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Research Article

Approximate Ad Hoc Parametric Solutions for Nonlinear First-Order PDEs Governing Two-Dimensional Steady Vector Fields

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Through a suitable ad hoc assumption, a nonlinear PDE governing a three-dimensional weak, irrotational, steady vector field is reduced to a system of two nonlinear ODEs: the first of which corresponds to the two-dimensional case, while the second involves also the third field component. By using several analytical tools as well as linear approximations based on the weakness of the field, the first equation is transformed to an Abel differential equation which is solved parametrically. Thus, we obtain the two components of the field as explicit functions of a parameter. The derived solution is applied to the two-dimensional small perturbation frictionless flow past solid surfaces with either sinusoidal or parabolic geometry, where the plane velocities are evaluated over the body's surface in the case of a subsonic flow.

1. Introduction

First-order PDEs, which mostly appear in fluid mechanics, describe the motion of ideal as well as of real fluids [1–3] and govern even the electrostatic plasma oscillation [4]. As is well known, there is no complete general theory concerning the derivation of exact analytical solutions for such equations. However, general solutions can be obtained for the quasilinear forms by means of the subsidiary Lagrange equations ([1, Section 2.6.a], Appendix A). We also mention Charpit's method for the general nonlinear case that yields to complete and general solutions [1, Section 2.6.b]. These solutions involve arbitrary functions of specific expressions of the dependent and independent variables. Furthermore, appropriate transformations of the dependent and (or) independent variables [1, Section 2.1], combined in several cases with the introduction of auxiliary functions (like stream functions), can occasionally linearize the original equation or more generally reduce it to a solvable form, like a quasilinear one, or even to a nonlinear ODE.

In our previous work [5], four simplified forms of the full two-dimensional nonlinear steady small perturbation equation in fluid mechanics [6] were treated analytically. As far as the three of the considered cases are concerned, closed form solutions have been derived for the two dependent variables of the equation, which represent the dimensionless velocities u, v of a perturbed frictionless flow past a solid body surface, while in the fourth case, a parametric solution was obtained with regard to these velocity resultants. We note that the components u, v are parallel to the x_1, x_2 axes of the Cartesian plane, respectively (see Figure 1 in Section 4, where a wavy surface is represented), with x_1 being the direction of the uniform velocity of the steady flow. The extracted closed form solutions provide v as a specific expression of u, as well as an equation for u involving an unknown arbitrary function. The analytical method was based on the introduction of a convenient *ad hoc assumption*, originally due to Pai [7], by means of which the original (simplified) equations, as well as the irrotanional relation, take a quasilinear form integrated by the Lagrange method. Thus, the above-mentioned solution (including the unknown function) for u is obtained, together with an ordinary differential equation, which, after a further analytical treatment, provides the exact or approximate (depending on the case) solutions $v(u)$. However, it should be mentioned that only in the first, more simplified case [5, Equation (9)] of the general equation, the unknown function can be defined by the use of the boundary condition of the problem, resulting in a transcendental equation for u (or v). Furthermore, no investigation has been performed in [5] with regard to the effectiveness of the obtained formulas (the expressions extracted in the application [5, Section 5] concerning the above-mentioned simplified case and the parametric solutions derived for one of the other examined cases [5, Equation (8)]) to evaluate the perturbed flow field.

In the present work (Section 2), we firstly treat a steady three-dimensional PDE concerning a general weak irrotational vector field. By taking into account the three irrotationality conditions and using the *ad hoc assumption* introduced in [5], the Lagrange method (see Appendix A) finally results in a system of two nonlinear ODEs for the two unknown functions introduced by the *ad hoc assumption.* These functions represent the field's components u_2, u_3, while the first component $u_1 = u$ stands for the independent variable. In Section 3, we proceed into the integration of the first ODE, which corresponds to the plane problem (u_1, u_2) (the second involves also u_3). The herein developed methodology consists of a *functional transformation* of the dependent variable, in combination with an appropriate *split* of the resulting equation by using an arbitrary function, which eventually is eliminated. By this technique, we finally derive an Abel equation, which admits a parametric solution. Thus, we obtain the field's components $u_1(= u)$ and u_2 as explicit expressions of a parameter τ. In several steps of the analysis developed in Section 3, the established, in Appendix C (linear), approximations based on the weakness of the field ($u_1 \ll 1$) have been used. Additionally, some limitations imposed by the analysis (see Cases P-1, P-2 in Appendix D) affect the domain of the physical parameter(s) of the problem, for which the extracted solution is valid.

Then in Section 4 we apply the obtained parametric solution in the plane case of the full small perturbation equation, simplified forms of which were investigated in [5]. Here, by combining the extracted parametric formulae with the boundary condition concerning the flow tangential to the solid surface, a transcendental equation is derived, involving τ, ξ_1, ξ_2, where ξ_1, ξ_2 represent the plane coordinates on the body's surface. Then, for a given pair (ξ_1, ξ_2), the solution of this equation yields $\tau(\xi_1, \xi_2)$, and hence the "surface" perturbed flow velocity field (u_1, u_2), can be evaluated (the perturbed velocity components u_1, u_2 refer to the

x_1, x_2 cartesian plane). Moreover, by expanding in Taylor series and taking into account the small perturbation, the perturbed velocities can be approximately obtained within a thin zone over the surface. In addition, under the mentioned limitations, we deduce that the obtained results hold true for subsonic flows as well.

Finally, by means of the extracted formulas, graphic representations of the perturbed field versus $x_1(= \xi_1)$ are obtained, concerning a sinusoidal as well as a parabolic boundary, and the results are compared to the solution of the linearized equation.

2. The Analytical Procedure

2.1. Transformation of the Governing Equations

Consider an irrotational field $\overline{u} = (u_1, u_2, u_3)$ satisfying the following PDE:

$$\left(A_0^{ij} + A_\kappa^{ij} u_\kappa + A_{\kappa\lambda}^{ij} u_\kappa u_\lambda\right) u_{i,j} + \left(A_0^{33} + A_3^{33} u_3\right) u_{3,3} = 0, \quad i, j, \kappa, \lambda = 1, 2, \tag{2.1}$$

where summation convention has been adopted and

$$A_{\kappa\lambda}^{ij} = A_{\lambda\kappa}^{ij}, \quad i, j, \kappa, \lambda = 1, 2,$$
$$u_{i,j} = \frac{\partial u_i}{\partial x_j}, \quad u_{3,3} = \frac{\partial u_3}{\partial x_3}, \quad i, j = 1, 2, \tag{2.2}$$

with (x_1, x_2, x_3) being the Cartesian space coordinates. Equation (2.1) is assumed dimensionless and properly scaled, while the coefficients A (with the respective upper and subindexes) represent constants or functions of one or more parameters. In this paper, we investigate the case where

$$A_{12}^{ii} = A_{\kappa\kappa}^{ij} = 0, \quad i \neq j, \ i, j, \kappa = 1, 2, \tag{2.3a}$$

as well as the case where

$$A_2^{ii} = A_0^{ij} = A_1^{ij} = 0, \quad i \neq j, \ i, j = 1, 2. \tag{2.3b}$$

However, the proposed solution can also be applied to cases where the coefficients involved in (2.3a) and (2.3b) are sufficiently small, so that the respective terms of (2.1) can be neglected in comparison with the others. Moreover, the field is supposed to be weak in the x_1 x_2 plane, that is,

$$u_i \ll 1, \quad i = 1, 2. \tag{2.4}$$

In fact the approximations (see Appendix C), used in certain steps of the analytical procedure, are based on the weakness of the field under consideration.

As a first step, we make the *ad hoc* assumption that the components u_2 and u_3 are functions of the component u_1, namely,

$$u_i = f_i(u_1), \quad i = 1, 2, 3, \tag{2.5}$$

and thus by substituting (2.5), (2.1) (taking into account (2.3a) and (2.3b)) becomes

$$R_1(u)u_{,1} + R_2(u)u_{,2} + R_3(u)u_{,3} = 0, \tag{2.6}$$

where u_1 has been replaced by u and

$$R_1(u) = A_0^{11} + A_1^{11}u + A_{11}^{11}u^2 + A_{22}^{11}f_2^2 + \left(A_2^{21}f_2 + 2A_{12}^{21}uf_2\right)f_2', \tag{2.7a}$$

$$R_2(u) = A_2^{12}f_2 + 2A_{12}^{12}uf_2 + \left(A_0^{22} + A_1^{22}u + A_{11}^{22}u^2 + A_{22}^{22}f_2^2\right)f_2', \tag{2.7b}$$

$$R_3(u) = \left(A_0^{33} + A_3^{33}f_3\right)f_3'. \tag{2.7c}$$

Here, the prime "$'$" denotes differentiation with respect to $u(f_i'(u), i = 2, 3)$.

