : 1 resonance. Our results exploit the concept of limiting phase trajectories (LPTs). The LPT, associated with full energy transfer, is, in certain sense, an alternative to nonlinear normal modes characterized by conservation of energy. We consider two benchmark examples. As a first example, we construct an LPT and examine the convergence to stationary oscillations for a Duffing oscillator subjected to resonance harmonic excitation. As a second example, we treat resonance oscillations in a system of two nonlinearly coupled oscillators. We demonstrate the reduction of the equations of motion to an equation of a single oscillator. It is shown that the most intense energy exchange and beating arise when motion of the equivalent oscillator is close to an LPT. Damped beating and the convergence to rest in a system with dissipation are demonstrated.

1. Introduction

The problem of passive irreversible transfer of mechanical energy (referred to as energy pumping) in oscillatory systems has been studied intensively over last decades; see, for example, [1, 2] for recent advances and references. In this case, the key role of transient process is evident, in contrast to great majority of conventional problems of nonlinear dynamics, in which the main attention has been given to nonlinear normal modes (NNMs), characterized by the conservation of energy. Recent studies [3, 4]; have shown that the NNM approach is effective in the case of weak energy exchange, while the concept of the limiting phase trajectories (LPT) can be used to describe intense energy exchange between weakly coupled oscillators or oscillatory chains.

The limiting phase trajectory (LPT) has been introduced [3] as a trajectory corresponding to oscillations with the most intensive energy exchange between weakly coupled oscillators or an oscillator and a source of energy; the transition from energy exchange to energy localization at one of the oscillators is associated with the disappearance of the LPT. Recently, the LPT ideas have been applied to the analysis of resonance-forced vibrations in a 1DOF and 2DOF nondissipative system [3–6]. An explicit expression of the LPT in a single oscillator [5, 6] is prohibitively difficult for practical utility. The purpose of the present paper is to show that explicit asymptotic solutions can be obtained for a class of problems that are associated with the maximum energy exchange, and introduce relevant techniques.

The paper is organized as follows. The first part is concerned with the construction of the LPT for the Duffing oscillator subject to 1 : 1 resonance harmonic excitation. In Section 2 we briefly reproduce the results of [5, 6]. We derive the averaged equations determining the slowly varying envelope and the phase of the nonstationary motion and then construct the LPT for different types of motion. Section 3 introduces the nonsmooth temporal transformations as an effective tool of the nonlinear analysis. Section 4 examines the transformations of the LPT into stationary motion in the weakly dissipated system.

In the remainder of the paper (Section 5), we analyse the dynamics of a 2DOF system. The system consists of a linear oscillator of mass M (the source of energy) coupled with a mass m (an energy sink) by a nonlinear spring with a weak linear component. Excitation is due to an initial impulse acting upon the mass M. It is shown that motion of the overall structure can be divided into two stages. The first stage is associated with the maximum energy exchange between the oscillators; here, motion is close to beating in a nondissipative system. At the second stage, trajectories of both masses in the damped system are approaching to rest. The task is to construct an explicit asymptotic solution describing both stages of motion for each oscillator. To this end, we reduce the equations of a 2DOF system to an equation of a single oscillator and then find beating oscillations characterizing the most intense energy exchange in a nondissipative system. A special attention is given to a difference between the dynamics of the Duffing oscillator and an equivalent oscillator in the presence of dissipation. While the Duffing oscillator is subjected to harmonic excitation, the 2DOF system is excited by an initial impulse applied to the linear oscillator. The response of the linear oscillator, exponentially vanishing at $t \to \infty$, stands for an external excitation for the nonlinear energy sink. We examine the transformation of beating to damped oscillations. We show that a solution of the system, linearized near the rest state, is sufficient to describe the second part of the trajectory.

2. Resonance Oscillations of the Duffing Oscillator

2.1. Main Equations

We investigate the transient response of the Duffing oscillator in the presence of resonance 1:1. The dimensionless equation of motion is

$$\frac{d^2u}{dt^2} + 2\varepsilon\gamma\frac{du}{dt} + u + 8\alpha\varepsilon u^3 = 2\varepsilon F\sin(1+\varepsilon s)t, \tag{2.1}$$

where $\varepsilon > 0$ is a small parameter. We recall that maximum energy pumping from the source of excitation into the oscillator takes place if the oscillator is initially at rest; this corresponds to the initial conditions

$$t = 0^{+}, \qquad u = 0, \qquad \frac{du}{dt} = 0. \tag{2.2}$$

An orbit satisfying conditions (2.2) is said to be *the limiting phase trajectory* [3].

In order to describe the nonlinear dynamics, we invoke a complex-valued transformation [7–9]. Introduce the variables ψ and ψ^{*}, such that

$$\begin{gathered} u = \frac{1}{2i}(\psi - \psi^{*}), \qquad v = \frac{1}{2}(\psi + \psi^{*}), \\ \psi = v + iu, \qquad \psi^{*} = v - iu, \end{gathered} \tag{2.3}$$

where $i = \sqrt{-1}$; the asterisk denotes complex conjugate. It will be shown that only one complex function is sufficient for a complete description of the dynamics. Inserting ψ, ψ^{*} from (2.3) into (2.1), a little algebra shows that (2.1) is equivalent to the following (still exact) equation of motion

$$\frac{d\psi}{dt} - i\psi + \varepsilon i\alpha\left(\psi - \psi^{*}\right)^{3} + \varepsilon\gamma\left(\psi + \psi^{*}\right) - 2\varepsilon F \sin(1 + \varepsilon s)t = 0. \tag{2.4}$$

Applying the multiple scales method [10, 11], we construct an approximate solution of (2.4) as an expansion

$$\begin{gathered} \psi(t, \varepsilon) = \psi_0(\tau_0, \tau_1) + \varepsilon\psi_1(\tau_0, \tau_1) + \cdots, \\ \frac{d}{dt} = \frac{\partial}{\partial\tau_0} + \varepsilon\frac{\partial}{\partial\tau_1}, \frac{d^2}{dt^2} = \frac{\partial^2}{\partial\tau_0^2} + 2\varepsilon\frac{\partial^2}{\partial\tau_0\partial\tau_1} + \cdots, \end{gathered} \tag{2.5}$$

where $\tau_0 = t$ and $\tau_1 = \varepsilon t$ are the fast and slow time-scales, respectively. A similar representation is valid for the function ψ^{*}. Then we substitute expressions (2.5) in (2.4) and equate the coefficients of like powers of ε. In the leading order approximation, we obtain

$$\begin{gathered} \frac{\partial\psi_0}{\partial\tau_0} - i\psi_0 = 0, \\ \psi_0(\tau_0, \tau_1) = \varphi_0(\tau_1)e^{i\tau_0} \end{gathered} \tag{2.6}$$

A slow function $\varphi_0(\tau_1)$ will be found at the next level of approximation. Equating the coefficients of order ε leads to

$$\frac{\partial\psi_1}{\partial\tau_0} + \frac{d\varphi_0}{d\tau_1}e^{i\tau_0} - \; i\psi_1 + i\alpha\left(\psi_0 - \psi_0^{*}\right)^{3} + \gamma\left(\psi_0 + \psi_0^{*}\right) + iF\left(e^{i(\tau_0 + s\tau_1)} - e^{-i(\tau_0 + s\tau_1)}\right) = 0, \tag{2.7}$$

In order to avoid the secular growth of $\varphi_1(\tau_0, \tau_1)$ in τ_0, that is, avoid a response not uniformly valid with increasing time, we eliminate resonance terms from (2.7). This yields the following equation for φ_0:

$$\frac{d\varphi_0}{d\tau_1} + \gamma\varphi_0 - 3i\alpha\left|\varphi_0\right|^2\varphi_0 = -iFe^{is\tau_1}, \quad \varphi_0(0) = 0. \tag{2.8}$$

Next we introduce the polar representation

$$\varphi_0 = ae^{i\delta}, \tag{2.9}$$

where a and δ represent a real amplitude and a real phase of the process $\varphi_0(\tau_1)$. Inserting (2.9) into (2.8) and setting separately the real and imaginary parts of the resulting equations equal to zero, (2.8) is transformed into the system

$$\begin{aligned} \frac{da}{d\tau_1} + \gamma a &= -F\sin\Delta, \\ a\frac{d\Delta}{d\tau_1} &= -sa + 3\alpha a^3 - F\cos\Delta, \end{aligned} \tag{2.10}$$

where $a > 0$, $\Delta = \delta - s\tau_1$. It now follows from (2.3), (2.9) that

$$u(t,\varepsilon) = a(\tau_1)\sin(t + \Delta(\tau_1)) + O(\varepsilon). \tag{2.11}$$

This means that the amplitude $a(\tau_1)$ and the phase $\Delta(\tau_1)$ completely determine the process $u(t,\varepsilon)$ (in the leading-order approximation). Note that $a = 0$ if the oscillator is not excited.

