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Research Article

Thermal Characterizations of Exponential Fin Systems

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Exponential fins are mathematically analyzed in this paper. Two types are considered: (i) straight exponential fins and (ii) pin exponential fins. The possibility of having increasing or decreasing cross-sectional areas is considered. Different thermal performance indicators are derived. The maximum ratio between the thermal efficiency of the exponential straight fin to that of the rectangular fin is found to be 1.58 at an effective thermal length of 2.0. This ratio is even larger when exponential fins are compared with triangular and parabolic straight fins. Moreover, the maximum ratio between the thermal efficiency of the exponential pin fin to that of the rectangular pin fin is found to be 1.17 at an effective thermal length of 1.5. However, exponential pin fins thermal efficiencies are found to be lower than those of triangular and parabolic pin fins. Moreover, exponential joint-fins may transfer more heat than rectangular joint-fins especially when differences between their senders and receivers portions dimensionless indices are very large. Finally, it is found that increasing the joint-fin exponential index may cause straight exponential joint-fins to transfer more heat than rectangular joint-fins.

1. Introduction

Enhancing heat transfer between solids and the adjoining fluids is one of the most important objectives in thermal engineering. Therefore, many methods were proposed to achieve this goal. Bergles [1, 2] classified these methods to active and passive methods. Active methods are those requiring external power to maintain their enhancement such as well stirring the fluid or vibrating the solid surface [3, 4]. On the other hand, the passive methods do not require external power to maintain the enhancement effect as when fins are utilized. Fins are widely used in industry, especially in heat exchanger and refrigeration industries [5–10]. Moreover, fins are used in cooling of large heat flux electronic devices as well as in cooling of gas turbine blades [11].

According to design aspects, fins can have simple designs such as rectangular, triangular, parabolic, hyperbolic, annular, and pin fins [9]. Complicated designs of fins such as spiral fins have been utilized [12, 13]. In addition, fins can be arranged uniformly on the solid surface [10]. In contrast, they can be arranged on the solid surface in complex networks as can be seen in the works of Alebrahim and Bejan [14], Almogbel and Bejan [15] and Khaled [16]. Moreover, fins can be further classified based on the number of the adjoining fluids interacting with their surfaces. Examples of works including fins surrounded by more than one adjoining fluid can be found in the works of Khaled [17, 18]. In addition, fins are usually attached to solid surfaces [5–13] but they may have roots in the solid walls [19]. To the best knowledge of the author, thermal characterization of exponential fin systems received almost negligible attention in the literature. Perhaps, this is due to the difficulty associated with manufacturing them in the past. However, the recent advancements in manufacturing technologies, which led to accurate micro- and nanosystems fabrications, may increase the opportunities of these passive systems to be implemented in industry.

In this paper, fins with exponentially varying cross-sectional areas are modeled and mathematically analyzed. Two types were considered: (i) exponential straight fins and (ii) exponential pin fins. The appropriate energy equations are solved, and the temperature distributions are found. Accordingly, different thermal performance indicators are calculated. The analysis is expanded to account for exponential joint-fins. Extensive parametric study is performed for the various controlling parameters in order to evaluate these kinds of systems.

2. Problem Formulation

It should be mentioned before starting the analysis that the following assumptions are considered:

(i) one-dimensional heat transfer analysis,

(ii) conduction and convection heat transfer rates being governed by the Fourier law and the Newtons law of cooling, respectively,

(iii) having, for exponential pin fins, $(dr/dx)^2 \ll 1.0$,

(iv) uniform heat transfer coefficient between the fin and the fluid stream.

2.1. Straight Fins with Exponentially Varying Widths

Consider a rectangular fin having a uniform thickness t that is much smaller than its width $H(x)$ and length L as shown in Figure 1. The fin width varies along the fin centerline axis (x-axis) according to the following relationship:

$$\overline{H}(\overline{x}) \equiv \frac{H(x/L)}{H_b} = e^{-bL\overline{x}}, \tag{2.1}$$

where $\overline{x} = x/L$ and b is a real number named as the exponential index. The quantity H_b represents the fin half-width at its base ($x = 0$). Note that, when $b > 0$, the analysis corresponds to the right portion of the joint-fin shown in Figure 1 while it corresponds to the left fin portion of the joint-fin when $b < 0$.

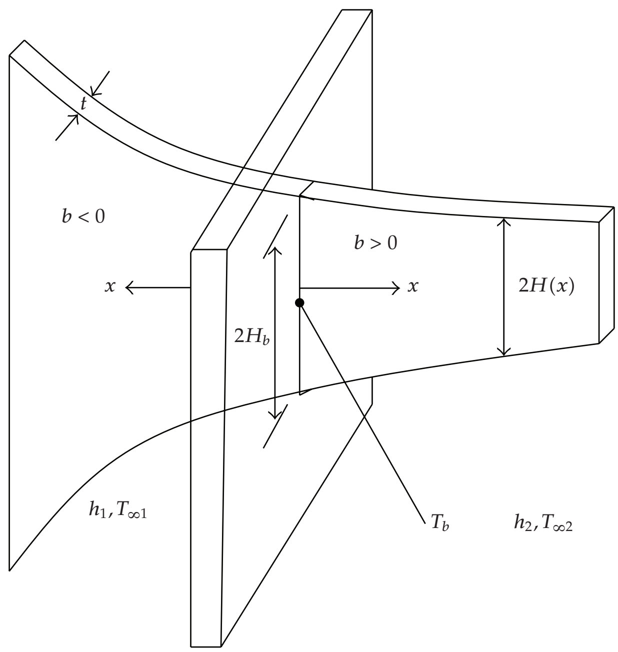

Figure 1: Schematic diagram for a straight exponential fin and exponential joint-fin and the system coordinates.