On the other hand, the irrotational condition of the field is written in the form

$$\nabla \times \overline{u} = \varepsilon_{kji}\frac{\partial u_i}{\partial x_j}\overline{e}_k = \overline{0}, \quad i, j, k = 1, 2, 3, \tag{2.8}$$

where ε_{kji} is the well-known Levi-Civita tensor and $\overline{e}_k$ represent the unit vectors corresponding to x_k, $k = 1, 2, 3$, respectively. By substituting the assumption (2.5) into (2.8), we arrive at the following three equations (u_1 is replaced by u):

$$f_3'u_{,2} - f_2'u_{,3} = 0, \tag{2.9a}$$

$$u_{,3} - f_3'u_{,2} = 0, \tag{2.9b}$$

$$f_2'u_{,1} - u_{,2} = 0. \tag{2.9c}$$

With respect to the physical relevance of (2.1), as well as of the constraints imposed above, we note the following. No "mixed" nonlinear terms involving the plane components u_1, u_2 together with u_3 are included in (2.1). Furthermore the restrictions (2.3a) and (2.3b) focus on cases where specific nonlinear terms are involved into the governing equation. More precisely, the procedure developed in this paper confronts nonlinear equations where the partial derivatives of the field components appear in products together with specific combinations of these components, of the first and the second degree. Indeed by (2.3a) and (2.3b), it is obvious that two groups of nonlinear terms are formed with respect to the variations of the plane components u_1, u_2, along their own axes ($u_{i,i}$) and the other axis ($u_{i,j}$, $i \neq j$). This can be clearly observed in the two-dimensional steady small perturbation equation of fluid mechanics, treated in Section 4 (4.1) as an application of the present analysis.

All the above notations, as well as the *ad hoc* assumption (2.5), outline a normalized structure as regards the behavior of the field in phase space, due to a regulated physical setup. In fact the small perturbation (4.1) is representative of the imposed restrictions, since the origin of the field (the perturbed velocities due to slight "geometric perturbations" of the body's surface) combined with the orientation of the uniform flow (with reference to the body—see Figure 1 in Section 4) can give rise to the specific nonlinear form of the governing equations (2.1), (2.3a), and (2.3b), as well as to the "weakness" and the *ad hoc* assumptions, (2.4) and (2.5), respectively.

2.2. Construction of Intermediate Integrals

Now, by integrating the correspondent to (2.6), (2.9a), (2.9b), and (2.9c) subsidiary Lagrange equations (see Appendix A), we, respectively, obtain the following general solutions:

$$(2.6) \Longrightarrow u = G\left[x_1 - \frac{R_1(u)}{R_2(u)}x_2, x_2 - \frac{R_2(u)}{R_3(u)}x_3\right], \tag{2.10}$$

$$(2.9\text{a}) \Longrightarrow u = G_1\left(x_1, x_2 + \frac{f_3'}{f_2'}x_3\right), \tag{2.11a}$$

$$(2.9\text{b}) \Longrightarrow u = G_2\left(x_2, x_1 + f_3'x_3\right), \tag{2.11b}$$

$$(2.9\text{c}) \longrightarrow u = G_3\left(x_3, x_1 + f_2'x_2\right), \tag{2.11c}$$

where G, G_1, G_2, and G_3 are arbitrary functions possessing continuous partial derivatives with respect to their arguments.

2.3. Reduction to a System of Nonlinear ODEs

In view of (2.10), (2.11a),(2.11b), and (2.11c), we construct a first set of relations by equating identically the functions G, G_1, G_2, and G_3 as well as their arguments. Thus, excluding the cases where in the extracted equations:

$$\begin{aligned} &x_1 = 0, \qquad x_2 = 0, \qquad x_3 = 0, \\ &x_1 = x_2, \qquad x_2 = x_3, \qquad x_1 = x_3, \end{aligned} \tag{2.12}$$

we eventually obtain the following systems.

Case 1 ($G_2 \equiv G_3$). We have

$$f_2' = 1 - \frac{x_1}{x_2}, \qquad f_3' = 1 - \frac{x_1}{x_3}. \tag{2.13}$$

Case 2 ($G \equiv G_1$). We have

$$\frac{R_1}{R_2} = \frac{x_1}{x_2} - 1 - \frac{f_3'}{f_2'}\frac{x_3}{x_2}, \qquad \frac{R_2}{R_3} = \frac{x_2}{x_3} - \frac{x_1}{x_3}. \tag{2.14}$$

Case 3 $(G \equiv G_2)$. We have

$$\frac{R_1}{R_2} = \frac{x_1}{x_2} - 1, \qquad \frac{R_2}{R_3} = \frac{x_2}{x_3} - \frac{x_1}{x_3} - f_3'. \tag{2.15}$$

Case 4 $(G \equiv G_3)$.

Subcase 1 $(G \equiv G_3)$. We have

$$\frac{R_1}{R_2} = \frac{x_1}{x_2} - \frac{x_3}{x_2}, \qquad \frac{R_2}{R_3} = \frac{x_2}{x_3} - \frac{x_1}{x_3} - f_2'\frac{x_2}{x_3}. \tag{2.16}$$

Subcase 2 $(G \equiv G_3)$. We have

$$\frac{R_1}{R_2} = -f_2', \qquad \frac{R_2}{R_3} = \frac{x_2}{x_3} - 1. \tag{2.17}$$

Subcases 1 and 2 are, respectively, derived by equating the arguments of G and G_3 in two possible combinations. Then, in order to obtain a system of equations not containing explicitly x_1, x_2, and x_3, we find that Cases 1, 3 and Subcase 2 are compatible to each other. Thus by combining their respective equations, we derive the following ODEs:

$$R_2(u) f_2'(u) + R_1(u) = 0, \tag{2.18}$$

$$R_3(u) f_3'(u) - R_3(u) f_2'(u) + R_1(u) + R_2(u) = 0. \tag{2.19}$$

Taking into account (2.7a), (2.7b), and (2.7c), we note that (2.18) contains only f_2 and f_2', and thus it constitutes the main equation, the manipulation of which is presented in the next section.

Therefore, the ordinary differential equations (2.18) and (2.19) represent the reduced forms of the partial differential equations (2.6), (2.9a), (2.9b), and (2.9c), via assumption (2.5). Then by substituting (2.7a), (2.7b) and replacing f_2 with y and u with x, (2.18) becomes

$$y_x'^2 + \rho_{22}(x) y^2 y_x'^2 + \rho_{11}(x) y y_x' + \rho_{20}(x) y^2 = \omega(x), \tag{2.20}$$

where y_x' denotes the derivative of $y(x)$ with respect to x and

$$\rho_{22}(x) = \frac{\alpha}{P(x)}, \quad \rho_{11}(x) = \frac{A_2 + A_3 x}{P(x)}, \quad \rho_{20}(x) = \frac{\beta}{P(x)}, \quad \omega(x) = \frac{P_1(x)}{P(x)}, \tag{2.21}$$

with

$$\alpha = A_{22}^{22}, \qquad \beta = A_{22}^{11}, \tag{2.22a}$$

$$P(x) = A_0^{22} + A_1^{22} x + A_{11}^{22} x^2, \qquad P_1(x) = -A_0^{11} - A_1^{11} x - A_{11}^{11} x^2. \tag{2.22b}$$

We note that A_2 and A_3 as well as all the other coefficients appearing in the next sessions are listed in Appendix E. Henceforth, the prime will denote differentiation with respect to the corresponding suffix.