2.2. Critical Parameters and LPTs of the Undamped Oscillator

In this section, we recall main definitions and results concerning the dynamics of the nondissipative system. In the absence of damping, system (2.10) is rewritten as

$$\begin{aligned} \frac{da}{d\tau_1} &= -F\sin\Delta, \\ a\frac{d\Delta}{d\tau_1} &= -sa + 3\alpha a^3 - F\cos\Delta. \end{aligned} \tag{2.12}$$

It is easy to prove that system (2.12) conserves the integral of motion

$$H = \frac{3}{4}\alpha a^4 - \frac{sa^2}{2} - Fa\cos\Delta = H_0, \tag{2.13}$$

where H_0 depends on initial conditions. In the phase plane, the LPT corresponds to the contour $H = 0$, as only in this case a system trajectory goes through the point $a = 0$. Taking $H_0 = 0$, we obtain the following expression:

$$H = a\left(\frac{3}{4}\alpha a^3 - \frac{s}{2}a - F\cos\Delta\right) = 0. \tag{2.14}$$

Formula (2.14) implies that the LPT has two branches, the first branch is $a = 0$; the second branch satisfies the cubic equation

$$\frac{3}{4}\alpha a^3 - \frac{s}{2}a - F\cos\Delta = 0. \tag{2.15}$$

Equality (2.15) determines the second initial condition $a(0^+) = 0$, $\cos\Delta(0^+) = 0$. We suppose that $\mathrm{d}a/\mathrm{d}\tau 1 > 0$ at $\tau_1 = 0^+$; under this assumption, $\Delta(0^+) = -\pi/2$. Hence the initial conditions for the LPT take the form

$$\tau_1 = 0^+, a(0^+) = 0, \Delta(0^+) = -\frac{\pi}{2}. \tag{2.16}$$

Throughout this paper, we write 0 instead of 0^+, except as otherwise noted.

Next we determine critical parameters of system (2.12). The steady states of (2.12) satisfies the equations

$$\frac{da}{d\tau_1} = 0, \qquad \frac{d\Delta}{d\tau_1} = 0. \tag{2.17}$$

The second equation is equivalent to the equality

$$-sa + 3\alpha a^3 - F\,\mathrm{sgn}(\cos\Delta) = 0, \tag{2.18}$$

where $\cos\Delta = \pm 1$. We analyze the properties of (2.18) considering the properties of its discriminant D_1

$$D_1 = \frac{1}{9\alpha^2}\left(\frac{F^2}{4} - \frac{s^3}{81\alpha}\right). \tag{2.19}$$

If $D_1 < 0$, (2.18) has 3 different real roots; if $D_1 > 0$, (2.18) has a single real and two complex conjugate roots; if $D_1 = 0$, two real roots will merge; see, for example, [12]. The latter condition gives the first critical value of the parameter α

$$\alpha_1^* = \frac{4s^3}{81F^2}. \tag{2.20}$$

A straightforward investigation proves that, if $\alpha > \alpha_1^*$ (strong nonlinearity), there exists only a single stable centre C_+: $(0, a_+)$ (Figure 1); if $\alpha < \alpha_1^*$ (weak nonlinearity), there exist two

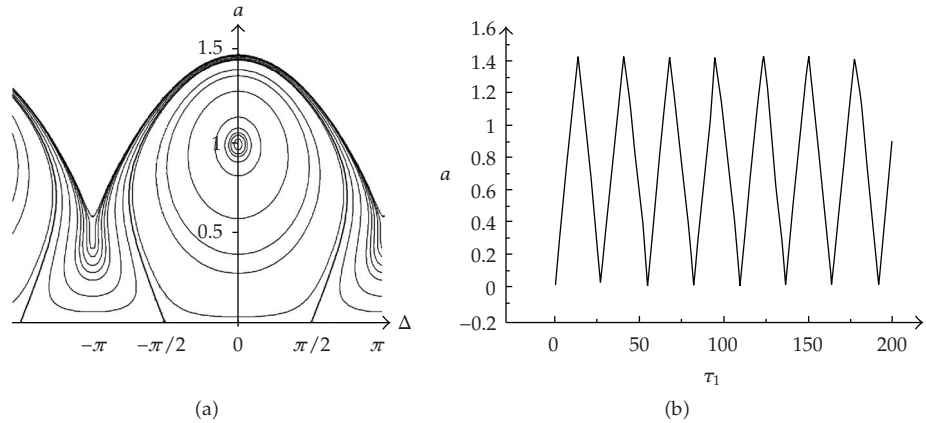


Figure 1: Phase portrait (a) and plot of $a(\tau_1)$ (b) for $s = 0.4$, $F = 0.13$, $\alpha = 0.187 = \alpha_1^*$.

stable centres C_-: $(-\pi, a_-)$, C_+: $(0, a_+)$, and an intermediate unstable hyperbolic point O: $(-\pi, a_0)$, see Figures 2 and 3.

We now suppose that $\alpha < \alpha_1^*$, that is the system may exhibit both types of oscillations, either near $\Delta = -\pi$, or near $\Delta = 0$ (Figures 2–4). In both cases, the LPT begins at $a = 0$, $\Delta = -\pi/2$ but the run of the LPT depends on the relationship between the parameters. In order to find a critical value $\alpha_2^* < \alpha_1^*$ ensuring the transition from small to large oscillations, we analyze (2.15). Consider the discriminant of (2.15)

$$D_2 = \frac{4}{\alpha^2}\left(F^2 - \frac{2s^3}{81\alpha}\right) \tag{2.21}$$

In the critical case $\alpha = \alpha_2^*$, an unstable hyperbolic point coincides with the maximum of the left branch of the LPT at $\Delta = -\pi$ (Figure 2). This means that $D_2 = 0$ at α_2^*, or

$$\alpha_2^* = \frac{2s^3}{81F^2} = \frac{\alpha_1^*}{2}, \tag{2.22}$$

which defines a boundary between small quasilinear ($\alpha < \alpha_2^*$) and large nonlinear ($\alpha > \alpha_2^*$) oscillations. In particular, for $s = 0.4$, $F = 0.13$ we obtain $\alpha_2^* = 0.0935$ (Figure 2).

Figures 3 and 4 are plotted for $\alpha < \alpha_2^*$ and $\alpha > \alpha_2^*$, respectively. In Figure 3(a), one can see the LPT encircling the center C_- of relatively small oscillations; Figure 4(a) shows the LPT encircling the centre C_+ of large oscillations; this case is associated with the maximum energy absorption. Figures 3(b) and 4(b) demonstrate the behavior of the function $a(\tau_1)$ corresponding to the respective branch of the LPT. Note that both branches of the LPT begin at the same point $a = 0$, $\Delta = -\pi/2$.

If $\alpha = \alpha_1^* = 2\alpha_2^*$, the above-mentioned coincidence of the stable and unstable points at $\Delta = -\pi$ results in the transformation of the phase portraits (Figure 1) and disappearance of the stable centre C_-. Figure 1(a) demonstrates a single stable fixed point at $\Delta = 0$.

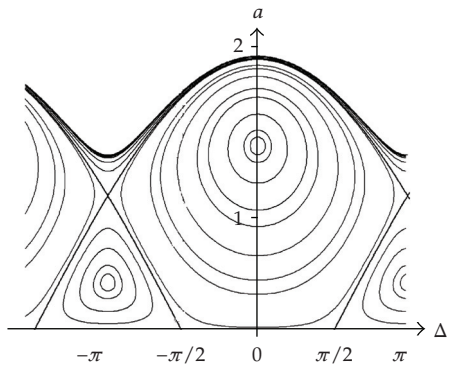


Figure 2: Passage from small to large oscillations: $\alpha = 0.0935 = \alpha_2^*$.

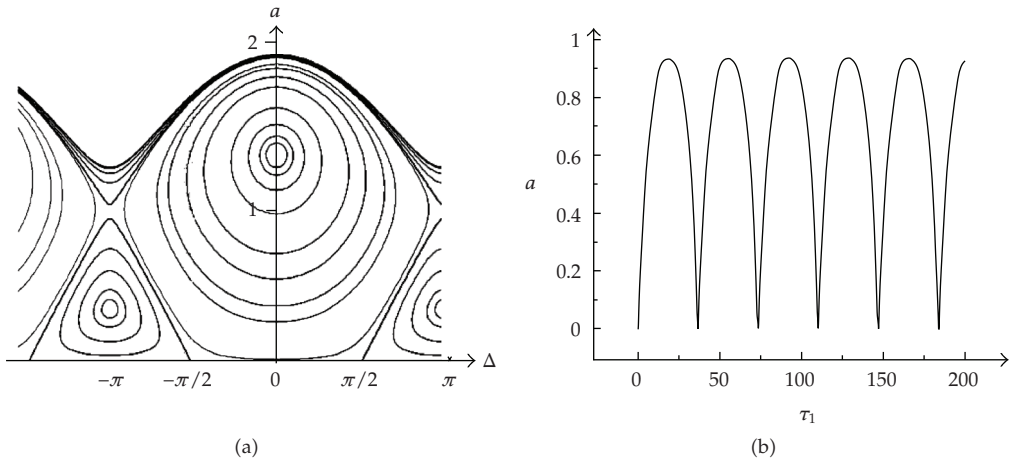


Figure 3: Phase portrait (a) and plot of $a(\tau_1)$ (b) for quasilinear oscillations: $s = 0.4$, $F = 0.13$, $\alpha = 0.093 < \alpha_2^*$.