The application of the energy equation [20] on a fin differential element results in the following differential equation:

$$\frac{d}{dx}\left(A_c\frac{dT}{dx}\right)-\frac{h}{k}\left(\frac{dA_s}{dx}\right)(T-T_\infty)=0, \tag{2.2}$$

where T, T_∞, k, and h are the fin temperature, free stream temperature, fin thermal conductivity, and the convection heat transfer coefficient between the fin and the fluid stream, respectively. The quantitiesA_c and A_s are the cross-sectional and the surface areas of the fin, respectively. Equation (2.2) has the following dimensionless form:

$$\frac{d}{d\overline{x}}\left(\overline{H}\frac{d\theta}{d\overline{x}}\right)-(mL)^2\overline{H}\theta=0, \tag{2.3}$$

where $m=\sqrt{2h/(kt)}$ and $\theta=(T(x)-T_\infty)/(T_b-T_\infty)$. The quantity m is called the fin index while T_b is the fin temperature at its base. Equation (2.3) prescribes the following general solution:

$$\theta(\overline{x})=C_1e^{s_1\overline{x}}+C_2{}^{s_2\overline{x}}, \tag{2.4}$$

where s_1 and s_2 are equal to

$$s_{1,2} = \frac{bL}{2}\left[1 \mp \sqrt{4X^2+1}\right] \quad \text{if } b>0,$$
$$s_{1,2} = \frac{bL}{2}\left[1 \pm \sqrt{4X^2+1}\right] \quad \text{if } b<0, \tag{2.5}$$

where $X = m/b$. The quantity X is named as the dimensionless exponential fin parameter. It represents the ratio of the fin index, m, to the exponential index, b. When $X \ll 1.0$, cross section gradients near the base are expected to be larger than the nearby temperature gradients. The opposite scenario occurs when $X \gg 1$. The boundary conditions for an adiabatic fin tip are given by

$$\theta(\overline{x}=0) = 1.0, \qquad \left.\frac{\partial\theta}{\partial\overline{x}}\right|_{\overline{x}=1} = 0.0. \tag{2.6}$$

As such, the dimensionless temperature distribution has the following form:

$$\theta(\overline{x}) = \frac{e^{s_1\overline{x}} - [(s_1e^{s_1})/(s_2e^{s_2})]e^{s_2\overline{x}}}{1-(s_1e^{s_1})/(s_2e^{s_2})}. \tag{2.7}$$

The rate of heat transfer through the fin is called the fin heat transfer rate. For this case, it is equal to

$$q_f = -kA_c\left.\frac{dT}{dx}\right|_{x=0} = \begin{cases} H_btkb\left\{-1+\sqrt{4X^2+1}\right\}(T_b-T_\infty)\Phi_1(s_1,s_2); & b>0, \\ -H_btkb\left\{1+\sqrt{4X^2+1}\right\}(T_b-T_\infty)\Phi_2(s_1,s_2); & b<0, \end{cases} \tag{2.8}$$

where Φ_1 and Φ_2 factors are smaller than unity. They are equal to

$$\Phi_1(s_1,s_2) = \frac{1-\operatorname{Exp}\left(-bL\sqrt{4X^2+1}\right)}{1-\left(\left(1-\sqrt{4X^2+1}\right)/\left(1+\sqrt{4X^2+1}\right)\right)\operatorname{Exp}\left(-bL\sqrt{4X^2+1}\right)}, \tag{2.9}$$

$$\Phi_2(s_1,s_2) = \frac{1-\operatorname{Exp}\left(bL\sqrt{4X^2+1}\right)}{1-\left(\left(1+\sqrt{4X^2+1}\right)/\left(1-\sqrt{4X^2+1}\right)\right)\operatorname{Exp}\left(bL\sqrt{4X^2+1}\right)}. \tag{2.10}$$

Utilizing (2.9) and (2.10), the fin lengths $L = (L_\infty)_1$ and $L = (L_\infty)_2$ that make Φ_1 and Φ_2 equal to 0.99, respectively, can be approximated by

$$\begin{aligned} m(L_\infty)_1 &\cong \frac{X}{\sqrt{4X^2+1}}\left(5.293 - \ln\left[1 + \frac{1}{\sqrt{4X^2+1}}\right]\right), \quad b > 0, \\ m(L_\infty)_2 &\cong \frac{-X}{\sqrt{4X^2+1}}\left(5.293 - \ln\left[1 - \frac{1}{\sqrt{4X^2+1}}\right]\right), \quad b < 0, \end{aligned} \tag{2.11}$$

where quantity mL_∞ is called the effective thermal length. This is because the fin material exists after $x = L_\infty$ encounters negligible heat transfer rates and should be removed. The fin thermal efficiency η_f is defined as the fin heat transfer rate divided by the fin heat transfer rate if the fin temperature is kept at T_b. For this case, it can have the following forms:

$$\eta_f \equiv \frac{0.99(q_f)_{\max}}{4h(T_b - T_\infty)\int_0^{L_\infty} H dx} = \begin{cases} \dfrac{0.99\left\{-1 + \sqrt{4X^2+1}\right\}}{2X^2\left[1 - \operatorname{Exp}(-b(L_\infty)_1)\right]}, & b > 0, \\[2ex] \dfrac{0.99\left\{1 + \sqrt{4X^2+1}\right\}}{2X^2\left[\operatorname{Exp}(-b(L_\infty)_2) - 1\right]}, & b < 0, \end{cases} \tag{2.12}$$

where $(q_f)_{\max}$ is the fin heat transfer rate when $L \gg L_\infty$. Now, define the fin performance indicator γ as the ratio of the fin heat transfer rate when $L > L_\infty$ to the fin heat transfer rate for a rectangular fin having a uniform width of $2H_b$, uniform thickness t and an infinite length. As such, γ is equal to

$$\gamma \equiv \frac{(q_f)_{\max}}{(q_f)\big|_{H=H_b}} = \frac{(q_f)_{\max}}{2H_b\sqrt{2hkt}(T_b - T_\infty)} = \begin{cases} \left(\dfrac{-1 + \sqrt{4X^2+1}}{2X}\right), & b > 0, \\[2ex] \left(\dfrac{1 + \sqrt{4X^2+1}}{2X}\right), & b < 0. \end{cases} \tag{2.13}$$

2.2. Pin Fins with Exponentially Varying Radii

Consider a pin fin of radius $r(x)$, as shown in Figure 2, that varies exponentially along the x-axis according to the following relationship:

$$\bar{r}(x) \equiv \frac{r(x)}{r_b} = e^{-bx}, \tag{2.14}$$

where $b > 0$. As such, (2.2) changes to:

$$\frac{d}{dx}\left(e^{-2bx}\frac{d\theta}{dx}\right) - m^2 e^{-bx}\theta = 0, \tag{2.15}$$