3. Integration of (2.20)

3.1. Transformation of (2.20)

Introducing transformation

$$y(x) = h[\xi(x)]f(x), \tag{3.1}$$

the left hand side of (2.20) results in a nonlinear expression involving h, h'_ξ, ξ'_x, f, and f'_x. Thus, by taking into account this expression and setting

$$f(x) = \exp\left(-\frac{\kappa(x)}{2}\right), \qquad \kappa(x) = \int \rho_{11}(x)dx, \tag{3.2}$$

$$\xi(x) = \int \rho_3^{1/2}(x)dx, \qquad \rho_3(x) = \frac{\rho_{11}^2}{4} - \rho_{20}, \tag{3.3}$$

with ρ_{11}, ρ_{20} as in (2.21), (2.20) takes the form

$$h_\xi'^2 + \rho_{22}f^2h^2h_\xi'^2 - \rho_{22}\rho_{11}f^2\rho_3^{-1/2}h^3h'_\xi + \frac{1}{4}\rho_{22}\rho_{11}^2f^2\rho_3^{-1}h^4 - h^2 = \frac{\omega}{f^2\rho_3}. \tag{3.4}$$

Then, by substituting

$$h^2(\xi) = s(\xi), \tag{3.5}$$

(3.4) becomes

$$\frac{s_\xi'^2}{s} + \rho_{22}f^2s_\xi'^2 - \left(2\rho_{22}\rho_{11}f^2\rho_3^{-1/2}ss'_\xi - \rho_{22}\rho_{11}^2f^2\rho_3^{-1}s^2 + 4s + \frac{4\omega}{f^2\rho_3}\right) = 0. \tag{3.6}$$

In addition, by substitution of (2.21) and (2.22b) into (3.3), we obtain

$$\rho_3(x) = \frac{P_2(x)}{4P^2(x)}, \qquad P_2(x) = A_4 + A_5x + A_6x^2, \tag{3.7}$$

with $P(x)$ as in (2.22b).

3.2. The Split of (3.6)

We now split (3.6) into the following system of equations:

$$\frac{s_{\xi}'^{2}}{s} + \rho_{22} f^2 s_{\xi}'^{2} = F(\xi), \tag{3.8a}$$

$$2\rho_{22}\rho_{11} f^2 \rho_3^{-1/2} s s_{\xi}' - \rho_{22}\rho_{11}^2 f^2 \rho_3^{-1} s^2 + 4s + \frac{4\omega}{f^2\rho_3} = F(\xi), \tag{3.8b}$$

where $F(\xi)$ is an unknown arbitrary function. Furthermore, after dividing (3.8b) by f^2 and setting

$$F(\xi) = \frac{4\omega(x)}{f^2(x)\rho_3(x)}\overline{G}(\xi), \quad x = x(\xi), \tag{3.9}$$

(3.8b) will be written as

$$\begin{aligned} &2\rho_{22}(x)\rho_{11}(x)\rho_3^{-1/2}(x) s s_{\xi}' \\ &\quad = \rho_{22}(x)\rho_{11}^2(x)\rho_3^{-1}(x)s^2 - \frac{4}{f^2(x)}s + \frac{4\omega(x)}{f^4(x)\rho_3(x)}\left[\overline{G}(\xi) - 1\right], \quad x = x(\xi), \end{aligned} \tag{3.10}$$

where $\overline{G}(\xi)$ represents now the unknown arbitrary function. We see that (3.10) is an Abel equation of the second kind, and thus by following the analysis presented in [8, Chapter 1, Section 3.4] and taking into account (3.2) and (3.3), it is reduced to a simpler Abel equation, namely,

$$\overline{z}\,\overline{z}_t' - \overline{z} = \frac{\omega(x)}{f(x)\rho_3(x)}\left[\overline{G}(\xi) - 1\right], \quad x = x[\xi(t)], \ \xi = \xi(t) \tag{3.11}$$

with

$$s(\xi) = \frac{\overline{z}[t(\xi)]}{f[x(\xi)]}, \qquad t(\xi) = -2\int \frac{\rho_3^{1/2}(x)d\xi}{\rho_{22}(x)\rho_{11}(x)f(x)}, \tag{3.12}$$

now, by differentiating s, given by (3.12), with respect to ξ and using (3.2), (3.3) (we consider the appropriate domains where $\xi(x)$ is invertible and hence $x_{\xi}' = \xi_x'^{-1}$) as well as the expression of t provided by (3.12), we obtain s_{ξ}', substitution of which into the left-hand side of (3.8a) results in

$$\left[\frac{1}{f(x)\overline{z}(t)} + \rho_{22}(x)\right]\left[\overline{z}_t' - \frac{\rho_{22}(x)\rho_{11}^2(x)f(x)}{4\rho_3(x)}\overline{z}(t)\right]^2 = \frac{\rho_{22}^2(x)\rho_{11}^2(x)\omega(x)}{\rho_3^2(x)}\overline{G}(\xi), \tag{3.13}$$

where (3.9) has been substituted for $F(\xi)$.

Equations (3.11) and (3.13) form a new system equivalent to that of (3.8a), (3.8b), obtained by splitting (3.6). The elimination of the arbitrary function $\overline{G}$ yields a nonlinear ODE, which represents the reduced form of (3.6). More precisely, after some algebra we extract

$$\left(3\overline{z} + \frac{1}{\rho_{22} f}\right)\overline{z}_t' - \frac{\rho_3}{\rho_{22}^2 \rho_{11}^2 f}\left(\frac{1}{f\overline{z}} + \rho_{22}\right)\overline{z}_t'^2 = \frac{\rho_{22}\rho_{11}^2 f}{8\rho_3}\overline{z}^2 + \left(2 + \frac{\rho_{11}^2}{8\rho_3}\right)\overline{z} - \frac{2\omega}{\rho_3 f}. \tag{3.14}$$

Furthermore, as far as the $\overline{z}_t'^2$-term is concerned, combination of (3.12) with (3.1) and (3.5) yields $\overline{z}(t) = y^2(x)/f(x)$, $x = x[\xi(t)]$. By differentiating with respect to t and taking into account certain relations obtained above, as well as that x, y, and y_x' represent $u, u_2 = f_2(u)$ and $f_2'(u)$, respectively, we conclude that $\overline{z}_t'$ is equal to $(\overline{\alpha} + \overline{\beta}u)(u_2 f_2' + \overline{\gamma}u_2^2 + \overline{\delta}uu_2^2)$, where $\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}$ represent expressions of the equation's coefficients. Therefore, when the plane field's components, as well as the variation of u_2 with respect to u, are very small compared with the unit (e.g., if they denote perturbed components in a small perturbation theory), we can perfectly consider

$$\overline{z}_t' \ll 1, \tag{3.15}$$

and hence we can neglect the $\overline{z}_t'^2$ term in the left-hand side of (3.14) in comparison with the others, as it is of $\mathcal{O}[\max\{u_2^4, u_2^2 f_2'^2\}]$. We should note here that in our previous work [5], after following a different analysis concerning two simplified forms of the full equation, an analogous to (3.15), but weaker approximation, has been applied, since the neglected term was of $\mathcal{O}(u_2^4/(4u^2))$, yielding less accurate results compared to the obtained herein solution of (3.14) especially when u takes smaller values than u_2. Moreover by means of (3.3) and (3.12), we have

$$\overline{t}(x) = t[\xi(x)] = -2\int \frac{\rho_3(x)dx}{\rho_{22}(x)\rho_{11}(x)f(x)}. \tag{3.16}$$

Thus, by writing

$$z(x) = \overline{z}\left[\overline{t}(x)\right], \tag{3.17}$$

neglecting the $\overline{z}_t'^2$ term and multiplying with $\overline{t}_x'$, then by using (3.16), (3.14) becomes

$$\left[3z + \frac{1}{\rho_{22}(x)f(x)}\right]z_x' = -\frac{\rho_{11}(x)}{4}z^2 - \frac{16\rho_3(x) + \rho_{11}^2(x)}{4\rho_{22}(x)\rho_{11}(x)f(x)}z + \frac{4\omega(x)}{\rho_{22}(x)\rho_{11}(x)f^2(x)}. \tag{3.18}$$

The above equation is also an Abel equation of the second kind and thus we proceed as in [8, Chapter 1, Section 3.4]. More precisely, by using the formulas (D.10a), (D.10b) (see Appendix D), after some algebra, we arrive at

$$qq_r' - q = -\frac{2}{3\alpha}\frac{F_{00} + F_{01}x + \mathcal{O}(x^2)}{M_0 + M_1 x + \mathcal{O}(x^2)}, \tag{3.19}$$

where

$$r(x) = \int \frac{F_{10} + F_{11}x + \mathcal{O}(x^2)}{Q_0 + Q_1 x + \mathcal{O}(x^2)} dx \tag{3.20}$$

and α being as in (2.22a). Now, by applying (C.4) (Appendix C) to both rational functions in the right-hand sides of (3.19) and (3.20), we obtain

$$qq'_r - q = B_2 + \overline{B}_3 x, \tag{3.21}$$

$$r(x) = \int (B_0 + B_1 x) dx = B_0 x + \mathcal{O}\left(x^2\right). \tag{3.22}$$