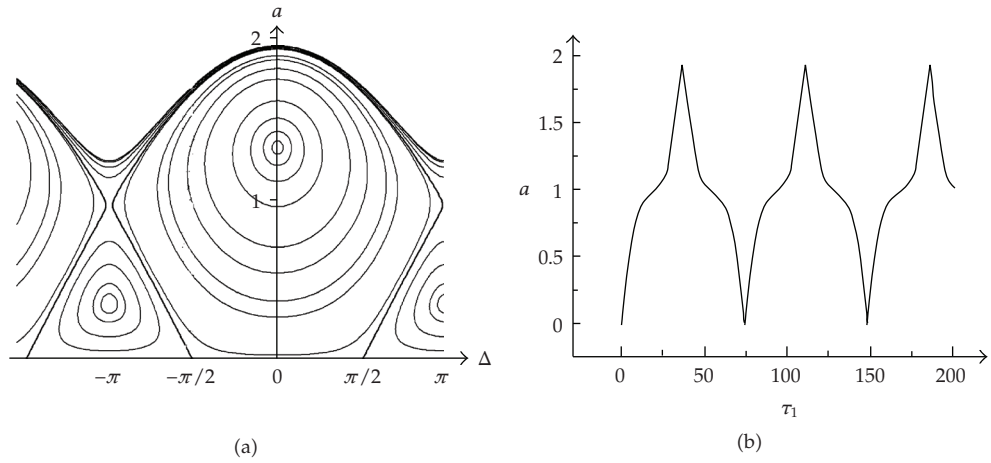


Figure 4: Phase portrait (a) and plot of $a(\tau_1)$ (b) for strongly nonlinear oscillations: $s = 0.4$, $F = 0.13$, $\alpha_2^* < \alpha = 0.094 < \alpha_1^*$.

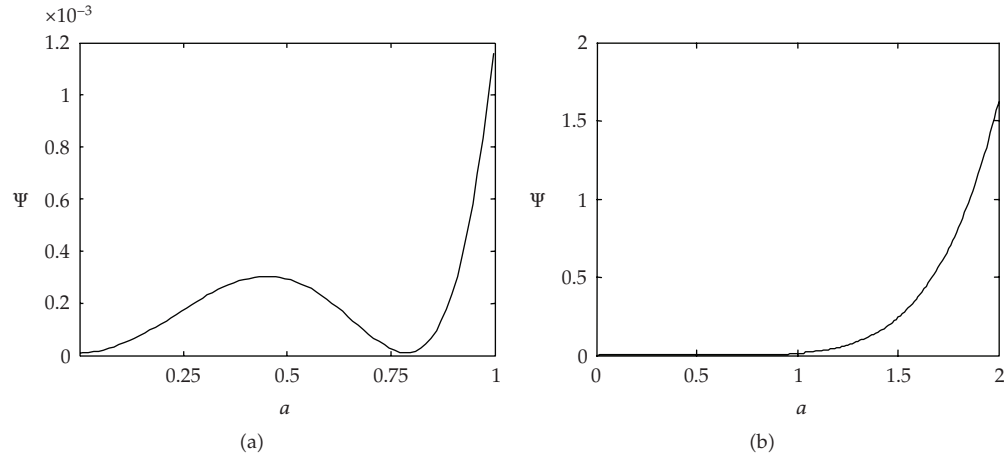


Figure 5: Potential $\Psi(a)$ in the small (a) and large (b) scales.

2.3. Reduction to the 2nd-Order Equation

For the further analysis, it is convenient to reduce the equation of the LPT to the second-order form. Using (2.15) to exclude Δ, we obtain the following equation:

$$\begin{aligned} \frac{d^2a}{d\tau_1^2} + f(a) &= 0, \\ a = 0, \qquad \frac{da}{d\tau_1} &= F \quad \text{at } \tau_1 = 0, \end{aligned} \tag{2.23}$$

where

$$\begin{aligned} f(a) &= \frac{a}{4}\left(\frac{3}{2}\alpha a^2 - s\right)\left(\frac{9}{2}\alpha a^2 - s\right) = \frac{d\Psi(a)}{da}, \\ \Psi(a) &= \frac{a^2}{8}\left(\frac{3}{2}\alpha a^2 - s\right)^2. \end{aligned} \tag{2.24}$$

Note that (2.23) can be treated as the equation of a conservative oscillator with potential $\Psi(a)$, yielding the integral of energy

$$E = \frac{1}{2}v^2 + \Psi(a) = \frac{1}{2}F^2, \quad v = \frac{da}{d\tau_1}. \tag{2.25}$$

Figure 5 depicts the potential $\Psi(a)$ in the small and large scales for system (3.1) with the parameters $\alpha = 1/3$, $s = 1/4$. The phase portrait is given in Figure 6.

The amplitude of oscillations A_0 is defined by the following equality (see Figure 6(b)):

$$E = \Psi(A_0) = \frac{1}{2}F^2. \tag{2.26}$$

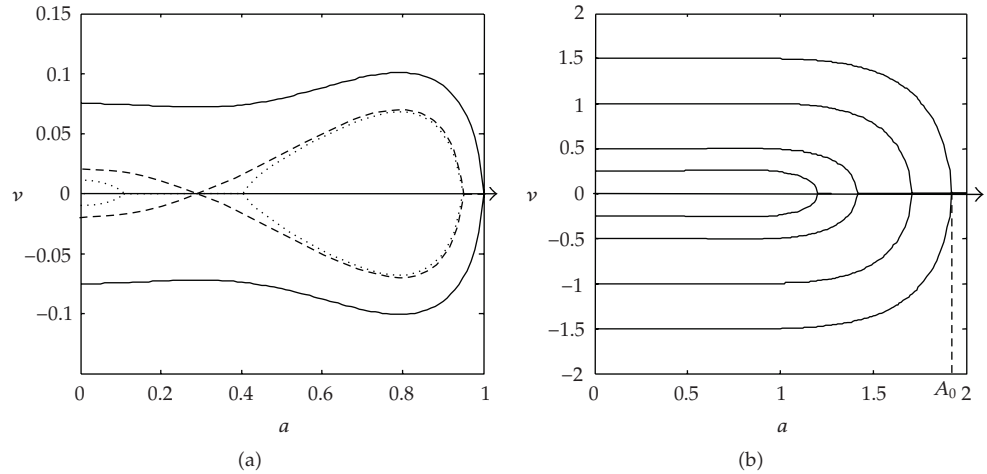


Figure 6: Phase portraits of (2.23) in the small (a) and large (b) scales.

From (2.25) we have $v = da/d\tau_1 = \pm\sqrt{F^2 - 2\Psi(a)}$. Thus, the half-period of oscillations is defined as

$$T_1(A_0) = \int_0^{A_0} \frac{da}{\sqrt{F^2 - 2\Psi(a)}}. \tag{2.27}$$

It follows from Figures 1 and 6 that a high-energy system is weakly sensitive to the shape of the potential; the orbit is close to the straight line until it reaches the wall of the potential well. This implies that motion of system (2.23) is similar to the dynamics of a particle moving with constant velocity between two motion-limiters. A connection between the smooth and vibro-impact modes of motion in a smooth nonlinear oscillator has been revealed in [13, 14]; a detailed exposition of this approach can be found in [2, Chapter 6].

The vibro-impact hypothesis suggests that the time T_1^* to reach A_0 is calculated as

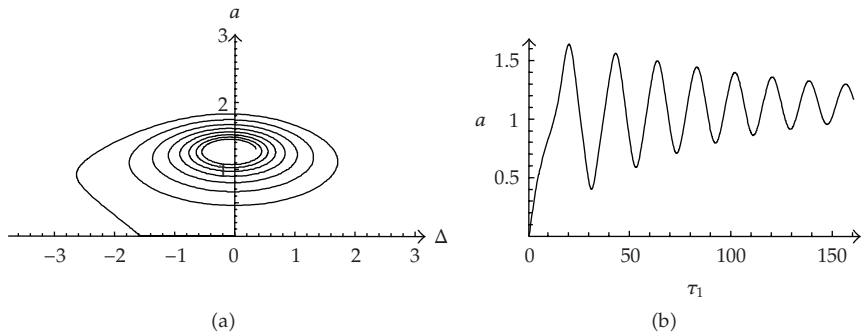
$$T_1^* = \frac{A_0}{F}. \tag{2.28}$$

Note that $T_1^* < T_1$, as the vibro-impact approximation ignores the deceleration of motion in the vicinity of A_0, when the velocity falls below the maximum level F. Formally, T_1^* can be found from (2.27) by letting $\Psi(a) = 0$.