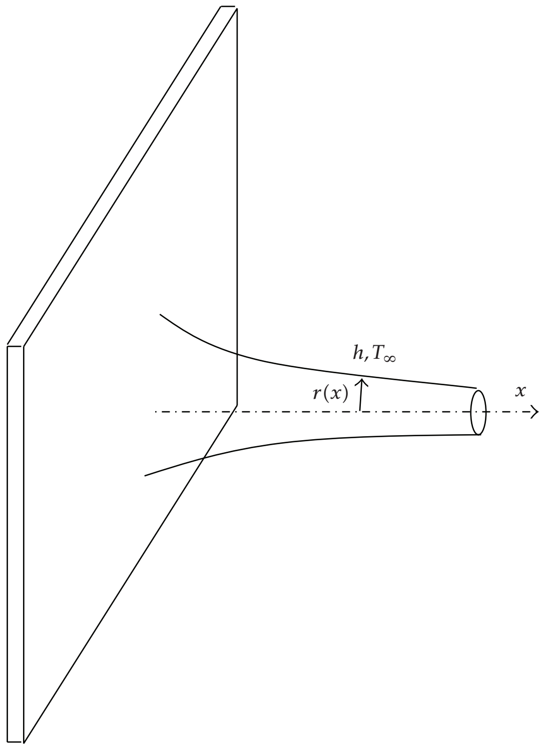

Figure 2: Schematic diagram for an exponential pin fin with $b < 0$ and the system coordinate.

where $m = \sqrt{2h/(kr_b)}$. It can be shown that the general solution of (2.15) is

$$\theta(x) = e^{mx/X}\left\{C_1 I_2\left(2Xe^{mx/(2X)}\right) + C_2 K_2\left(2Xe^{mx/(2X)}\right)\right\}, \tag{2.16}$$

where $X = m/b$. As such, the fin heat transfer rate is

$$\begin{aligned} q_{\mathrm{f}} = -k\pi r_b^2 (T_b - T_\infty) b \{ & C_1 [I_2(2X) + 0.5X(I_1(2X) + I_3(2X))] \\ & + C_2 [K_2(2X) - 0.5X(K_1(2X) + K_3(2X))]\}. \end{aligned} \tag{2.17}$$

For a fin with an infinite length ($L \to \infty$), the constants C_1 and C_2 are given by

$$C_1 = 0, \quad C_2 = \frac{1}{K_2(2X)}. \tag{2.18}$$

This is because $e^{mx/(2X)}$ approaches infinity as x approaches infinity; hence $I_2(2Xe^{mx/(2X)})$ approaches infinity. Thus, C_1 is equal to zero.

For adiabatic fin tips, boundary conditions given by (2.6) should be satisfied. Accordingly, the constants C_1 and C_2 are equal to

$$C_1 = (I_2(2X) - K_2(2X)C_3)^{-1},$$
$$C_3 = \frac{I_2\left(2Xe^{mL/(2X)}\right) + 0.5Xe^{mL/(2X)}\left\{I_1\left(2Xe^{mL/(2X)}\right) + I_3\left(2Xe^{mL/(2X)}\right)\right\}}{K_2\left(2Xe^{mL/(2X)}\right) - 0.5Xe^{mL/(2X)}\left\{K_1\left(2Xe^{mL/(2X)}\right) + K_3\left(2Xe^{mL/(2X)}\right)\right\}}, \tag{2.19}$$
$$C_2 = -C_1C_3.$$

The fin efficiency η_f for $b > 0$ can be found to be equal to

$$\eta_f \equiv \frac{0.99(q_f)_{\max}}{2\pi r_b h(T_b - T_\infty)\int_0^{L_\infty}\overline{r}(x)dx}$$
$$= \frac{0.99}{(1 - e^{-mL_\infty/X})}\left(\frac{1}{X^2}\right)\left\{\frac{X}{2}\left[\frac{K_1(2X)}{K_2(2X)} + \frac{K_3(2X)}{K_2(2X)}\right] - 1\right\}, \quad b > 0, \tag{2.20}$$

where mL_∞ is obtained from the solution of the following equation:

$$\frac{(q_f)_{\text{Ins. Tip}}}{(q_f)_{L\to\infty}} = \phi\left(X, \frac{mL_\infty}{X}\right) = 0.99. \tag{2.21}$$

The fin performance indicator γ for this case is defined as the ratio of the fin heat transfer rate when $L \gg L_\infty$ to that of a rectangular pin fin having a uniform radius of r_b and an infinite length. It is equal to the following:

$$\gamma_1 \equiv \frac{(q_f)_{\max}}{(q_f)\big|_{r=r_b}} = \frac{(q_f)_{\max}}{\pi r_b\sqrt{2hkr_b}(T_b - T_\infty)}$$
$$= \left(\frac{1}{X}\right)\left\{\frac{X}{2}\left[\frac{K_1(2X)}{K_2(2X)} + \frac{K_3(2X)}{K_2(2X)}\right] - 1\right\}, \quad b > 0. \tag{2.22}$$

For cases when $b < 0$; X is replaced with $-X$, and the constants C_1 and C_2 for a fin with infinite length are replaced by

$$C_1 = \frac{1}{I_2(-2X)}, \quad C_2 = 0, \quad b < 0. \tag{2.23}$$

As such, the fin thermal efficiency η_f and the indicator γ when $b < 0$ change to

$$\eta_f = \frac{0.99}{(e^{-mL_\infty/X} - 1)}\left(\frac{1}{X^2}\right)\left\{\frac{-X}{2}\left[\frac{I_1(-2X)}{I_2(-2X)} + \frac{I_3(-2X)}{I_2(-2X)}\right] + 1\right\}, \quad b < 0,$$
$$\gamma_2 \equiv \frac{q_f}{(q_f)\big|_{r=r_b}} = \left(\frac{-1}{X}\right)\left\{\frac{-X}{2}\left[\frac{I_1(-2X)}{I_2(-2X)} + \frac{I_3(-2X)}{I_2(-2X)}\right] + 1\right\}, \quad b < 0. \tag{2.24}$$

2.3. Pin Fins with Exponentially Decaying Temperature Distribution

Consider a pin fin having a given fin temperature distribution that varies exponentially with x according to the following relationship:

$$\theta(\overline{x}) \equiv \frac{T(x) - T_\infty}{T_b - T_\infty} = e^{-ax_o\overline{x}}, \tag{2.25}$$

where $\overline{x} = x/x_o$ and a is the exponential index. The dimensionless form of the energy equation has the following form

$$\frac{d}{d\overline{x}}\left(\overline{r}^2(x)\frac{d\theta}{d\overline{x}}\right) - (mx_o)^2\overline{r}(x)\theta = 0, \tag{2.26}$$

where $m = \sqrt{2h/(kr_b)}$ and $\overline{r}(x) = r(x)/r_b$. By substituting (2.25) in (2.26), a differential equation of first order constructed. It has the form

$$\frac{d\overline{r}}{\overline{r} - X^2} = \frac{(ax_0)}{2}d\overline{x}, \tag{2.27}$$

where $X = m/a$. The solution to (2.27) is given by

$$\overline{r}(\overline{x}) = X^2 + \left[1 - X^2\right]\mathrm{Exp}\left(\frac{ax_0}{2}\overline{x}\right). \tag{2.28}$$