Finally, substitution of $x = r/B_0$ into (3.21) yields

$$qq'_r - q = B_2 + B_3 r. \tag{3.23}$$

Moreover, by the followed procedure (see [8]), we have that

$$z(x) = \frac{q[r(x)]}{E} - \frac{P(x)}{3\alpha f(x)} \tag{3.24}$$

with $P(x)$ as in (2.22b) and E given by (D.10b) (see Appendix D). Finally, the Abel equation (3.23) is solved parametrically (Appendix B, formulas (B.7)) as

$$r = \frac{C}{B_3} \tau \Gamma^{-1/2}(\tau) e^{-I(\tau)/2} - \frac{B_2}{B_3}, \qquad q = C\Gamma^{-1/2}(\tau) e^{-I(\tau)/2}, \tag{3.25}$$

where

$$\Gamma(\tau) = \tau^2 + \tau - B_3, \qquad I(\tau) = \int \frac{d\tau}{\Gamma(\tau)}. \tag{3.26}$$

In the above relations τ represents the parameter while C is an arbitrary constant. Now, by substituting

$$\Gamma^{-1/2}(\tau) e^{-I(\tau)/2} = \Omega(\tau) \tag{3.27}$$

and taking into account (3.22), the above parametric solution takes the form

$$x = B_4 + B_5 C \tau \Omega(\tau), \quad q = C\Omega(\tau). \tag{3.28}$$

All the coefficients appearing through the analysis are listed in Appendix E.

3.3. The Parametric Solution for the Field's Components u_1, u_2

By combining (3.1), (3.5), (3.12), (3.17), and (3.24), we obtain

$$y^2(x) = \frac{q[r(x)]f(x)}{E} - \frac{P(x)}{3\alpha}. \tag{3.29}$$

Approximating linearly $P(x)$, namely,

$$P(x) = A_0^{22} + A_1^{22}x + \mathcal{O}\left(x^2\right) \tag{3.30}$$

and substituting q from (3.28), as well as f and E from (D.10a), (D.10b) (Appendix D), then (3.29) yields

$$3\alpha y^2 = \frac{\left[B_6 + C\tau\ \Omega(B_7 + B_8/\tau) + C^2\tau^2\Omega^2(B_9 + B_{10}/\tau) + C^3\tau^3\Omega^3(B_{11} + B_{12}/\tau) + B_{13}C^4\tau^4\Omega^4\right]}{(c_0 + c_1C\tau\ \Omega + c_2C^2\tau^2\Omega^2)}. \tag{3.31}$$

Furthermore, by solving the first part of (3.28) for $C\tau\Omega(\tau)$, we have

$$C\tau\Omega(\tau) = b_0 + b_1x. \tag{3.32}$$

Now, approximating linearly the powers of $C\tau\Omega(\tau)$ involved into (3.31), that is

$$\begin{aligned} C^2\tau^2\Omega^2 &= b_0^2 + 2b_0b_1x + \mathcal{O}\left(x^2\right), \\ C^3\tau^3\Omega^3 &= b_0^3 + 3b_0^2b_1x + \mathcal{O}\left(x^2\right), \\ C^4\tau^4\Omega^4 &= b_0^4 + 4b_0^3b_1x + \mathcal{O}\left(x^2\right) \end{aligned} \tag{3.33}$$

and substituting (3.32) and (3.33) into (3.31), then by replacing x with $u = u_1$ and y with $u_2 = f_2(u)$ and taking also into account (3.28), we conclude that

$$u(x_1, x_2, x_3) = \varphi_1(\tau) = B_4 + B_5C\tau\Omega(\tau), \tag{3.34a}$$

$$u_2^2(x_1, x_2, x_3) = \varphi_2^2(\tau, u) = \frac{1}{3\alpha}\frac{b_2 + b_3u + (b_4 + b_5u)(1/\tau)}{c_3 + c_4u}, \tag{3.34b}$$

with Ω as in (3.27) and α given by (2.22a). Equations (3.34a), (3.34b) constitute the approximate analytical parametric solution of the problem for u_1, u_2. As far as the component u_3 is concerned, combination of (2.18) and (2.19) results in

$$R_3f_2' + R_2\left(f_2' - 1\right) - R_3f_3' = 0, \qquad u_3 = f_3(u). \tag{3.35}$$

The above equation can be simplified a little if we neglect the last term in the left-hand side (it is of the form $(a + bf_3)f_3'^2$) by considering $f_3' \ll 1$. Anyhow we will not investigate (3.35) in this work.

Moreover, in order to evaluate the constant C involved into the parametric solution (3.34a), (3.34b), we need a boundary condition, that is, to locate to a point $\overline{x}_0 = (x_{10}, x_{20}, x_{30})$ where the field components $u_0 = u(\overline{x}_0)$, $u_{20} = u_2(\overline{x}_0)$ are known. Then, by solving (3.34b) for τ, we extract the corresponding value of the parameter $\tau_0 = \tau(\overline{x}_0)$, and finally, by using (3.34a), we arrive at

$$C = \frac{u_0 - B_4}{B_5 \tau_0 \Omega(\tau_0)}. \tag{3.36}$$

In the next section we apply the derived solution in the two-dimensional case of a flow past bodies with specific boundaries.

4. Parametric Solution for a 2-D Flow

As an application of the parametric solution obtained above for the plane case of (2.1), we consider the full nonlinear PDE governing the two-dimensional ($u_3 = 0$, $x_3 = 0$) steady small perturbation frictionless flow past a solid body surface [6], namely,

$$\begin{aligned}
&\left[1 - M^2 - (\gamma+1)M^2 u_1 - \frac{1}{2}(\gamma+1)M^2 u_1^2 - \frac{1}{2}(\gamma-1)M^2 u_2^2\right] u_{1,1} \\
&\quad + \left[1 - (\gamma-1)M^2 u_1 - \frac{1}{2}(\gamma-1)M^2 u_1^2 - \frac{1}{2}(\gamma+1)M^2 u_2^2\right] u_{2,2} \\
&\quad - M^2 (u_2 + u_1 u_2)(u_{1,2} + u_{2,1}) = 0,
\end{aligned} \tag{4.1}$$

where M is the correspondent to the uniform flow Mach number, which stands for the physical parameter of the problem, and γ is the ratio of the specific heats usually taken equal to 1.4; hence, the respective (dimensionless) coefficients A_0^{ij}, A_κ^{ij} and $A_{\kappa\lambda}^{ij}$ of (2.1) (the $u_{3,3}$ term vanishes) are given by

$$\begin{gathered}
A_0^{11} = 1 - M^2, \quad A_1^{11} = -2.4M^2, \quad A_{11}^{11} = -1.2M^2, \quad A_{22}^{11} = -0.2M^2, \\
A_2^{12} = A_2^{21} = -M^2, \quad A_{12}^{12} = A_{12}^{21} = -\frac{M^2}{2}, \\
A_0^{22} = 1, \quad A_1^{22} = -0.4M^2, \quad A_{11}^{22} = -0.2M^2, \quad A_{22}^{22} = -1.2M^2.
\end{gathered} \tag{4.2}$$

Relations (2.3a), (2.3b) also hold true. As mentioned in Section 2, the above equation represents a highly appropriate case, where the physical relevance of the imposed constraints (2.3a)-(2.3b)–(2.5) can be explained by a normalized physical background like the one generated by a uniform flow passing over a slightly "perturbed" surface, according to a specific geometry (see Figure 1, and the applications at the end of this section). Here, u_1, u_2 represent the dimensionless perturbation velocity components along the x_1, x_2 axes (see

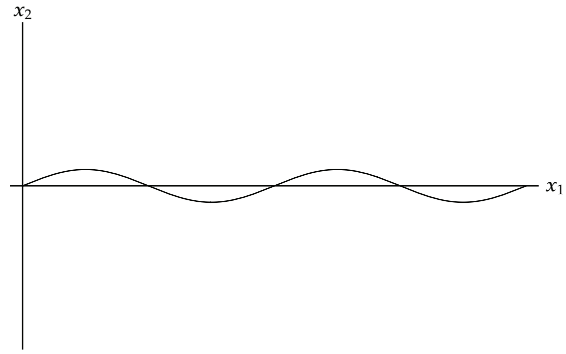

Figure 1: Orientation of the perturbed plane field with respect to the body's surface.