3. Analysis of the Transient Dynamics

In what follows, we consider the dynamics of a weakly damped oscillator with strong nonlinearity ($\alpha > \alpha^*$); it is often the case of particular interest.

As seen in Figure 7, the damped system exhibits strongly nonlinear behavior on the time interval $[0, T_1^*]$; an instant T_1^* corresponds to the first maximum of the function $a(\tau_1)$. After that motion becomes similar to smooth oscillations about the stationary point. This allows separating the transient dynamics into two stages. While on the interval $0 \le \tau_1 \le T_1^*$



(a) (b)

Figure 7: Phase portrait (a) and plot of $a(\tau_1)$ (b) for strongly nonlinear oscillations in a dissipative system.

motion is close to the LPT of the undamped system, at the second stage, $\tau_1 \geq T_1^*$, motion is similar to quasilinear oscillations.

In the remainder of this section we investigate a segment of the trajectory on the interval $[0, T_1^*]$. The task is to calculate an instant T_1^* and the values $a(T_1^*)$ and $\Delta(T_1^*)$ determining the starting point for the second interval of motion. For simplicity, we assume that dissipation is sufficiently small and may be ignored on the interval $[0, T_1^*]$. This allows one to approximate the first part of the trajectory by a corresponding segment of the LPT of the nondissipative system.

As mentioned above, the dynamics of a strongly nonlinear oscillator is similar to free motion of a particle moving with constant velocity between two motion-limiters. This allows us to employ the method of nonsmooth transformations [1, 2] in the study of strongly nonlinear oscillations.

At the first step, we introduce nonsmooth functions $\tau(\phi)$ and $e(\phi) = d\tau/d\phi$ defined as follows:

$$\tau = \tau(\phi) = \frac{2}{\pi}\left|\arcsin\left(\sin\frac{\pi\phi}{2}\right)\right|,$$
$$e(\phi) = 1, \quad 0 < \phi \leq 1, \quad e(\phi) = -1, \quad 1 < \phi \leq 2, \tag{3.1}$$

where $\phi = \Omega\tau_1$, the frequency Ω will be found below. Plots of functions (3.1) are given in Figure 8. In a general setting, the solution of (2.12) is constructed in the form

$$a(\tau_1) = X_1(\tau) + e(\phi)Y_1(\tau), \qquad \Delta(\tau_1) = X_2(\tau) + e(\phi)Y_2(\tau),$$
$$\frac{d}{d\tau_1} = \Omega\left(e\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\phi}\right). \tag{3.2}$$

We recall that $\partial e/\partial\phi = \delta(\phi - n)$, where $\delta(\phi - n)$ is Dirac's delta-function, $n = 1, 2, \ldots$. We exclude δ-singularity by requiring

$$Y_{1,2} = 0 \quad \text{at } \tau = 1, 2, \ldots. \tag{3.3}$$

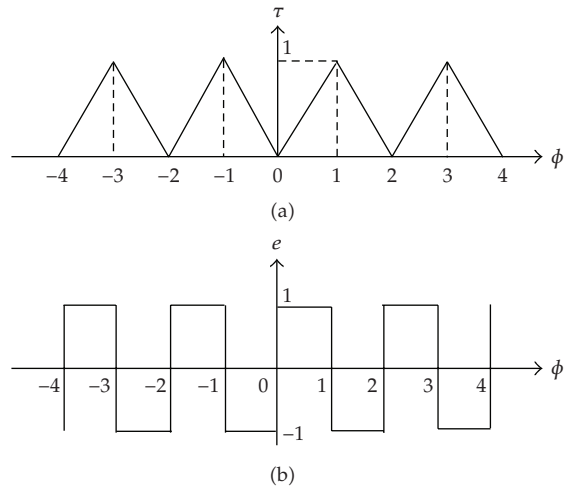


Figure 8: Functions $\tau(\phi)$ and $e(\phi)$.

This implies that

$$\begin{aligned}\frac{da}{d\tau_1} &= \Omega\left(e\frac{\partial X_1}{\partial \tau} + \frac{\partial Y_1}{\partial \tau}\right),\\ \frac{d\Delta}{d\tau_1} &= \Omega\left(e\frac{\partial X_2}{\partial \tau} + \frac{\partial Y_2}{\partial \tau}\right)\end{aligned} \tag{3.4}$$

provided $Y_{1,2} = 0$ at $\tau = 1, 2, \ldots$. To derive the equations for X_i, Y_i, $i = 1, 2$, we insert (3.4) into (2.12) and separate the terms with and without e. This yields the set of equations

$$\begin{aligned}\Omega\frac{\partial Y_1}{\partial \tau} &= -F\sin X_2\cos Y_2,\\ \Omega\frac{\partial X_1}{\partial \tau} &= -F\sin Y_2\cos X_2,\\ \Omega\left(X_1\frac{\partial Y_2}{\partial \tau} + Y_1\frac{\partial X_2}{\partial \tau}\right) + sX_1 - 3\alpha X_1^3 - 9\alpha X_1 Y_1^2 &= -F\cos Y_2\cos X_2,\\ \Omega\left(X_1\frac{\partial X_2}{\partial \tau} + Y_1\frac{\partial Y_2}{\partial \tau}\right) &= F\sin Y_2\sin X_2.\end{aligned} \tag{3.5}$$

It is easy to prove that (3.5) are satisfied by $Y_1 = 0$, $X_2 = 0$. Under these conditions, the variables X_1 and Y_2 satisfy the equations similar to (2.12)

$$\begin{aligned}\Omega\frac{\partial X_1}{\partial \tau} + F\sin Y_2 &= 0,\\ \Omega X_1\frac{\partial Y_2}{\partial \tau} + sX_1 - 3\alpha X_1^3 + F\cos Y_2 &= 0,\end{aligned} \tag{3.6}$$

with the initial conditions $X_1 = 0$, $Y_2 = -\pi/2$ at $\tau = 0$. System (3.6) is integrable, yielding the integral of motion similar to (2.14)

$$\frac{3}{4}\alpha X_1^3 - \frac{s}{2}X_1 - F\cos Y_2 = 0. \tag{3.7}$$

Then, it follows from (3.3) and (3.6) that

$$\frac{\partial X_1}{\partial \tau} = 0 \quad \text{at } Y_2 = 0,\ \tau = 1. \tag{3.8}$$

It is worth noting that equalities (3.8) have a clear physical meaning: *they represent the condition of maximum of* X_1 *at* $Y_2 = 0$.

For the further analysis, it is convenient to transfer (3.6) into the second-order form. Using (3.7) to exclude Y_2, the resulting equation and the initial conditions are written as

$$\begin{aligned} &\Omega_2 \frac{\partial^2 X_1}{\partial \tau^2} + f(X_1) = 0, \\ &X_1 = 0, \qquad \Omega \frac{dX_1}{d\tau} = F, \end{aligned} \tag{3.9}$$

where $f(X_1)$ is defined by (2.24). A precise solution of (3.9), expressed in terms of elliptic functions, is prohibitively difficult for practical utility [6, 13]. In order to highlight the substantial dynamical features, the solution to (3.6), (3.9) is expressed in terms of successive approximations

$$X_1 = x_0 + x_1 + \cdots, \qquad Y_2 = y_0 + y_1 + \cdots, \qquad \Omega = \Omega_0(1 + \varepsilon_1 + \cdots), \tag{3.10}$$

where it is assumed that $|x_1(\tau| \ll |x_0(\tau)|, |y_1(\tau| \ll |y_0(\tau)|, \varepsilon_1 \ll 1$ on an interval of interest. The validity of this assumption will be tested below by numerical simulations. Since the vibro-impact approximation is insensitive to the presence of the potential, the function x_0 is chosen as the solution of the equation

$$\frac{\partial^2 x_0}{\partial \tau^2} = 0 \tag{3.11}$$

with the initial conditions $x_0 = 0$, $\Omega_0 \partial x_0/\partial \tau = F$ at $\tau = 0$. It follows from (3.11) that

$$x_0(\tau) = A_0\tau, \qquad A_0\Omega_0 = F. \tag{3.12}$$

From (3.1) and the maximum condition we have

$$\begin{aligned} &\Omega_0 T_1^* = 1, \quad T_1^* = \Omega_0^{-1} = \frac{A_0}{F}, \\ &x_0(\tau) = F\tau_1, \quad 0 \le \tau_1 \le T_1^*, x_0\left(\tau\left(T_1^*\right)\right) = x_0(1) = A_0. \end{aligned} \tag{3.13}$$

We now recall that system (3.9) is conservative; it possesses the integral of energy

$$E_1 = \frac{1}{2}V_1^2 + \frac{1}{\Omega^2}\Psi(X_1) = \frac{1}{2\Omega^2}F^2, \quad V_1 = \frac{dX_1}{d\tau}, \tag{3.14}$$

where $\Psi(X_1)$ is defined by (2.24). By analogy with (2.26), we obtain

$$\Psi(X_1) = \frac{1}{2}F^2 \quad \text{at } \tau = 1. \tag{3.15}$$

Inserting (3.10), (3.12) into (3.15) and ignoring small terms, we then have the equation to determine A_0

$$\Psi(A_0) = \frac{1}{2}F^2. \tag{3.16}$$

Given A_0, we obtain from $T_1^* = A_0/F$ (see (2.28)).