For engineering problems, $\overline{r}(\overline{x})$ cannot be negative and it should intersect with fin centerline at $\overline{x} = 1$ when $X > 1.0$. As such, x_0 is found to be equal to

$$mx_0 = (2X)\ln\left[\frac{X^2}{X^2 - 1}\right], \quad X > 1.0. \tag{2.29}$$

In situations when $0 < X < 1.0$, mx_0 is minimally equal to $4.605X$ ($x_0 = 4.605/a$; $X < 1.0$). Under this constraint, the heat transfer rate at the fin tip ($x = x_0$) is always 0.01 times the fin heat transfer rate. The rate of heat transfer through the fin base is equal to

$$q_f = -k_f A_c \left.\frac{dT}{dx}\right|_{x=0} = \pi r_b^2 ka(T_b - T_\infty). \tag{2.30}$$

As such, the fin thermal efficiency and the performance indicator for this case are equal to

$$\eta_{\mathrm{f}} = \frac{q_f}{2\pi r_b x_0 h(T_b - T_\infty)\int_0^1 \overline{r}(\overline{x})d\overline{x}} = \begin{cases} \dfrac{1}{2X^2}\left\{X^2 \ln\left[\dfrac{X^2}{X^2-1}\right] - 1\right\}^{-1}, & X > 1.0, \\ \dfrac{1}{2X^2(9.0 - 6.697X^2)}, & 0 \le X \le 1.0, \end{cases} \tag{2.31}$$

$$\gamma_1 = \frac{q_f}{\pi r_b\sqrt{2hkr_b}(T_b - T_\infty)} = \frac{1}{X}.$$

2.4. Exponential Joint-Fins

Consider an infinite exponential fin joining two different fluid streams separated by a wall of negligible thickness such as a pipe wall. The convection coefficient between the fin and the fluid stream of the heat source side (side with maximum free stream temperature $T_{\infty 1}$) is h_1. This coefficient is h_2 for the heat sink side (side with $T_{\infty 2} < T_{\infty 1}$) as illustrated in Figure 1. The joint-fin portion on the source side is named as the "joint-fin receiver portion" while the other portion is named as the "joint-fin sender portion". The heat transfer rates through a straight exponential joint-fin $(q_f)_s$, pin exponential joint-fin $(q_f)_P$, and the pin joint-fin with exponential decaying temperature $(q_f)_T$ are given by the following equations:

$$(q_f)_s = -\gamma_2 2H_b\sqrt{2h_1 kt}(T_b - T_{\infty 1}) = \gamma_1 2H_b\sqrt{2h_2 kt}(T_b - T_{\infty 2}), \tag{2.32}$$

$$(q_f)_P = -\gamma_2 \pi r_b\sqrt{2h_1 kr_b}(T_b - T_{\infty 1}) = \gamma_1 \pi r_b\sqrt{2h_2 kr_b}(T_b - T_{\infty 2}), \tag{2.33}$$

$$(q_f)_T = -\gamma_1 \pi r_b\sqrt{2h_1 kr_b}(T_b - T_{\infty 1}) = \gamma_1 \pi r_b\sqrt{2h_2 kr_b}(T_b - T_{\infty 2}), \tag{2.34}$$

where the exponential index for the joint-fin receiver portion is considered to be negative, $b < 0$, while that for the sender portion is positive, $b > 0$. This is only for cases represented by (2.32) and (2.33).

By solving (2.32)–(2.34), the temperature at the joint-fin base ($x = 0$) can be calculated. They are equal to

$$(T_b)_s = \frac{\left[\sqrt{4X_1^2+1}+1\right]T_{\infty 1} + \left[\sqrt{4X_2^2+1}-1\right]T_{\infty 2}}{\sqrt{4X_1^2+1}+\sqrt{4X_2^2+1}}, \tag{2.35}$$

$$(T_b)_P = \frac{M \times T_{\infty 1} + N \times T_{\infty 2}}{M + N}, \tag{2.36}$$

$$(T_b)_T = \frac{T_{\infty 1} + T_{\infty 2}}{2}, \tag{2.37}$$

where M and N are given by

$$M = \left[\left(\frac{X_1}{X_2}\right)\left\{\frac{I_1(2X_1)}{2I_2(2X_1)} + \frac{I_3(2X_1)}{2I_2(2X_1)}\right\} + \frac{1}{X_2}\right],$$
$$N = \left[\frac{K_1(2X_2)}{2K_2(2X_2)} + \frac{K_3(2X_2)}{2K_2(2X_2)} - \frac{1}{X_2}\right]. \tag{2.38}$$

By substituting (2.35)–(2.37) in (2.32)–(2.34), the joint-fin heat transfer rates reduce to the following forms:

$$(q_f)_s = 2H_b\sqrt{2h_1kt}(T_{\infty 1} - T_{\infty 2})\frac{\left[\sqrt{4X_1^2+1}+1\right]\left[\sqrt{4X_2^2+1}-1\right]}{2X_1\left\{\sqrt{4X_1^2+1}+\sqrt{4X_2^2+1}\right\}},$$
$$(q_f)_s = \pi r_b\sqrt{2h_1kr_b}(T_{\infty 1} - T_{\infty 2})\frac{N}{M(M+N)}\left(\frac{X_2}{X_1}\right),$$
$$(q_f)_T = \pi r_b\sqrt{2h_1kr_b}\frac{(T_{\infty 1} - T_{\infty 2})}{2X_1}. \tag{2.39}$$

Define the joint-fin performance indicator γ_3 as the ratio of the joint-fin maximum heat transfer rate to maximum heat transfer rate through a rectangular joint-fin with uniform cross-section ($b = 0$). It is mathematically defined as

$$\gamma_3 \equiv \frac{q_f}{\left[(q_f)|_{A_c=\text{constant}}\right]_{\text{Joint-fin}}}. \tag{2.40}$$

The heat transfer rate through the joint fin when $b = 0$ is obtainable from [17]. It is equal to

$$\left[(q_f)|_{H=H_b}\right]_{\text{Joint-fin}} = \frac{\sqrt{h_1kPA_c}}{1+\sqrt{h_1/h_2}}(T_{\infty 1} - T_{\infty 2}) \tag{2.41}$$