Figure 1), normalized by the uniform velocity of the steady flow, which is parallel to the x_1 direction in the physical plane.

A wavy surface (projection in the x_1x_2 plane) is shown in Figure 1, as a representative case able to produce small plane perturbations in the velocity field (the surface is supposed to have very small amplitude). Moreover, the irrotationality condition (2.9c) holds true and (2.10) and (2.11c) become

$$u = G\left[x_1 - \frac{R_1(u)}{R_2(u)}x_2\right], \tag{4.3}$$

$$u = G_3\left(x_1 + f_2' x_2\right), \tag{4.4}$$

respectively, where G, G_3 denote arbitrary functions and R_1, R_2 are as in (2.7a),(2.7b) (we mention that $u = u_1$). Obviously in the two-dimensional case, (2.9a),(2.9b) and (2.11a),(2.11b) become identities. Comparison between (4.3) and (4.4) results in (2.18).

If we refer now to the proper conditions restricted by the analysis (see Appendix D), we extract that the discriminant Δ of $P(u)$ (x has been replaced by u) is always positive ($\Delta > 0$), and, moreover, since $A_{11}^{22} < 0$, by obtaining the roots of $P(u)$, considering the respective to the Cases P-1 and P-2 intervals for u and assuming $u \leq 0.1$ ($u \ll 1$) as well, then a restriction to the domain of M is derived. More precisely, we find that formulae (3.34a), (3.34b) are valid for

$$M \leq \begin{cases} 0.71\left(u = 10^{-1}\right), \\ 0.74\left(u = 5 \times 10^{-2}\right), \\ 0.78\left(u = 10^{-2}\right), \\ 0.79\left(u = 10^{-4}\right). \end{cases} \tag{4.5}$$

Thus, for the specific 2-D steady flow field, the obtained approximate solution can be applied only to subsonic flows. Moreover, as far as the integral $I(\tau) = \int d\tau/\Gamma(\tau)$, involved into the function $\Omega(\tau)$, is concerned (see (3.26), (3.27)), the discriminant δ of $\Gamma(\tau)$ is evaluated negative ($\delta < 0$), and therefore the integral I is obtained as

$$-\frac{I(\tau)}{2} = -\frac{1}{\sqrt{-\delta}} \arctan \frac{1+2\tau}{\sqrt{-\delta}}. \tag{4.6}$$

Now, in order to construct an appropriate procedure to obtain $\tau(x_1, x_2)$, we consider the well-known boundary condition (see [6, page 208])

$$\overline{u} \cdot \nabla \varphi = 0, \tag{4.7}$$

where $\overline{u} = (1 + u(\xi_1, \xi_2), u_2(\xi_1, \xi_2))$ is the total dimensionless velocity vector of the flow at the solid surface, while $\varphi(\xi_1, \xi_2) = 0$ represents the equation of the *"surface line"*, that is, the section of the body's surface with the x_1x_2 plane. Here, ξ_1, ξ_2 denote the plane coordinates on this line with $\xi_1 \in [0, L]$, L being the body's length, and $|\xi_2| \ll 1$. Condition (4.7) states that at the surface of the body the direction of the flow must be tangential to the surface line. Developing (4.7), we arrive at

$$(1 + u(\xi_1, \xi_2))\varphi_{,\,\xi_1} + u_2(\xi_1, \xi_2)\varphi_{,\,\xi_2} = 0, \tag{4.8}$$

where by neglecting $u(u \ll 1)$, we obtain

$$u_2(\xi_1, \xi_2) = -\frac{\varphi_{,\,\xi_1}}{\varphi_{,\,\xi_2}} = \frac{d\xi_2}{d\xi_1} = g(\xi_1, \xi_2). \tag{4.9}$$

By squaring (4.9) and substituting u_2 and u by their parametric expressions (3.34b),(3.34a), we derive a transcendental equation for τ, namely,

$$\varphi_2^2[\tau, \varphi_1(\tau)] = g^2(\xi_1, \xi_2). \tag{4.10}$$

Thus, for a given pair (ξ_1, ξ_2) on the *surface line*, the solution of (4.10) results in $\tau(\xi_1, \xi_2)$, substitution of which into (3.34a), (3.34b) yields the perturbed velocity vector (u_1, u_2) of the flow at (ξ_1, ξ_2). In fact, in the case of the flow under consideration, only the perturbed velocity u is evaluated by, use of the extracted parametric solution, since due to (4.9) u_2 simply expresses approximately the slope of the *surface line*. Furthermore, assuming that the functions $u_i(x_1, x_2)$, $i = 1, 2$ are analytic inside a domain located on any line $x_1(= \xi_1) =$constant with $x_1 \in [0, L]$ (L represents the body's length) and x_2, slightly different from $\xi_2(x_2 > \xi_2)$, by developing in Taylor series around (ξ_1, ξ_2), we have

$$u_i(\xi_1, x_2) = u_i(\xi_1, \xi_2) + \frac{\partial u_i}{\partial x_2}(x_2 - \xi_2) + \frac{1}{2}\frac{\partial^2 u_i}{\partial x_2^2}(x_2 - \xi_2)^2 + \cdots, \quad i = 1, 2. \tag{4.11}$$

Taking into account the small perturbation theory (the derivatives involved into the series (4.11), as well as ξ_2, are very small compared to unity) and also that x_2 lies close enough to ξ_2, so that $(x_2 - \xi_2) \ll 1$, all the terms after the first in the right-hand side of (4.11) can

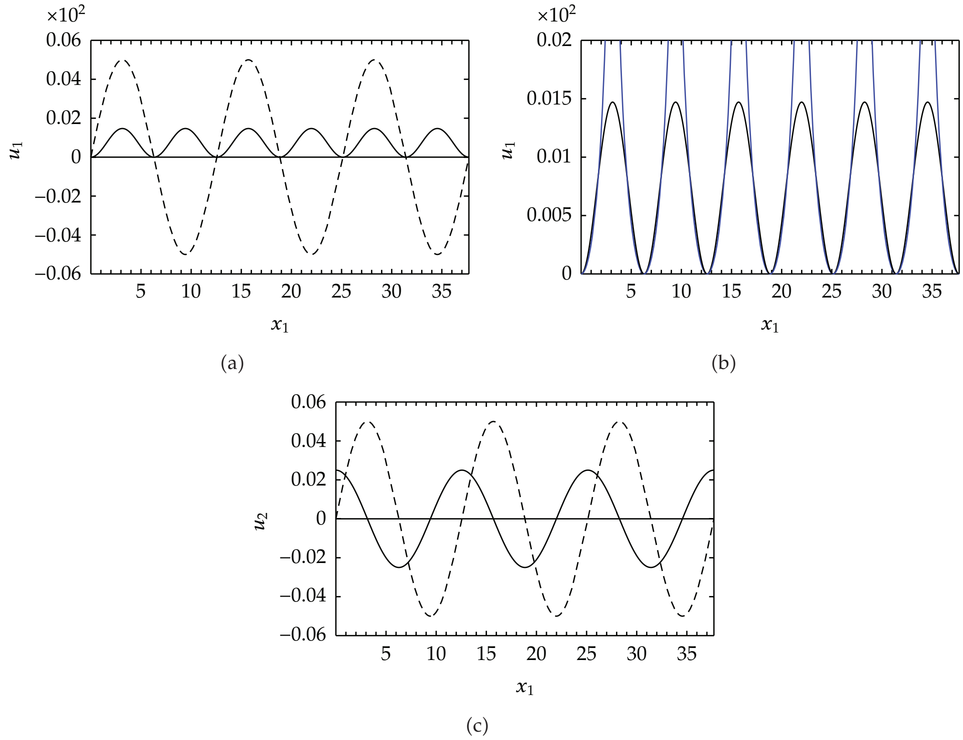

Figure 2: (a) $u_1(\times 10^2)$ versus x_1, $M = 0.7$, sinusoidal shape: $a_s = 0.05$, $b = 0.5$; (b) $u_1(\times 10^2)$ versus x_1, $M = 0.7$, sinusoidal shape: $a_s = 0.05$, $b = 0.5$, Ad hoc parametric solution (black line)—linearized equation (blue line); (c) u_2 versus x_1, $M = 0.7$, sinusoidal shape: $a_s = 0.05$, $b = 0.5$.