The approximation x_1 is governed by the following equation:

$$\begin{aligned} \Omega_0^2 \frac{\partial^2 x_1}{\partial \tau^2} &= -f(x_0), \\ x_1(\tau) &= -\Omega_0^{-2}\int_0^\tau (\tau - \xi) f(A_0\xi)\mathrm{d}\xi, \end{aligned} \tag{3.17}$$

Given A_0 and Ω_0, formula (3.17) yields

$$x_1(\tau) = -\frac{A_0\tau^3}{4\Omega_0^2}\left[\frac{9\alpha^2}{56}(A_0\tau)^4 - \frac{3s\alpha}{5}(A_0\tau)^2 + \frac{s^2}{6}\right]. \tag{3.18}$$

We now find the function $Y_2(\tau)$. Arguing as above, we construct $Y_2 = y_0 + y_1$, where y_0 can be found from the first equation (3.6), in which we let $X_1 = x_0$. This yields

$$y_0 = -\arcsin\left(\frac{F}{A_0\Omega_0}\right) = -\frac{\pi}{2}, \quad 0 < \tau < 1. \tag{3.19}$$

The term $y_1(\tau)$ is defined by the second equation (3.6). As before, we take $X_1 = x_0$ and exclude $\cos Y_2$ by (3.7) to get

$$\begin{aligned} \frac{\partial y_1}{\partial \tau} &= \frac{1}{\Omega_0}\left(-\frac{s}{2} + \frac{9}{4}\alpha A_0^2\tau^2\right), \\ y_1(\tau, t_0) &= \frac{1}{\Omega_0}\left(-\frac{s\tau}{2} + \frac{3}{4}\alpha A_0^2\tau^3\right). \end{aligned} \tag{3.20}$$

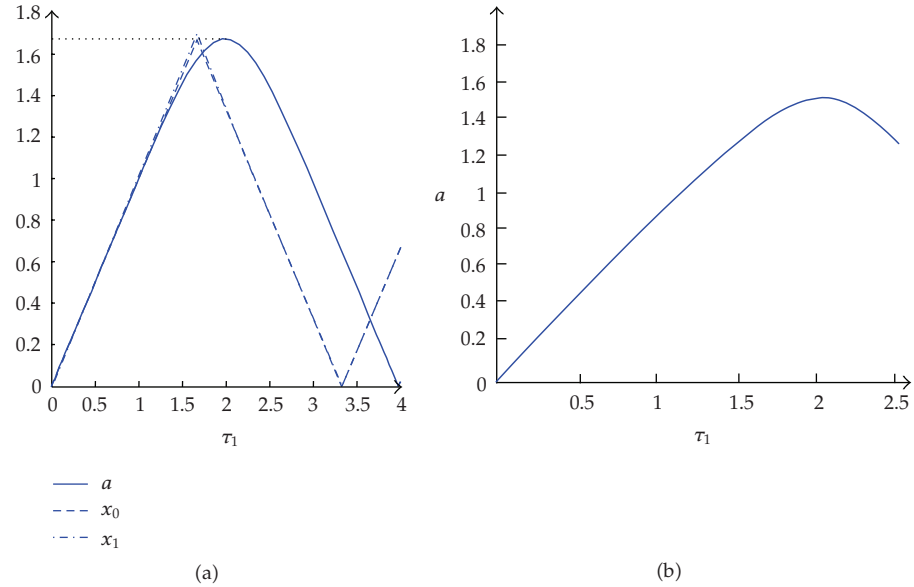


Figure 9: LPT of system (2.12) (a) and solution $a(\tau_1)$ of system (2.10) (b) solid line—numerical solution; dashed line—the leading-order approximation x_0; dot-and-dash line—the first order approximation $x_0 + x_1$.

As an example, we calculate the LPT for system (2.12) with the parameters

$$\gamma = 0, \qquad s = 0.2, \qquad \alpha = 0.333, \qquad F = 1 \tag{3.21}$$

and compare the results with the numerical solution for system (2.10), in which $\gamma = 0.04$.

Calculations by formulas (3.13), (3.16) give $A_0 = 1.67$, $T_1^* = \Omega_0^{-1} = 167$. Thus we have the maximum $M = X_1(T_1^*) \approx 1.67$ at $T_1^* \approx 1.67$ in the leading-order approximation and $M_1 \approx 1.67$ at $T_1^* \approx 2$ for the numerical solution $a(\tau_1)$ (Figure 9(a)); for system (2.10) with $\gamma = 0.05$ the numerical solution gives the maximum $M \approx 1.56$ at $T_1^* \approx 2.1$. This confirms that small dissipation can be ignored over the interval $0 \le \tau \le T^*$.

It is easy to check by a straightforward calculation that the correction x_1 is negligible. In a similar way, one can evaluate the small term y_1.

4. Quasilinear Oscillations

In this section, we examine quasilinear oscillations on the second interval of motion, $\tau > 1$. It is easy to see that an orbit of the dissipative system tends to its steady state as $\tau \to \infty$. The steady state O: (a_0, Δ_0) for system (2.10) is determined by the equality

$$a^2\left[\left(s - 3\alpha a^2\right)^2 + \gamma^2\right] = F^2, \tag{4.1}$$

or, for sufficiently small γ,

$$\begin{aligned} \gamma a_0 &= -F\sin\Delta_0, \qquad & s a_0 - 3\alpha a_0^3 &= -F\cos\Delta_0, \\ \Delta_0 &\approx -\frac{\gamma a_0}{F} + O\left(\gamma^3\right), \qquad & a_0\left(s - 3\alpha a_0^2\right) &= -F + O\left(\gamma^2\right). \end{aligned} \tag{4.2}$$

Let $\xi = a - a_0$, $\beta = \Delta - \Delta_0$ denote deviations from the steady state. In addition, we must impose the matching conditions

$$a_0 + \xi = x_0^*, \qquad \frac{d\xi}{d\tau_1} = 0 \quad \text{at } \tau_1 = T_1^*, \tag{4.3}$$

where $x_0^* = x_0(T_1^*)$, T_1^* is determined by (2.28).

We suppose that the contribution of nonlinear force in oscillations near O is relatively small. Under this assumption, one can consider the system linearized near O

$$\begin{aligned} \frac{d\xi}{d\tau_1} + F\beta &= -\gamma\xi, \\ \frac{d\beta}{d\tau_1} - \frac{k_1}{a_0}\xi &= -\gamma\beta, \end{aligned} \tag{4.4}$$

where $k_1 = 9\alpha a_0^2 - s$. If $k_1 > 0$, the solution of system (4.4) takes the form

$$\xi(z) = c_0 e^{-\gamma(\tau_1 - T_1^*)}\cos\kappa(\tau_1 - T_1^*), \qquad \beta(z) = r c_0 e^{-\gamma(\tau_1 - T_1^*)}\sin\kappa(\tau_1 - T_1^*), \quad \tau_1 - T_1^* > 0, \tag{4.5}$$

where we denote $c_0 = x_0^* - a_0$, $\kappa^2 = F k_1/a_0 > 0$, $r = \kappa/F$. In particular, taking the parameters (3.21) we find $x_0^* = 1.46$, $a_0 = 1.065$, $\Delta_0 = 0.1$, $k_1 = 3.2$, and, therefore, $c_0 = 0.395$, $\kappa = \sqrt{3}$.

Figure 10 demonstrates a good agreement between a numerical solution of (2.10) with parameters (3.21) (solid line) and an approximate solution found by matching the segment (3.12) (dot-and-dash) with the solution (4.5) of the linearized systems (dash) at the point T_1^*. Despite a certain discrepancy in the initial interval of motion, the numerical and analytic solutions approach closely as τ_1 increases. This implies that a simplified model (3.12), (3.16) matched with solution (4.5) suffices to describe a complicated near-resonance dynamics.

Arguing as above, one can obtain the solution in case $k_1 < 0$. Denoting $k_2 = F|k_1|/a_0$ and assuming $\gamma \ll k$, we find a solution similar to (4.5) with $\cosh(k(\tau_1 - T_1^*))$ and $\sinh(k(\tau_1 - T_1^*))$ in place of $\cos(\kappa(\tau_1 - T_1^*))$ and $\sin(\kappa(\tau_1 - T_1^*))$, respectively.

We now correlate numerical and analytic results. As seen in Figure 11, the first maximum of the slowly varying envelope of the process $u(t,\varepsilon)$ equals $M_1 \approx 1.5$; it is reached at the instant $t^* \approx 20$, or $T_1^* \approx 2$; the second maximum $M_2 \approx 1.4$ is at $t^* \approx 60$, $T_1^* \approx 6$, the third maximum $M_3 \approx 1.3$ is at $t^* \approx 100$, $T_1^* \approx 10$, and so forth. When these results are compared with that of Figure 10, it is apparent that the numerically constructed envelope is in a good agreement with the asymptotic approximations of the function $a(\tau_1)$.