As such, γ_3 can be written in the following forms:

$$(\gamma_3)_s = \frac{1}{2}\left(\frac{1}{X_1}+\frac{1}{X_2}\right)\frac{\left[\sqrt{4X_1^2+1}+1\right]\left[\sqrt{4X_2^2+1}-1\right]}{\left\{\sqrt{4X_1^2+1}+\sqrt{4X_2^2+1}\right\}},$$
$$(\gamma_3)_P = \left\{\frac{N}{M(M+N)}\right\}\left(\frac{X_2}{X_1}+1\right),$$
$$(\gamma_3)_T = \frac{X_2+X_1}{2X_1X_2}. \tag{2.42}$$

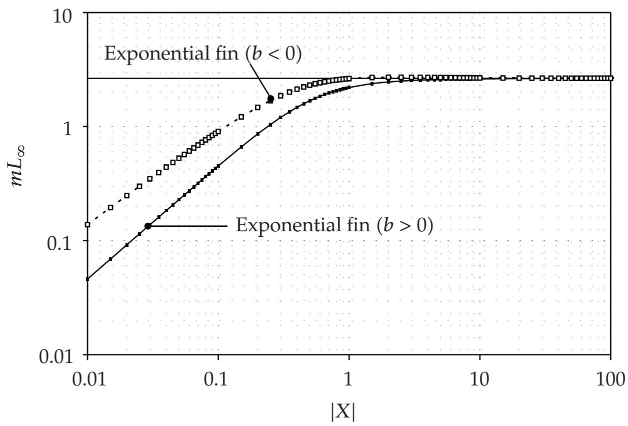

Figure 3: Effect of the fin dimensionless parameter X on the effective thermal length mL_∞ for straight exponential fins with $b > 0$ and $b < 0$ ($m = \sqrt{2h/(kt)}$).

3. Discussion of the Results

Figure 3 illustrates the effects of the fin dimensionless parameter X on the effective thermal length mL_∞ for a straight exponential fin. When $b > 0$, mL_∞ increases as X increases. It also increases as X increases for the other case ($b > 0$) until X reaches almost unity. For both cases, mL_∞ approaches to an asymptotic value of 2.65 as $X \to \infty$. Similar findings can be noticed for exponential pin fins except that, when $b < 0$, mL_∞ increases as X increases until X reaches almost the value of 1.7 as shown in Figure 4. On the other hand, mx_0 decreases as X increases for pin fins with exponential decaying temperature when $X > 1.0$. For exponential pin fins, the effective thermal lengths mL_∞ values shown in Figure 4 are correlated to the parameter X by the following correlations:

$$mL_\infty = 0.8233\left(\frac{X^{0.8804} - 0.0047}{0.2945X^{0.8934} + 0.2428}\right), \quad b > 0, \tag{3.1}$$

$$mL_\infty = 0.7237\left(\frac{X^{0.6906} + 3.3301X^{1.4019} + 0.3939X^{0.6902} - 0.0302}{0.9547X^{1.3923} + e^{-0.6311X^{1.9309}} - 0.7425}\right), \quad b < 0. \tag{3.2}$$

These correlations were obtained using the least square method by utilizing a specialized iterative statistical software. The maximum percentage error between correlations (3.1), and (3.2) and the results shown in Figure 4 are found to be 7.5% and 13% at $X = 0.01$ when $b > 0$, and $b < 0$, respectively.

Figure 5 shows the relation between the effective thermal length mL_∞ on the fin thermal efficiency η_f for a straight exponential fin. It is seen that η_f when $b > 0$ is greater than the fin thermal efficiency of rectangular, triangular, and parabolic straight fins having the same thermal length. However, the latter thermal efficiencies are greater than the fin thermal efficiency for the straight exponential fin when $b < 0$. It can be shown using Figure 5 that the maximum ratio between the thermal efficiency of the exponential straight fin to that of the rectangular fin is 1.58 at an effective thermal length of 2.0. For pin exponential fins with $b > 0$, η_f is found to be higher than η_f for the rectangular pin fins and lower than those

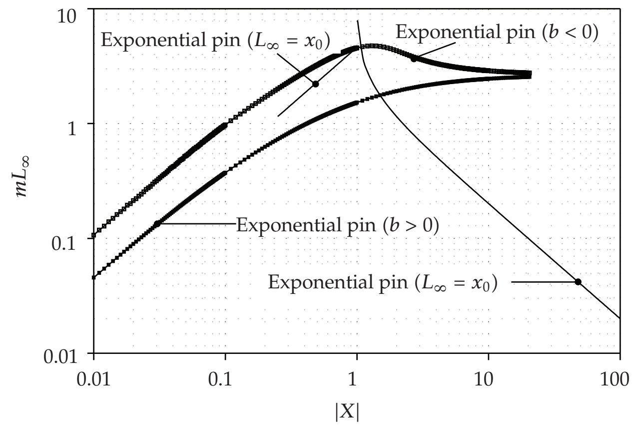

Figure 4: Effect of the fin dimensionless parameter X on the effective thermal length mL_∞ for exponential pin fins with $b > 0$, $b < 0$ and $L_\infty = x_0$ $(m = \sqrt{2h/(kr_b)})$.

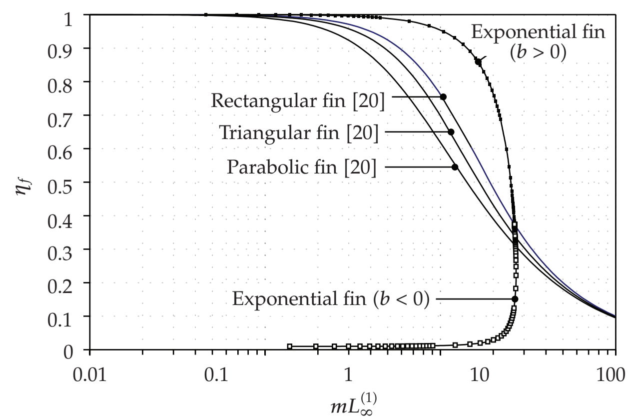

Figure 5: Effect of the fin dimensionless parameter mL_∞ on the fin efficiency η_f for straight exponential fins with $b > 0$, or $b < 0$ ($m = \sqrt{2h/(kt)}$; other than exponential fin mL_∞ is replaced with mL).

for triangular and parabolic pin fins having the same thermal length as shown in Figure 6. It is recommended to operate pin exponential fins, $b < 0$, at smaller values of mL_∞ as their efficiencies increase as mL_∞ decreases as can be seen from Figure 6. In addition, the maximum ratio between the thermal efficiency of the exponential pin fin to that of the rectangular pin fin is found to be 1.17 at an effective thermal length of 1.5.