be neglected. Thus, we can approximately evaluate the perturbed plane flow field inside a thin zone over the body's surface. Obviously, the thickness of this zone depends on the order of magnitude of ξ_2. For example, if the boundary has a sinusoidal shape (one of the cases considered below), that is, $\xi_2 = a\sin(b\xi_1)$, $\xi_1 \in [0, L]$, $a, b \in (0,1)$ and if we take $a = 0.05$, then within the domain $\{(x_1, x_2) : x_1 \in [0, L],\ x_2 \in (\xi_2, \xi_2 + 2a)\}$ (a plane zone of thickness $2a$ (measured in the x_2-direction) with parallel sinusoidal boundaries), the error in (4.11) is $\mathcal{O}(x_2 - \xi_2) \le 10^{-2}$. Therefore, the above approximation is valid inside a zone over the solid surface of thickness less or equal to $2a(= 0.1)$. In addition, in order to obtain $\tau_0 = \tau(x_{10}, x_{20})$ and C (see the end of Section 3), the axes origin is used which is located at the point where the flow arrives at the body surface and consequently $(x_{10}, x_{20}) = (0,0)$, $(u_0, u_{20}) = (0, g(0,0))(u_0 = u(0,0),\ u_{20} = u_2(0,0))$, where g is given by (4.9). Therefore, by means of (3.34b) and (3.36) we conclude that

$$\tau(0,0) = \tau_0 = \frac{b_4}{3\alpha c_3 g^2(0,0) - b_2}, \qquad C = -\frac{B_4}{B_5 \tau_0 \Omega(\tau_0)} \tag{4.12}$$

with Ω provided by (3.26) and (3.27), where the integral I is evaluated by (4.6).

The derived solution, constructed by relations (3.34a), (3.34b), and (4.10), is applied to the two-dimensional steady frictionless flow past a boundary of sinusoidal (wavy wall), as well as of a parabolic shape. The problem is governed by (4.1). Especially for the "sinusoidal"

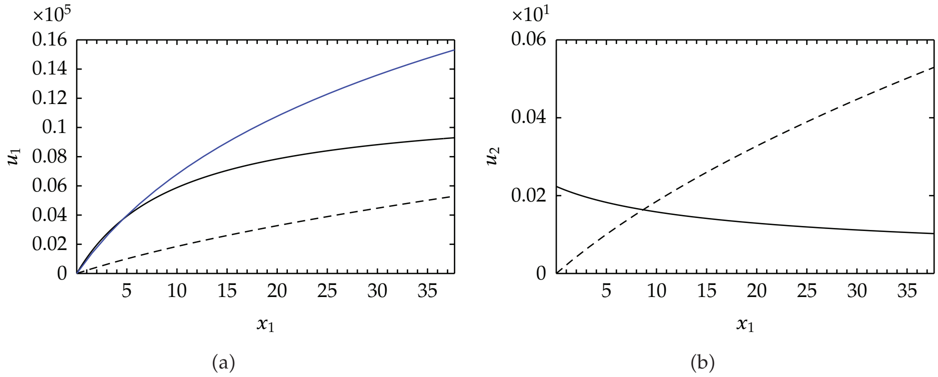

Figure 3: Solid lines: ad hoc parametric solution (black)—linearized equation (blue). (a) $u_1(\times 10^4)$ versus x_1, $M = 0.7$, parabolic shape: $a_p = 5 \times 10^3$, $\xi_{10} = 10$; (b) $u_2(\times 10)$ versus x_1, $M = 0.7$, parabolic shape: $a_p = 5 \times 10^3$, $\xi_{10} = 10$.

boundary problem, implicit solutions in the form of transcendental equations have been extracted in [5, Section 5, Equation (72) and (74), (77)], for the more simplified case of (2.1), where the only nonzero coefficients were A_0^{11}, A_1^{11}, and A_0^{22} [5, Equation (9)]. Here, boundary condition (4.7) holds with

$$\varphi_s(\xi_1, \xi_2) = \xi_2 - a_s \sin(b\xi_1) = 0, \quad a_s, b \in (0,1), \tag{4.13a}$$

$$\varphi_p(\xi_1, \xi_2) = a_p(\xi_2 + \xi_{20})^2 - \xi_1 - \xi_{10} = 0, \quad a_p, \xi_{10}, \xi_2 > 0, \ \xi_{20} = \left(\xi_{10}/a_p\right)^{1/2}, \tag{4.13b}$$

where φ_s and φ_p describe the sinusoidal and parabolic form of the surface, respectively, while a_s and a_p denote the amplitude and the curvature of the *surface line* in the cases under consideration. The low magnitude of a_s and the large magnitude of a_p allow the small perturbation theory to be applied. Additionally, in both (4.13a), (4.13b), we have $\xi_1 (= x_1) \in [0, L]$, where L stands for the assumed body's length, while in the "sinusoidal" case, the wavelength of the wavy surface is equal to $2\pi/b$.

As far as the graphs exhibited below are concerned, the "dashed" line represents the sinusoidal or the parabolic boundary, with geometries: $a_s = 0.05$, $b = 0.5$ (Figures 2(a), 2(c)—(4.13a)) and $a_p = 5 \times 10^3$, $\xi_{10} = 10$ (Figures 3(a), 3(b), (4.13b)). Moreover, the solid blue line in Figures 2(b) and 3(a) has been obtained as the solution for u_1 of the linearized form of (4.1), where the slope of the solid surface has been substituted for the component u_2. In both geometries, the body's length L is taken equal to 12π (three wavelengths in the wavy case) and the correspondent to the uniform unperturbed flow Mach number is set equal to 0.7. We note that by changing the values of the geometric parameters involved in (4.13a) and (4.13b), as well as the value of the Mach number, the perturbed field presents qualitatively similar graphs to those obtained here. Finally, as mentioned above the perturbed velocity u_2 is obtained as the slope of the surface.

In Figure 2(b), we note that the linear approximation is excellent throughout the body's length except in small intervals centered at the picks of the sinusoidal surface with radius approximately equal to $\pi/6(((2k+1)\pi - \pi/6,\ (2k+1)\pi + \pi/6),\ k = 0, 1, \ldots)$. Outside these locations the maximum error of the linear approximation (with respect to the ad hoc solution) is approximately equal to 6×10^{-5}, while inside these intervals the difference between the two solutions increases with x_1 moving towards the pick. Furthermore, concerning the comparison of the solutions in the case of the parabolic surface (Figure 3(a)), we find that for the considered body's length, the maximum error of the linear approximation is approximately equal to 1.5×10^{-6} (the error increasing with x_1).

5. Summary and Conclusion

In this paper an *ad hoc* analytical parametric solution has been obtained, concerning a nonlinear PDE governing a two-dimensional steady irrotational vector field. However, in Section 2 of this work the three-dimensional case is treated. As a result, we obtain a system of two (nonlinear) ODEs being equivalent to that of the original PDEs (including the irrotationality conditions). The analytical tools have been used in order to integrate the first ODE (concerning the two-dimensional case), in combination with linear approximations of certain polynomial and rational expressions, succeeded in transforming the above equation to a parametrically solvable Abel form. In particular, as established in Section 3, the "splitting" technique proved excellent in manipulating and transforming strongly nonlinear ODEs to integrable equations, and hence it may be considered representative of the general pattern of the analysis. Thus, we believe that the developed methodology, possibly modified, extended and enriched with more analytical techniques, can be a powerful tool of research on nonlinear problems in mechanics and physics.