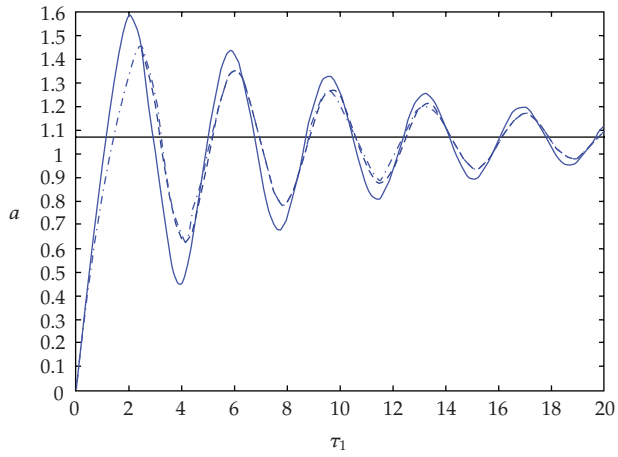


Figure 10: Transient dynamics of system (2.10): numerics (solid); segment (3.12) (dot-and-dash); solution (4.5) (dash).

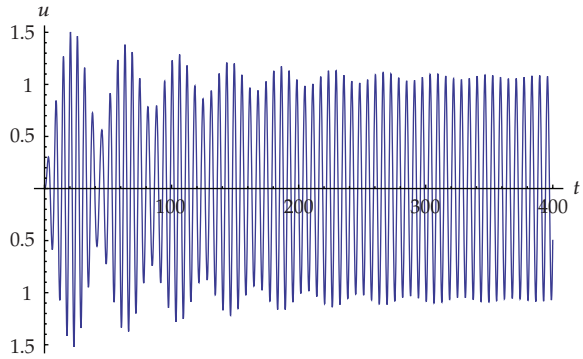


Figure 11: Numerical integration of (2.1): $\varepsilon = 0.1$, $s = 0.2$, $\alpha = 0.333$, $F = 1$, $\gamma = 0.05$.

5. Dynamics of a 2DOF System

5.1. Reduction of a 2DOF System to a Single Oscillator

In this section we present a reduction of the dynamical equations of a 2DOF system to an equation of a single oscillator. The system consists of a linear oscillator of mass M (the source of energy) coupled with a mass m (an energy sink) by a nonlinear spring with a weak linear component. For brevity, we consider the nonlinear spring with cubic nonlinearity. In this case, the equations of motion and the initial conditions have the following form:

$$\begin{aligned} M\frac{d^2x_1}{dt^2} + \tilde{\gamma}\frac{dx_1}{dt} + k_1x_1 + k_3(x_1 - x_2)^3 + D(x_1 - x_2) + \tilde{\eta}\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right) = 0, \\ m\frac{d^2x_2}{dt^2} - k_3(x_1 - x_2)^3 - D(x_1 - x_2) - \tilde{\eta}\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right) = 0, \end{aligned} \tag{5.1}$$

with the initial conditions

$$t = 0: \quad x_1 = x_2 = 0: \quad \frac{dx_1}{dt} = V_0 > 0, \quad \frac{dx_2}{dt} = 0. \tag{5.2}$$

Here x_1 and x_2 are the displacements of the masses M and m, respectively; $k_1 > 0$ is the stiffness of linear spring; $k_3 > 0$ is the coefficient of nonlinear coupling between the linear oscillator and the sink; D is the coefficient of linear coupling ($D < 0$ corresponds to a system with multiple states of equilibrium); the coefficients $\tilde{\gamma}$ and $\tilde{\eta}$ characterize dissipation in the linear oscillator and the coupling, respectively. For simplicity, we let $\tilde{\gamma} = 0$. Note that energy transfer cannot be activated in a nonexited system. In the absence of external forcing, it requires nonzero initial conditions for at least a single unit.

In what follows we assume that $m/M = \varepsilon^2 \ll 1$. Then, we introduce the dimensionless time variable $\tau_0 = \omega_0 t$, where $\omega_0 = \sqrt{k_1/M}$. In these notations, system (5.1) becomes

$$\begin{gathered}
\frac{d^2x_1}{d\tau_0^2} + x_1 + \varepsilon^2 c(x_1 - x_2)^3 + \varepsilon^3 d(x_1 - x_2) + \varepsilon^3 \eta\left(\frac{dx_1}{d\tau_0} - \frac{dx_2}{d\tau_0}\right) = 0, \\
\frac{d^2x_2}{d\tau_0^2} - c(x_1 - x_2)^3 - \varepsilon d(x_1 - x_2) - \varepsilon\eta\left(\frac{dx_1}{d\tau_0} - \frac{dx_2}{d\tau_0}\right) = 0, \\
\tau_0 = 0: \quad x_1 = x_2 = 0; \quad v_1 = \varepsilon v_0, \quad v_2 = 0,
\end{gathered} \tag{5.3}$$

where we denote

$$\varepsilon^2 c = \frac{k_3}{k_1}, \qquad \varepsilon^3 d = \frac{D}{k_1}, \qquad \varepsilon^3 \eta = \frac{\tilde{\eta}}{\sqrt{k_1 M}}, \qquad \varepsilon v_0 = \frac{V_0}{\omega_0}, \qquad vi = \frac{dx_i}{d\tau_0}. \tag{5.4}$$

In addition, we consider the relative displacement $u = x_2 - x_1$, $du/d\tau_0 = v$ satisfying the equation

$$\frac{d^2u}{d\tau_0^2} + \frac{d^2x_1}{d\tau_0^2} + \varepsilon\eta\frac{du}{d\tau_0} + cu^3 + \varepsilon du = 0, \tag{5.5}$$

with the initial conditions $\tau_0 = 0$: $u = 0$, $v = -\varepsilon\, v_0$. Using (5.3), (5.5), the variable x_1 can be excluded. We recall that oscillations in the damped system vanish at rest O_1: ($x_1 = x_2 = 0$, $v_1 = v_2 = 0$) as $\tau_0 \to \infty$. This implies that the effect of dissipation, whatever small it might be, must be considered in the approximate solution; otherwise, the convergence to O_1 is ignored. Hence the solution of the first equation (5.3) should be written as

$$\begin{gathered}
x_1(\tau_0) = \varepsilon v_0 h_\varepsilon(\tau_0)\sin\tau_0 - \varepsilon^2 c\int_0^{\tau_0} h_\varepsilon(\tau_0 - s)\sin(\tau_0 - s)u^3(s)ds + \varepsilon^3 \ldots, \\
h_\varepsilon(\tau_0) = e^{-(\varepsilon^3\eta/2)\tau_0},
\end{gathered} \tag{5.6}$$

and, by (5.6),

$$\frac{d^2x_1}{d\tau_0^2} = -\varepsilon v_0 h_\varepsilon(\tau_0)\sin\tau_0 - \varepsilon^2 c u^3(\tau_0) + \varepsilon^2 c\int_0^{\tau_0} h_\varepsilon(\tau_0 - s)\sin(\tau_0 - s)u^3(s)ds + \varepsilon^3\cdots. \tag{5.7}$$

Next, we insert (5.7) into (5.5) and ignore small terms insubstantial for the asymptotic analysis. As a result, we obtain the following equation:

$$\begin{aligned}\frac{d^2u}{d\tau_0^2} + \varepsilon\eta\frac{du}{d\tau_0} + cu^3 + \varepsilon du &= \varepsilon v_0 h_\varepsilon(\tau_0)\sin\tau_0 - \varepsilon c_\varepsilon(\tau_0),\\ c_\varepsilon(\tau_0) = \varepsilon c I_\varepsilon,\quad I_\varepsilon &= \int_0^{\tau_0} h_\varepsilon(\tau_0 - s)\sin(\tau_0 - s)u^3(s)ds,\end{aligned} \tag{5.8}$$

with the initial conditions $u(0) = 0$, $v(0) = -\varepsilon\, v_0$. We will show that, under conditions of 1:1 resonance, $I_\varepsilon(\tau_0) \sim \varepsilon^{-1}$, $c_\varepsilon(\tau_0) \sim 1$; this means that the integral term should be taken into consideration in the asymptotic analysis.

Formula (5.8) represents a nonhomogeneous integro-differential equation with respect to u, that is above transformations reduce the original 2DOF system to a single oscillator of a more complicated structure. The initial condition $v(0) = -\varepsilon\, v_0$ implies that initially the system is close to rest, and the trajectory of system approaches the LPT of (5.8). Thus the task is to construct the LPT for the integro-differential equation (5.8).