Exponential straight or pin fins having increasing cross-sectional areas ($b < 0$) always exhibit higher fin heat transfer rates relative to rectangular straight or pin fins as can be seen from Figures 7 and 8. However, γ_1 values for those having decreasing cross-sectional areas ($b > 0$) are always smaller than unity as shown in Figures 7 and 8. Exponential joint-fins are found to transfer more heat than rectangular joint-fins fins at smaller values of X_1 and larger values of X_2 as can be seen from Figures 9 and 10. On the other hand, pin joint-fins with exponentially decaying temperatures were found to be preferable over rectangular pin joint-fins at smaller X_1 and X_2 values as shown in Figure 11.

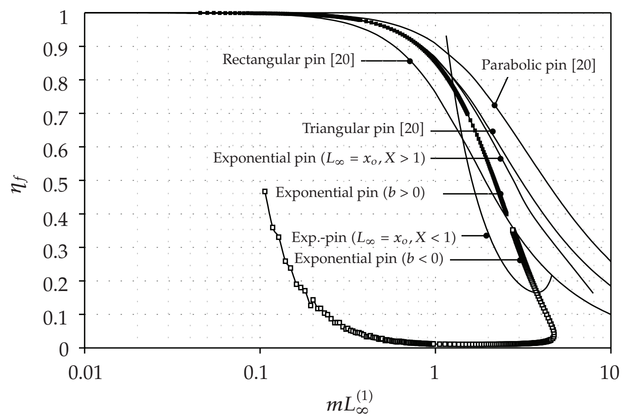

Figure 6: Effect of the fin dimensionless parameter mL_∞ on the fin efficiency η_f for exponential pin fins with $b > 0$, $b < 0$ and $L_\infty = x_0$ ($m = \sqrt{2h/(kr_b)}$; other than exponential fin mL_∞ is replaced with mL).

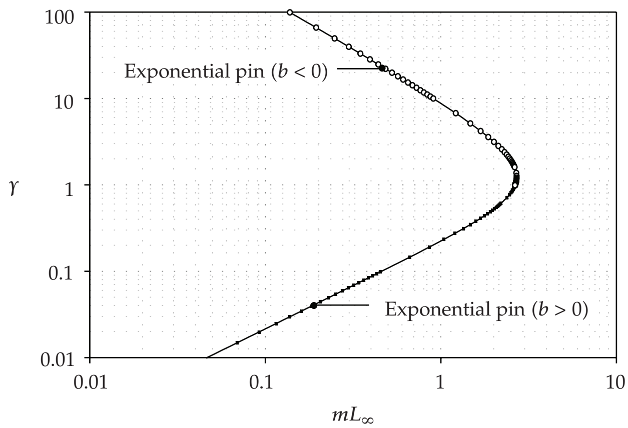

Figure 7: Effect of the fin dimensionless parameter mL_∞ on the performance indicator γ for straight exponential fins with $b > 0$ and $b < 0$ ($m = \sqrt{2h/(kt)}$).

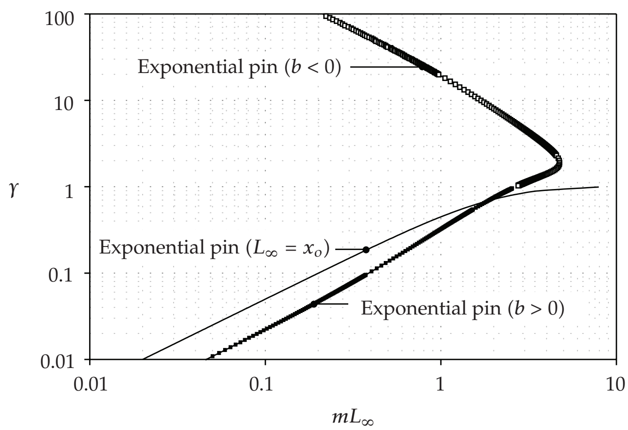

Figure 8: Effect of the fin dimensionless parameter mL_∞ on the performance indicator γ for exponential pin fins with $b > 0$, $b < 0$ and $L_\infty = x_0$ ($m = \sqrt{2h/(kr_b)}$).

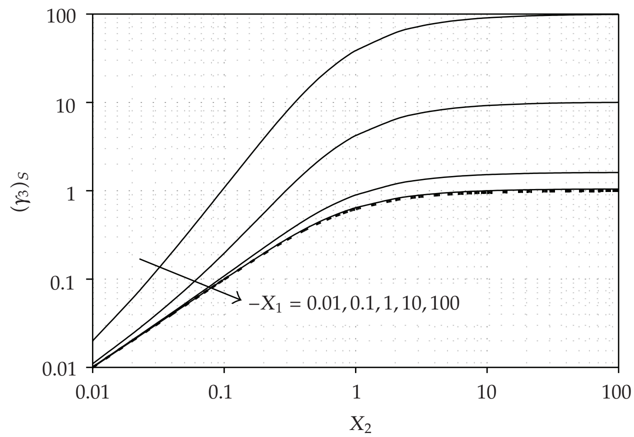

Figure 9: Effect of the parameters X_1 and X_2 on the performance indicator $(\gamma_3)_s$ for a straight exponential joint-fin with one side having $b < 0$ and the other side having $b > 0$.

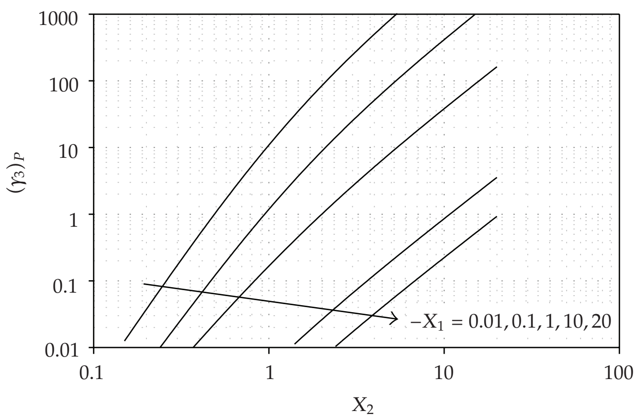

Figure 10: Effect of the parameters X_1 and X_2 on the performance indicators $(\gamma_3)_P$ for an exponential pin joint-fin with one side having $b < 0$ and the other side having $b > 0$.