Appendices

A. Lagrange Method for Quasilinear PDEs of First Order

According to this method, a general solution of the quasilinear equation

$$\begin{gathered} H_1(x_1,x_2,x_3,u)u_{,1} + H_2(x_1,x_2,x_3,u)u_{,2} + H_3(x_1,x_2,x_3,u)u_{,3} = R(x_1,x_2,x_3,u), \\ u = u(x_1,x_2,x_3), \quad u_{,i} = \frac{\partial u}{\partial x_i}, \quad i = 1,2,3 \end{gathered} \tag{A.1}$$

has the form

$$G(w_1,w_2,w_3) = 0, \tag{A.2}$$

where

$$w_1(x_1,x_2,x_3,u) = a, \qquad w_2(x_1,x_2,x_3,u) = b, \qquad w_3(x_1,x_2,x_3,u) = c, \tag{A.3}$$

with a, b, c being constants, are solutions of the subsidiary Lagrange equations

$$\frac{dx_1}{H_1} = \frac{dx_2}{H_2} = \frac{dx_3}{H_3} = \frac{du}{R} \tag{A.4}$$

and G is an arbitrary function possessing continuous partial derivatives with respect to its arguments.

B. Analytical Parametric Solution of the Equation $yy'_x - y = Ax + B$

It is well known that the general ODE of the first order

$$F\left(x, y, y'_x\right) = 0 \tag{B.1}$$

can accept a parametric solution of the form

$$x = x(t), \qquad y = y(t), \tag{B.2}$$

in case where the following system can be integrated, namely,

$$\frac{dx}{dt} = -\frac{F_{,t}}{F_{,x} + tF_{,y}}, \tag{B.3a}$$

$$\frac{dy}{dt} = t\frac{dx}{dt} = -\frac{tF_{,t}}{F_{,x} + tF_{,y}}, \tag{B.3b}$$

where the notation $F'_x = dF/dx$, $F_{,x} = \partial F/\partial x$ has been adopted. The above system is obtained by the substitution of $y'_x = t$ and differentiation of (B.1) with respect to t. In particular, if t can be eliminated from (B.2), then a closed-form solution of (B.1) is extracted.

Therefore, as far as the Abel equation $yy'_x - y = Ax + B$ is concerned, since it is solvable for x, that is,

$$x = \frac{t-1}{A}y - \frac{B}{A}, \tag{B.4}$$

then (B.3b) is considered, namely, ($F(x, y, t) = yt - y - Ax - B = 0$)

$$\frac{dy}{dt} = -\frac{ty}{t^2 - t - A}. \tag{B.5}$$

Integration of (B.5) in combination with (B.4) results in

$$x = \frac{C}{A}(t-1)\exp\left(-\int \frac{tdt}{t^2 - t - A}\right) - \frac{B}{A}, \qquad y = C\exp\left(-\int \frac{tdt}{t^2 - t - A}\right) \tag{B.6}$$

with C being an arbitrary constant. Moreover, by substituting $\tau = t-1$ and taking into account [19, Integral 2.175.1], the parametric solution (B.6) takes the form

$$x = \frac{C}{A}\tau\left(\tau^2 + \tau - A\right)^{-1/2} \exp\left(-\frac{1}{2}\int \frac{d\tau}{\tau^2 + \tau - A}\right) - \frac{B}{A},$$
$$y = C\left(\tau^2 + \tau - A\right)^{-1/2} \exp\left(-\frac{1}{2}\int \frac{d\tau}{\tau^2 + \tau - A}\right). \tag{B.7}$$

C. Approximations due to the Weakness of the Field

The weakness of the field under consideration, especially of the $u_1(= u)$ coordinate, that is, $u \ll 1$, allows us to establish the following approximations.

(1) We linearly approximate all the polynomials $p\ (x)$ (x represents u) of degree greater or equal than two, namely,

$$p(x) = a + bx + \mathcal{O}\left(x^2\right). \tag{C.1}$$

(2) Considering the ratio of binomials

$$p_1(x) = \frac{\alpha + \beta x}{\gamma + \delta x}, \tag{C.2}$$

we evaluate

$$p_1(x) = \frac{(\alpha + \beta x)(\gamma - \delta x)}{\gamma^2 - \delta^2 x^2} = \frac{\alpha\gamma + (\beta\gamma - \alpha\delta)x + \mathcal{O}(x^2)}{\gamma^2 + \mathcal{O}(x^2)}, \tag{C.3}$$

and therefore we obtain

$$p_1(x) \cong \frac{\alpha}{\gamma} + \frac{1}{\gamma}\left(\beta - \frac{\alpha\delta}{\gamma}\right)x. \tag{C.4}$$

D. Expressions for $f(x)$ and $E(x)$

In this appendix, we extract appropriate formulas for the function $f(x)$, appearing in (3.2), as well as for the function $E(x) = \exp(\kappa(x)/12)$, involved into the reduction procedure of the Abel equation (3.18) [8, Chapter 1, Section 3.4]. Thus, by considering the function $\kappa(x) = \int \rho_{11}(x)dx$, given from (3.2) and substituting ρ_{11} from (2.21), by means of [9, Expression 2.175.1], we arrive at

$$\kappa(x) = A_7 \ln[P(x)] + A_8 \int \frac{dx}{P(x)}, \quad P(x) = A_0^{22} + A_1^{22}x + A_{11}^{22}x^2. \tag{D.1}$$

The coefficients involved in various expressions appearing in this appendix are listed in Appendix E. We mention that all these coefficients (appeared through the analytical procedure in this work) are functions of the physical parameter(s), of the problem. Therefore, for this (these) parameter(s) taking values such that the discriminant Δ of $P(x)$ becomes positive ($\Delta = (A_1^{22})^2 - 4A_0^{22}A_{11}^{22}$) if ρ_1, ρ_2 represent the roots of $P(\rho_{1,2} = (-A_1^{22} \pm \sqrt{\Delta}2A_{11}^{22})$ and considering the following cases:

Case P-1

$$\begin{aligned} &\Delta > 0, \quad A_{11}^{22} < 0(\rho_1 < \rho_2) : \rho_1 < x < \frac{\rho_2 - \varepsilon}{3}, \\ &\Delta > 0, \quad A_{11}^{22} > 0(\rho_2 < \rho_1) : \frac{\rho_2 + \varepsilon}{3} < x < \rho_1. \end{aligned} \tag{D.2}$$

Case P-2

$$\begin{aligned} &\Delta > 0, \quad A_{11}^{22} < 0 : x < \rho_1 \quad \text{or } x > \rho_2 + \varepsilon, \\ &\Delta > 0, \quad A_{11}^{22} > 0 : x < \rho_2 - \varepsilon \quad \text{or } x > \rho_1 \end{aligned} \tag{D.3}$$

with $\varepsilon = \sqrt{\Delta}/|A_{11}^{22}|$, then by elementary algebra (using [9, Expression 2.172]) we can easily prove the following Lemma.

Lemma D.1. *If Case P-1 or P-2 is valid, then the integral $\int dx/P$ can be written in the form:*

$$\int \frac{dx}{P(x)} = \frac{1}{\sqrt{\Delta}} \ln[1 - \lambda(x)], \quad |\lambda(x)| < 1 \tag{D.4}$$

with

$$\lambda(x) = \frac{\lambda_0 + \lambda_1 x}{\mu_0 + \mu_1 x}. \tag{D.5}$$

The coefficients λ_0, λ_1 are different as regards these two cases, while μ_0, μ_1 are common (see Appendix E). Thus, substituting (D.4) into (D.1), we obtain

$$\begin{aligned} f(x) &= \exp\left(-\frac{\kappa(x)}{2}\right) = P^{A_9}(x)[1 - \lambda(x)]^{A_{10}}, \\ E &= \exp\left(\frac{\kappa(x)}{12}\right) = P^{a_9}(x)[1 - \lambda(x)]^{a_{10}}. \end{aligned} \tag{D.6}$$

Then by writing P in the form $P(x) = A_0^{22}(1 + A_{11}x + \overline{A}_{11}x^2)$ and developing in power series (assuming that $|A_{11}x + \overline{A}_{11}x^2| < 1$) up to the first order, we take

$$P^{A_9}(x) = A_{12} + A_{13}x + \mathcal{O}\left(x^2\right), \qquad P^{a_9}(x)a_{12} + a_{13}x + \mathcal{O}\left(x^2\right). \tag{D.7}$$

Furthermore, by applying (C.4) to (D.5), we arrive at

$$\lambda(x) = \lambda_2 + \lambda_3 x, \qquad \lambda^2(x) = \lambda_2^2 + 2\lambda_2\lambda_3 x + \mathcal{O}\left(x^2\right). \tag{D.8}$$