5.2. Equations of the Resonance Dynamics

To study the system subject to 1:1 resonance, we rewrite (5.8) in the form

$$\begin{aligned}\frac{du}{d\tau_0} - v &= 0,\\ \frac{dv}{d\tau_0} + (1 + 2\varepsilon\sigma)u + \varepsilon\mu\left(cu^3 - u\right) + \varepsilon\left(\eta v - v_0 h_\varepsilon(\tau_0)\sin\tau_0\right) + \varepsilon c_\varepsilon\ (\tau_0) &= 0,\\ u(0) = 0,\quad v(0) &= -\varepsilon v_0.\end{aligned} \tag{5.9}$$

In (5.9), we denote $\mu = 1/\varepsilon$, $\sigma = d/2$. The resonance conditions imply that the parenthetical expression with factor $\varepsilon\mu$ is relatively small compared to all other terms of order 1.

As in Section 2, we use complex-valued transformations (2.5) and the multiple time-scale method. Inserting (2.5) into (5.9), we then have

$$\begin{aligned}&\frac{d\psi}{d\tau_0} - i(1 + \varepsilon\sigma)\psi + \varepsilon\mu i\left[\frac{c}{8}\left(\psi - \psi^*\right)^3 + \frac{1}{2}\psi - \psi^*\right]\\ &\quad + \varepsilon\left(\frac{\eta}{2}\left(\psi + \psi^*\right) - i\sigma\psi^* - v_0 h_\varepsilon(\tau_0)\sin\tau_0\right) - \varepsilon C_\varepsilon(\tau_0) = 0,\\ &C_\varepsilon(\tau_0) = \varepsilon i\frac{c}{8}\int_0^{\tau_0} h_\varepsilon(\tau_0 - s)\sin(\tau_0 - s)\left(\psi - \psi^*\right)^3 ds\end{aligned} \tag{5.10}$$

Then we construct an approximate solution of (5.10) in terms of expansions (2.5) with the slow and fast tine scales $\tau_0 = t$, $\tau_1 = \varepsilon t$, respectively.

For the resonance effect to be considered in a proper way, the leading-order equation and its solution should be

$$\begin{aligned} \frac{\partial \psi_0}{\partial \tau_0} - i(1+\varepsilon\sigma)\psi_0 = 0, \\ \psi_0(\tau_0, \tau_1) = \varphi_0(\tau_1)e^{i(1+\varepsilon\sigma)\tau_0}. \end{aligned} \tag{5.11}$$

The function $\varphi_0(\tau_1)$ can be found from the equation

$$\begin{aligned} &\frac{\partial \psi_1}{\partial \tau_0} + \frac{d\varphi_0}{d\tau_1}e^{i(1+\varepsilon\sigma)\tau_0} - i(1+\varepsilon\sigma)\psi_1 + i\mu\left[\frac{c}{8}\left(\psi_0 - \psi_0^*\right)^3 + \frac{1}{2}\psi_0 - \psi_0^*\right] \\ &\quad + \eta\,\frac{\psi + \psi^*}{2} - i\sigma\psi^* - v_0 h_\varepsilon(\tau_0)\sin\tau_0 + C_{0\varepsilon}(\tau_0) = 0, \end{aligned} \tag{5.12}$$

where $C_{0\varepsilon}(\tau_0) = \varepsilon i(c/8)\int_0^{\tau_0} h_\varepsilon(\tau_0 - s)\sin(\tau_0 - s)(\psi_0 - \psi_0^*)^3 ds$.

To avoid secularity, we separate the resonance terms including $e^{i(1+\varepsilon\sigma)\tau_0}$ and then equate the sum to zero. First, we evaluate $C_{0\varepsilon}(\tau_0)$. To do so, we present the cubic term as $(\psi_0 - \psi_0^*)^3 = -3|\varphi_0|^2\varphi_0 e^{i(1+\varepsilon\sigma)\tau_0} + \textit{nonresonance terms}$, and then write $C_{0\varepsilon}(\tau_0)$ in the form

$$\begin{aligned} C_{0\varepsilon}(\tau_0) &= -\varepsilon\frac{3c}{16}\left|\varphi_0(\tau_1)\right|^2\varphi_0(\tau_1)\int_0^{\tau_0} e^{-(\varepsilon^3\eta/2)(\tau_0 - s)}\left(e^{i(\tau_0 - s)} - e^{-i(\tau_0 - s)}\right)e^{i(1+\varepsilon\sigma)s}ds + \cdots \\ &= -\varepsilon\frac{3c}{16}\left|\varphi_0(\tau_1)\right|^2\varphi_0(\tau_1)[S_{1\varepsilon}(\tau_0) - S_{2\varepsilon}(\tau_0)] + \cdots, \end{aligned} \tag{5.13}$$

where the nonresonance terms are omitted. Here we denote

$$\begin{aligned} S_{1\varepsilon}(\tau_0) &= \int_0^{\tau_0} e^{-(\varepsilon^3\eta/2)(\tau_0 - s)}e^{i(\tau_0 - s)}e^{i(1+\varepsilon\sigma)s}ds = \frac{e^{i(1+\varepsilon\sigma)\tau_0}}{\varepsilon\left(i\sigma + \varepsilon^2\eta\right)}\left(1 - e^{-[i\sigma + (\varepsilon^3\eta/2)]\tau_0}\right), \\ S_{2\varepsilon}(\tau_0) &= \int_0^{\tau_0} e^{-(\varepsilon^3\eta/2)(\tau_0 - s)}e^{-i(\tau_0 - s)}e^{i(1+\varepsilon\sigma)s}ds = e^{[i - (\varepsilon^3\eta/2)]\tau_0}\int_0^{\tau_0} e^{\varepsilon[i(2+\sigma) + \varepsilon^2\eta/2]s}ds \sim O(1). \end{aligned} \tag{5.14}$$

Ignoring S_2 compared with S_1, we calculate

$$C_{0\varepsilon}(\tau_0) = -i\left(\frac{3c}{16\sigma}\right)\left|\varphi_0(\tau_1)\right|^2\varphi_0(\tau_1)\left(1 - h_{1\varepsilon}(\tau_1)e^{-i\sigma\tau_1}\right)e^{i(1+\varepsilon\sigma)\tau_0} + \text{nonresonance terms}, \tag{5.15}$$

$$h_{1\varepsilon}(\tau_1) = e^{-(\varepsilon^2\eta/2)\tau_1}. \tag{5.16}$$

If we sum the resonance constituents in all other terms of (5.12) and then equate the total sum to zero, we obtain the equation

$$\begin{aligned}\frac{\partial\varphi_0}{\partial\tau_1} + i\mu\left(-\frac{3c}{8}\left|\varphi_0\right|^2\varphi_0 + \frac{1}{2}\varphi_0\right) + \frac{\eta}{2}\varphi_0 + \frac{i}{2}v_0 h_{1\varepsilon}(\tau_1)e^{-i\sigma\tau_1} \\ - i\frac{3c}{16\sigma}\left|\varphi_0(\tau_1)\right|^2\varphi_0(\tau_1)\left(1 - h_{1\varepsilon}(\tau_1)e^{-i\sigma\tau_1}\right) = 0.\end{aligned} \tag{5.17}$$

Then we insert the polar representation $\varphi_0 = ae^{i\delta}$ into (5.17) and set separately the real and imaginary parts of the resulting equations equal to zero. In these transformations, the last term in (5.17) can be omitted if $2\mu\sigma = \mu d > 1$. Under this assumption, we obtain

$$\begin{aligned}\frac{da}{d\tau_1} &= -\gamma a - h_{1\varepsilon}(\tau_1)F\sin\Delta, \\ a\frac{d\Delta}{d\tau_1} &= \mu a\left(-s + \alpha a^2\right) - h_{1\varepsilon}(\tau_1)F\cos\Delta,\end{aligned} \tag{5.18}$$

where $\Delta = \delta + \sigma\,\tau_1$, and $\sigma = d/2$, $\gamma = \eta/2$, $\alpha = 3c/8$, $s = (1/2)(1 - \varepsilon d)$, $F = v_0/2$.

By analogy with (2.16), we accept the initial conditions

$$a(0) = 0, \qquad \Delta(0) = -\frac{\pi}{2}. \tag{5.19}$$

In the absence of dissipation ($\eta = 0$), system (5.18) takes the form

$$\begin{aligned}\frac{da}{d\tau_1} &= -F\sin\Delta, \\ a\frac{d\Delta}{d\tau_1} &= \mu a\left(-s + \alpha a^2\right) - F\cos\Delta,\end{aligned} \tag{5.20}$$

which is very similar to (2.10). Critical parameters, stable centers, and the LPT for (5.20) are derived in the same way as in Section 3. However, as mentioned above, steady state positions of dissipated systems (2.10) and system (5.18) are different. This reflects the fact that, while (2.10) is subjected to persistent harmonic excitation, the effect of an initial impulse exponentially decreases with time. Therefore, the first segment of the trajectory (5.18) can be approximated by a corresponding solution of (5.20) but the second segments convergences to zero as $\tau_1 = \infty$.