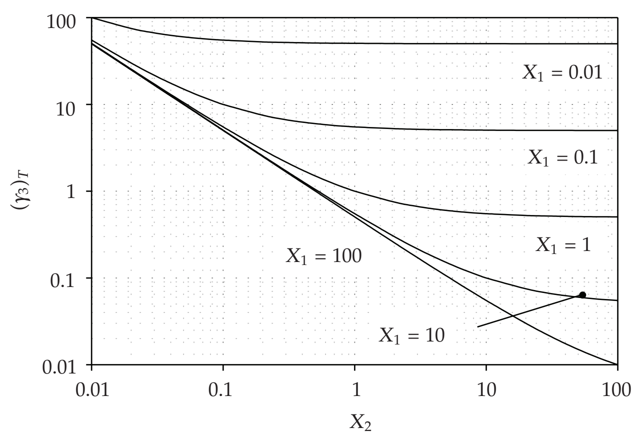

Figure 11: Effect of the parameters X_1 and X_2 on the performance indicators $(\gamma_3)_T$ for an exponential pin joint-fin having an exponential decaying temperature distribution.

The effect of increasing the exponential index b on γ_3 can be illustrated using Figures 9 and 10, for example, increasing b by a factor of 10 while maintaining the other parameters results in reductions in both X_1 and X_2 values by a factor of 0.1, for example, if $X_1 = 10$ and $X_2 = 10$. This produces $(\gamma_3)_s = 1.52$ and $(\gamma_3)_P = 0.857$. Increasing b by factor of 10 changes the joint-fin performance indicators $(\gamma_3)_s = 0.894$ and $(\gamma_3)_P = 0.167$ which are smaller than the initial values. In contrast, initially selecting $X_1 = 0.1$ and $X_2 = 10$ which produce $(\gamma_3)_s = 9.22$ and $(\gamma_3)_P = 414$ results in final $X_1 = 0.01$ and $X_2 = 1.0$ which lead to $(\gamma_3)_s = 38.6$ and $(\gamma_3)_P = 10.91$. As such, we can conclude that only $(\gamma_3)_s$ may increase as b increases when $X_2 - X_1$ is relatively large.

4. Conclusions

Exponential fin systems were modeled and mathematically analyzed in this work. The possibility of having decreasing or increasing cross-sectional areas was considered. Rectangular and circular cross-sectional areas are considered. Special thermal performance indicators were derived. The maximum ratio between the thermal efficiency of the exponential straight fin to that of the rectangular fin was found to be 1.58 at an effective thermal length of 2.0. This ratio was found to be larger when the exponential fin was compared with triangular and parabolic fins. Meanwhile, the maximum ratio between the thermal efficiency of the exponential pin fin to that of the rectangular pin fin was found to be 1.17 at an effective thermal length of 1.5. However, exponential pin thermal efficiency was found to be lower than those of triangular and parabolic pin fins. In addition, exponential joint-fins may transfer more heat than rectangular joint-fins especially when differences between their senders and receivers portions dimensionless indices are very large. Finally, the summary of the closed-form solutions and correlations reported in this work as compared to those of rectangular, triangular, and parabolic fin systems are summarized in Table 1.

Nomenclature

a, b: Exponential functions indices
H: Half-fin width
H_b: Half-fin width at its base
h: Convection heat transfer coefficient between the fin and the fluid stream
h_1: Convection heat transfer coefficient for the joint-fin source side
h_2: Convection heat transfer coefficient for the joint-fin sink side
$I_n(x)$: Modified Bessel functions of the first kind of order n
$K_n(x)$: Modified Bessel functions of the second kind of order n
k: Fin thermal conductivity
L: Fin length
L_∞: Effective fin length
m: Fin thermal index
q_f: Fin heat transfer rate
r: Pin fin radius
r_b: Pin fin radius at its base
T: Fin temperature
T_b: Fin base temperature
T_∞: Free stream temperature of the adjoining fluid
$T_{\infty 1}$: Free stream temperature of the source side adjoining fluid

Table 1: Efficiencies of exponential fins compared to efficiencies of common fins.

Fin type	Cross-sectional area Perimeter	mL_∞	η_f
Rectangular straight or pin fins with insulated tips(i) [20]	$\frac{2Ht}{4H}$, $\frac{\pi r_b^2}{2\pi r_b}$	2.65	$\frac{\tanh(mL)}{(mL)}$
Triangular straight fin(i) [20]	$\frac{H0.5t_o\{1-x/L\}}{2H}$	—	$\frac{1}{(mL)}\frac{I_1(2mL)}{I_0(2mL)}$
Triangular pin fin(ii) [20]	$\frac{\pi r_b^2\left\{1-\left(\frac{x}{L}\right)\right\}^2}{2\pi r_b\left\{1-\left(\frac{x}{L}\right)\right\}}$	—	$\frac{2}{(mL)}\frac{I_2(2mL)}{I_1(2mL)}$
Parabolic straight fin(i) [20]	$\frac{H0.5t_o\{1-(x/L)^2\}}{2H}$	—	$\frac{2}{(4(mL)^2+1)^{0.5}+1}$f
Parabolic pin fin(ii) [20]	$\frac{\pi r_b^2\left\{1-\left(\frac{x}{L}\right)^2\right\}^2}{2\pi r_b\left\{1-\left(\frac{x}{L}\right)^2\right\}}$	—	$\frac{2}{([4/9](mL)^2+1)^{0.5}+1}$
Exponential straight fin(i)(1)	$\frac{2H_b e^{-bx} t}{4H_b e^{-bx}}$	$\left(\frac{X}{\sqrt{4X^2+1}}\right)\left(5.293-\ln\left[1+\left(\frac{1}{\sqrt{4X^2+1}}\right)\right]\right);\ b>0,$ $\left(\frac{-X}{\sqrt{4X^2+1}}\right)\left(5.293-\ln\left[1-\left(\frac{1}{\sqrt{4X^2+1}}\right)\right]\right);\ b<0$	$\frac{0.99\{-1+\sqrt{4X^2+1}\}}{2X^2[1-\mathrm{Exp}(-b(L_\infty)_1)]},\ b>0$ $\frac{0.99\{1+\sqrt{4X^2+1}\}}{2X^2[\mathrm{Exp}(-b(L_\infty)_2)-1]},\ b<0$

Table 1: Continued.