Then developing $[1 - \lambda(x)]^{A_{10}}$ and $[1 - \lambda(x)]^{a_{10}}$ up to the second order and substituting (D.8), we conclude that

$$[1 - \lambda(x)]^{A_{10}} = \lambda_4 + \lambda_5 x + \mathcal{O}\left(x^2\right), \qquad [1 - \lambda(x)]^{a_{10}} = \mu_4 + \mu_5 x + \mathcal{O}\left(x^2\right). \tag{D.9}$$

Finally, substitution of (D.7) and (D.9) into (D.6) results in

$$f\ (x) = (A\ _{12} + A\ _{13}x)(\lambda_4 + \lambda_5 x), \tag{D.10a}$$

$$E = (a_{12} + a_{13}x)(\mu_4 + \mu_5 x). \tag{D.10b}$$

E. List of Coefficients

$$\alpha = A_{22}^{22}, \quad \beta = A_{22}^{11}, \quad A_2 = A_2^{12} + A_2^{21}, \quad A_3 = 2\left(A_{12}^{12} + A_{12}^{21}\right), \quad A_4 = A_2^2 - 4\beta A_0^{22},$$

$$A_5 = 2A_2A_3 - 4\beta A_1^{22}, \quad A_6 = {A_3}^2 - 4\beta A_{11}^{22}, \quad A_7 = \frac{A_3}{2A_{11}^{22}}, \quad A_8 = A_2 - \frac{A_3A_1^{22}}{2A_{11}^{22}},$$

$$A_9 = -\frac{A_7}{2}, \quad a_9 = \frac{A_7}{12}, \quad A_{10} = -\frac{A_8}{2\sqrt{\Delta}}, \quad a_{10} = \frac{A_8}{12\sqrt{\Delta}},$$

$$A_{11} = \frac{A_1^{22}}{A_0^{22}}, \quad \overline{A}_{11} = \frac{A_{11}^{22}}{A_0^{22}}, \quad A_{12} = \left(A_0^{22}\right)^{A_9}, \quad A_{13} = A_9A_{11}A_{12},$$

$$a_{12} = \left(A_0^{22}\right)^{a_9}, \quad a_{13} = a_9A_{11}a_{12},$$

$$\mu_0 = A_{221} + \sqrt{\Delta}, \qquad \mu_1 = 2A_{11}^{22},$$

$$\lambda_0 = 2A_1^{22}, \qquad \lambda_1 = 4A_{11}^{22}, \quad (Case\ P-1)$$

$$\lambda_0 = 2\sqrt{\Delta}, \qquad \lambda_1 = 0, \quad (Case\ P-2)$$

$$\lambda_2 = \frac{\lambda_0}{\mu_0}, \quad \lambda_3 = \frac{\lambda_1 - \lambda_2\mu_1}{\mu_0}, \quad \lambda_4 = 1 - A_{10}\lambda_2\left(1 - \frac{A_{10}-1}{2}\lambda_2\right), \quad \lambda_5 = -A_{10}\lambda_3(1-(A_{10}-1)\lambda_2),$$

$$\mu_4 = 1 - a_{10}\lambda_2\left(1 - \frac{a_{10}-1}{2}\lambda_2\right), \qquad \mu_5 = -a_{10}\lambda_3(1-(a_{10}-1)\lambda_2),$$

$$F_{10} = a_{12}\left(5A_2^2 - 12A_4 + 12A_2A_1^{22}\right)\mu_4,$$

$$F_{11} = 2a_{12}\left[-6A_5 + 6A_3A_1^{22} + A_2\left(5A_3 + 12A_{11}^{22}\right)\right]\mu_4 + \left(\frac{a_{13}}{a_{12}} + \frac{\mu_5}{\mu_4}\right)F_{10},$$

$$Q_0 = 36\alpha A_2A_{12}\lambda_4, \qquad Q_1 = 36\alpha(A_3A_{12}\lambda_4 + A_2A_{13}\lambda_4 + A_2A_{12}\lambda_5),$$

$$F_{00} = a_{12}A_0^{22}\left(A_2^2 + 6A_4 - 72\alpha A_0^{11}\right)\mu_4,$$

$$F_{01} = 2a_{12}A_0^{22}\left(A_2A_3 + 3A_5 - 36\alpha A_1^{11}\right)\mu_4 + \left(\frac{a_{13}}{a_{12}} + \frac{\mu_5}{\mu_4} + \frac{A_1^{22}}{A_0^{22}}\right)F_{00},$$

$$M_0 = \frac{A_{12}\lambda_4}{a_{12}\mu_4}F_{10}, \quad M_1 = \frac{A_{13}\lambda_4 + A_{12}\lambda_5}{a_{12}\mu_4}F_{10} + 2A_{12}\left(-6A_5 + 6A_3A_1^{22} + 5A_2A_3 + 12A_2A_{11}^{22}\right)\lambda_4,$$

$$B_0 = \frac{F_{10}}{Q_0}, \quad B_1 = \frac{F_{11} - B_0Q_1}{Q_0}, \quad B_2 = -\frac{2}{3\alpha}\frac{F_{00}}{M_0}, \quad \overline{B}_3 = -\frac{2}{3\alpha}\frac{1}{M_0}\left(F_{01} - \frac{F_{00}M_1}{M_0}\right),$$

$$B_3 = \frac{\overline{B}_3}{B_0}, \qquad B_4 = -\frac{B_2}{\overline{B}_3}, \qquad B_5 = \frac{1}{\overline{B}_3},$$

$$B_6 = -(a_{12} + a_{13}B_4)\left(A_0^{22} + A_1^{22}B_4 + A_{11}^{22}{B_4}^2\right)(\mu_4 + B_4\mu_5),$$

$$B_7 = -B_5\Big[a_{12}\left(A_1^{22} + 2A_{11}^{22}B_4\right)\mu_4 + \left(A_0^{22} + 2A_1^{22}B_4 + 3A_{11}^{22}B_4^2\right)(a_{13}\mu_4 + a_{12}\mu_5)$$
$$+a_{13}B_4\left(2A_0^{22} + 3A_1^{22}B_4 + 4A_{11}^{22}B_4^2\right)\mu_5\Big],$$

$$B_8 = 3\alpha(A_{12} + A_{13}B_4)(\lambda_4 + B_4\lambda_5),$$

$$B_9 = -B_5^2\left[a_{12}A_{11}^{22}\mu_4 + \left(A_1^{22} + 3A_{11}^{22}B_4\right)(a_{13}\mu_4 + a_{12}\mu_5) + a_{13}\left(A_0^{22} + 3A_1^{22}B_4 + 6a_{13}A_{11}^{22}B_4^2\right)\mu_5\right],$$

$$B_{10} = 3\alpha B_5(A_{13}\lambda_4 + A_{12}\lambda_5 + 2A_{13}B_4\lambda_5),$$

$$B_{11} = -B_5^3\left[a_{13}A_{11}^{22}\mu_4 + \left(a_{13}A_1^{22} + a_{12}A_{11}^{22} + 4a_{13}A_{11}^{22}B_4\right)\mu_5\right],$$

$$B_{12} = 3\alpha A_{13}B_5^2\lambda_5, \quad B_{13} = -a_{13}A_{11}^{22}{B_5}^4\mu_5,$$

$$c_0 = (a_{12} + a_{13}B_4)(\mu_4 + B_4\mu_5), \quad c_1 = B_5\left[a_{13}\mu_4 + (a_{12} + 2a_{13}B_4)\mu_5\right], \quad c_2 = a_{13}B_5^2\mu_5,$$

$$b_0 = -\frac{B_4}{B_5}, \qquad b_1 = \frac{1}{B_5}, \qquad b_2 = B_6 + b_0 B_7 + b_0^2 B_9 + b_0^3 B_{11} + b_0^4 B_{13},$$

$$b_3 = b_1 \left(B_7 + 2b_0 B_9 + 3b_0^2 B_{11} + 4b_0^3 B_{13} \right), \qquad b_4 = b_0 B_8 + b_0^2 B_{10} + b_0^3 B_{12},$$

$$b_5 = b_1 \left(B_8 + 2b_0 B_{10} + 3b_0^2 B_{12} \right),$$

$$c_3 = c_0 + b_0 c_1 + b_0^2 c_2, \qquad c_4 = b_1 (c_1 + 2b_0 c_2). \tag{E.1}$$

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