5.3. Dynamical Analysis of the Oscillator

5.3.1. Beating in a Nondissipative System

We now compare analytic and numerical results. We recall that numerical and experimental studies [3, 4] have shown the effective energy exchange for sets of parameters. Following [3, 4], we choose $\varepsilon = 0.316$ ($\varepsilon^2 = 0.1$), $c = 0.8$, $d = 0.632$ ($\varepsilon d = 0.2$), $v_0 = 2.215$ ($\varepsilon v_0 = 0.7$). Note that in this case $\mu d = 2$, and integral terms in (5.17) can be ignored.

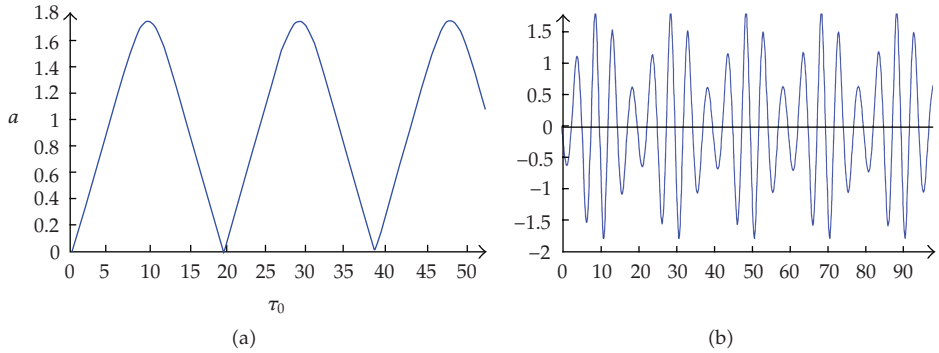


Figure 12: LPT of system (5.20) (a) and beating solution $u(\tau_0)$ of (5.8).

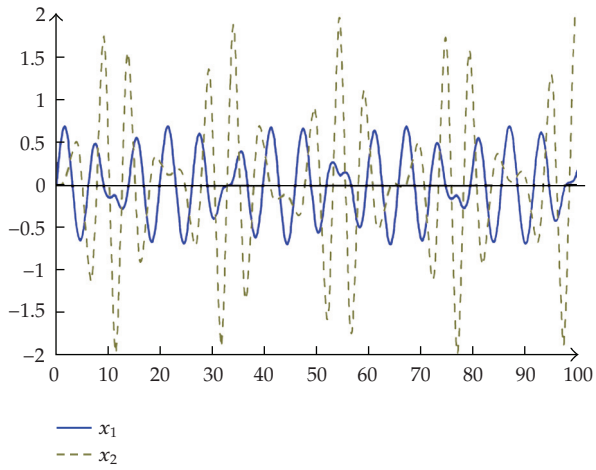


Figure 13: Plots of $x_1(\tau_0)$ (solid) and $x_2(\tau_0)$ (dash).

The LPT (Figure 12(a)) is calculated as a solution of (5.20) with initial conditions $a(0) = 0$, $\Delta(0) = -\pi/2$. One can see that the LPT approximates the envelope of the process $u_2(\tau_0)$ (Figure 13) with an error about 10%. Thus we may conclude that the observed intense energy exchange is associated with motion over the LPT.

We now calculate the critical parameter α_1^*. Here we have $\mu^2 = 10$, $F = 1.1075$, $\varepsilon d = 0.2$, $s = 0.4$, and, by (2.20)

$$\alpha_1^* = \frac{3}{8}, \qquad c_1^* = \frac{4\mu^2 s^3}{27F^2}, \qquad c_1^* = \frac{32\mu^2 s^3}{81F^2} = 0.205. \tag{5.21}$$

Since the accepted value c $0.8 > c_1^* =$, the resonance exchange takes place in the domain of large oscillations demonstrated in Figure 1.

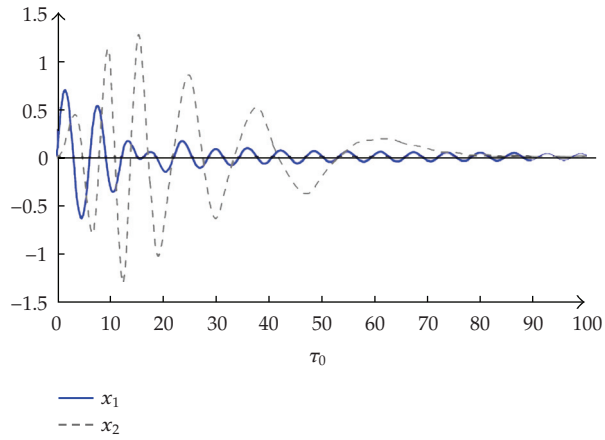


Figure 14: Plots of $x_1(t)$ (solid) and $x_2(t)$ (dash) in the damped system.

5.3.2. The Transient Dynamics

Here we consider the resonance dynamics of a weakly dissipated system. We recall that, if $\tau_1 \to \infty$, the transient process in system (5.18) does not turn into a stationary process; it vanishes at rest O: $\{a = 0\}$. To describe transient oscillations, we use the same arguments as in Section 3. The trajectory is separated into two parts; motion is assumed to be close to strongly nonlinear undamped oscillations over the interval $[0, \tau_1^*]$; then the orbit approaches the point O with an exponentially decreasing amplitude of oscillations.

We note that the contribution of nonlinear force in oscillations near O is negligible. In particular, this implies that the resonance construction of Section 5 is incorrect if τ_1 is large enough. In order to present the linearised dynamics, one can consider system (5.3) linearized near the point O_1: $(x_1 = x_2 = 0, v_1 = v_2 = 0)$. Let ξ_1 and ξ_2 denote small deviations of x_1, x_2 from O_1. A corresponding linearized system takes the form

$$\begin{aligned} \frac{d^2\xi_1}{d\tau_0^2} + \xi_1 + \varepsilon^3\left[d(\xi_1 - \xi_2) + \eta\left(\frac{d\xi_1}{dt} - \frac{d\xi_2}{dt}\right)\right] = 0, \\ \frac{d^2\xi_2}{d\tau_0^2} - \varepsilon d(\xi_1 - \xi_2) - \varepsilon\eta\left(\frac{d\xi_1}{dt} - \frac{d\xi_2}{dt}\right) = 0. \end{aligned} \tag{5.22}$$

In addition, we impose the matching constraints at $\tau_0 = \tau_0^* = \tau_1^*/\varepsilon$, namely,

$$\xi_i = x_i, \qquad \frac{d\xi_i}{d\tau_0} = \frac{dx_i}{d\tau_0}, \tag{5.23}$$

where x_i denotes the corresponding solution over $[0, \tau_0^*]$.

Formally, one can reproduce above transformations and reduce system (5.22) to a single integro-differential equation. However, a system of low dimensionality enables a straightforward investigation.

By definition, $x_1(\tau_0^*) \approx 0$ but $|x_2(\tau_0^*)|$ is close to maximum, that is, $dx_2/d\tau_0 = d\xi_2/d\tau_0 \approx 0$ at τ_0^*. The solution can be presented in the form

$$\begin{aligned} \xi_1(\tau_0) &= \varepsilon v_0 e^{-\varepsilon^3 \eta(\tau_0 - \tau_0^*)/2} \sin(\tau_0 - \tau_0^*), \\ \xi_2(\tau_0) &= C_0 e^{-\varepsilon\eta(\tau_0 - \tau_0^*)/2} \cos\left[\sqrt{\varepsilon d}(\tau_0 - \tau_0^*)\right] + \varepsilon \ldots, \end{aligned} \tag{5.24}$$

where $\xi_1(\tau_0)$ is defined by (5.6) with the negligible integral term (see discussion after (5.17)); $\xi_2(\tau_0)$ is obtained as a solution of the second equation in (5.22) provided $\xi_1(\tau_0) \sim O(\varepsilon)$. It is seen from (5.24) that the amplitude of $\xi_1(\tau_0)$ is much lesser than the amplitude of $\xi_2(\tau_0)$ but the period and the rate of decay for $\xi_2(\tau_0)$ vastly exceed similar parameters for $\xi_1(\tau_0)$). Omitting detailed calculation, we note that the results of numerical simulation (Figure 14) agree with the conclusions following from (5.23), (5.24). The parameters of simulations are $\varepsilon^2 = 0.1$, $c = 0.8$, $\varepsilon d = 0.2$, $\varepsilon\eta = 0.2$, $\varepsilon v_0 = 0.7$.

6. Conclusion

In this paper, we have extended the concept of the limiting phase trajectories (LPTs) to dissipative oscillatory systems. Using this concept, we have constructed an approximate solution describing the maximum energy exchange between coupled oscillators. The solution consists of two parts: on an initial interval, the trajectory is close to the LPT of the undamped system; then motion becomes similar to quasilinear oscillations, approaching an asymptotically stable state. We have demonstrated a good agreement between numerical and approximate analytical solutions for two typical examples, namely, the Duffing oscillator with harmonic excitation and a system of two coupled oscillators excited by an initial impulse. In addition, we have developed a procedure for reducing a 2DOF system to a single oscillator. This allows us to obtain an approximate analytic solution describing the energy exchange and beating with complete energy transfer in a 2DOF system.

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