Fin type	Cross-sectional area Perimeter	mL_∞	η_f
Exponential pin fin[(ii)(1)]	$\pi r_b^2 e^{-2bx}$ $2\pi r_b e^{-bx}$	$0.8233\left(\dfrac{X^{0.8804}-0.0047}{0.2945X^{0.8934}+0.2428}\right);\ b>0,$ $0.7237\left(\dfrac{X^{0.6906}+3.3301X^{1.4019}+0.3939X^{0.6902}-0.0302}{0.9547X^{1.3923}+e^{-0.6311X^{1.9309}}-0.7425}\right);\ b<0$	$\dfrac{0.99}{(1-e^{-mL_\infty/X})} \times\dfrac{1}{X^2}\left(\dfrac{X}{2}\right)\left\{\left[\dfrac{K_1(2X)}{K_2(2X)}+\left(\dfrac{K_3(2X)}{K_2(2X)}\right)\right]\right\}-1,\ b>0$ $\dfrac{0.99}{(e^{-mL_\infty/X}-1)}\times\dfrac{1}{X^2}\left\{\dfrac{-X}{2}\left[\dfrac{I_1(-2X)}{I_2(-2X)}+\dfrac{I_3(-2X)}{I_2(-2X)}\right]+1\right\},\ b<0$
Exponential decaying temperature-pin fin[(ii)(2)]	$\pi r_b^2\{X^2+[1-X^2] \times e^{-0.5ax_ox}\}^2$ $2\pi r_b\{X^2+[1-X^2] \times e^{-0.5ax_ox}\}$	$(2X)\ln\left[\dfrac{X^2}{(X^2-1)}\right];\ X>1.0$ $4.605X;\ X\le 1.0$	$\dfrac{1}{2X^2}\left\{X^2\ln\left[\dfrac{X^2}{(X^2-1)}\right]-1\right\}^{-1},\ X>1.0$ $\dfrac{1}{2X^2}(9.0-6.697X^2),\ X\le 1.0$

(i) $m=\sqrt{2h/(kt)}$; (ii) $m=\sqrt{2h/(kr_b)}$; (1) $X=m/b$; (2) $X=m/a$.

$T_{\infty 2}$: Free stream temperature of the sink side adjoining fluid
t: Fin thickness
X: Dimensionless exponential fin parameter
X_1: Dimensionless exponential parameter of the receiver fin portion
X_2: Dimensionless exponential parameter of the sender fin portion
x: Coordinate axis along the fin centerline
x_0: Pin fin length for exponential fins with exponentially decaying temperature
$\bar{x}$: Dimensionless x-coordinate.

Greek Symbols

θ: Dimensionless fin temperature
$\gamma_{1,2}$: Fin second thermal performance indicators
γ_3: Joint-fin thermal performance indicator
η_f: Fin thermal efficiency.

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References

[1] A. E. Bergles, *Handbook of Heat Transfer*, McGraw-Hill, New York, NY, USA, 3rd edition, 1998.
[2] A. E. Bergles, “The implications and challenges of enhanced heat tranfer for the chemical process industries,” *Chemical Engineering Research and Design*, vol. 79, no. 4, pp. 437–444, 2001.
[3] E. I. Nesis, A. F. Shatalov, and N. P. Karmatskii, “Dependence of the heat transfer coefficient on the vibration amplitude and frequency of a vertical thin heater,” *Journal of Engineering Physics and Thermophysics*, vol. 67, no. 1-2, pp. 696–698, 1994.
[4] J. K. Hagge and G. H. Junkhan, “Experimental study of a method of mechanical augmentation of convective heat transfer in air,” HTL3, ISU-ERI-Ames-74158, Iowa State University, Ames, Iowa, USA, 1975.
[5] W. M. Kays, “Pin-fin heat-exchanger surfaces,” *Journal of Heat Transfer*, vol. 77, pp. 471–483, 1955.
[6] D. O. Kern and A. D. Kraus, *Extended Surface Heat Transfer*, McGraw-Hill, New York, NY, USA, 1972.
[7] A. D. Kraus, “sixty-five years of extended surface technology (1922–1987),” *Applied Mechanical Review*, vol. 41, no. 9, pp. 321–364, 1988.
[8] P. J. Schenider, *Conduction Heat Transfer*, Addison-Wesley, Reading, Mass, USA, 1955.
[9] A. D. Kraus, A. Aziz, and J. R. Welty, *Extended Surface Heat Transfer*, John Wiley & Sons, New York, NY, USA, 2001.
[10] S. Kakaç and H. Liu, *Heat Exchangers: Selection, Rating, and Thermal Design*, CRC Press, Boca Raton, Fla, USA, 2001.
[11] N. Sahiti, A. Lemouedda, D. Stojkovic, F. Durst, and E. Franz, “Performance comparison of pin fin in-duct flow arrays with various pin cross-sections,” *Applied Thermal Engineering*, vol. 26, no. 11-12, pp. 1176–1192, 2006.
[12] A. Nuntaphan, T. Kiatsiriroat, and C. C. Wang, “Air side performance at low Reynolds number of cross-flow heat exchanger using crimped spiral fins,” *International Communications in Heat and Mass Transfer*, vol. 32, no. 1-2, pp. 151–165, 2005.
[13] R. M. Manglik and A. E. Bergles, “Swirl flow heat transfer and pressure drop with twisted-tape inserts,” *Advances in Heat Transfer*, vol. 36, p. 183, 2002.
[14] A. Alebrahim and A. Bejan, “Constructal trees of circular fins for conductive and convective heat transfer,” *International Journal of Heat and Mass Transfer*, vol. 42, no. 19, pp. 3585–3597, 1999.

[15] M. Almogbel and A. Bejan, “Cylindrical trees of pin fins,” *International Journal of Heat and Mass Transfer*, vol. 43, no. 23, pp. 4285–4297, 2000.
[16] A.-R. A. Khaled, “Heat transfer enhancement in hairy fin systems,” *Applied Thermal Engineering*, vol. 27, no. 1, pp. 250–257, 2007.
[17] A.-R. A. Khaled, “Maximizing heat transfer through joint fin systems,” *Journal of Heat Transfer*, vol. 128, no. 2, pp. 203–206, 2006.
[18] A.-R. A. Khaled, “Analysis of heat transfer through Bi-convection fins,” *International Journal of Thermal Sciences*, vol. 48, no. 1, pp. 122–132, 2009.
[19] A.-R. A. Khaled, “Heat transfer analysis through solar and rooted fins,” *Journal of Heat Transfer*, vol. 130, no. 7, Article ID 074503, 4 pages, 2008.
[20] F. P. Incorpera, D. P. DeWitt, T. L. Bergman, and A. S. Lavine, *Fundamentals of Heat and Mass Transfer*, John Wiley & Sons, New York, NY, USA, 6th edition, 2006.