Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2011, Article ID 291028, 18 pages
doi:10.1155/2011/291028

Research Article

A Class of Negatively Fractal Dimensional Gaussian Random Functions

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Received 4 October 2010; Accepted 15 November 2010

Academic Editor: Cristian Toma

Let $x(t)$ be a locally self-similar Gaussian random function. Denote by $r_{xx}(\tau)$ the autocorrelation function (ACF) of $x(t)$. For $x(t)$ that is sufficiently smooth on $(0, \infty)$, there is an asymptotic expression given by $r_{xx}(0) - r_{xx}(\tau) \sim c|\tau|^{\alpha}$ for $|\tau| \to 0$, where c is a constant and α is the fractal index of $x(t)$. If the above is true, the fractal dimension of $x(t)$, denoted by D, is given by $D = D(\alpha) = 2 - \alpha/2$. Conventionally, α is strictly restricted to $0 < \alpha \leq 2$ so as to make sure that $D \in [1, 2)$. The generalized Cauchy (GC) process is an instance of this type of random functions. Another instance is fractional Brownian motion (fBm) and its increment process, that is, fractional Gaussian noise (fGn), which strictly follow the case of $D \in [1, 2)$ or $0 < \alpha \leq 2$. In this paper, I claim that the fractal index α of $x(t)$ may be relaxed to the range $\alpha > 0$ as long as its ACF keeps valid for $\alpha > 0$. With this claim, I extend the GC process to allow $\alpha > 0$ and call this extension, for simplicity, the extended GC (EGC for short) process. I will address that there are dimensions $0 \leq D(\alpha) < 1$ for $2 < \alpha \leq 4$ and further $D(\alpha) < 0$ for $4 < \alpha$ for the EGC processes. I will explain that $x(t)$ with $1 \leq D < 2$ is locally rougher than that with $0 \leq D < 1$. Moreover, $x(t)$ with $D < 0$ is locally smoother than that with $0 \leq D < 1$. The local smoothest $x(t)$ occurs in the limit $D \to -\infty$. The focus of this paper is on the fractal dimensions of random functions. The EGC processes presented in this paper can be either long-range dependent (LRD) or short-range dependent (SRD). Though applications of such class of random functions for $D < 1$ remain unknown, I will demonstrate the realizations of the EGC processes for $D < 1$. The above result regarding negatively fractal dimension on random functions can be further extended to describe a class of random fields with negative dimensions, which are also briefed in this paper.

1. Introduction

Conventionally, for a time series, or a random function $x(t)$, such as fully developed ocean wave series, we need not discuss its fractal dimension, as can be judged from the power spectra discussed by Massel [1], the Specialist Committee on Waves of the 23rd ITTC [2], Li [3]. However, in some cases, we have to consider the fractal dimension of a random function, such as oceanic monthly temperature; see Alvarez-Ramirez et al. [4]. As a matter of fact, time series with fractal dimensions are observed in many fields of sciences and technologies; see, for example, Beran [5], Mandelbrot [6], Korvin [7], West and Deering [8], Schreiber [9], Abry et al. [10], Werner [11], Levy-Vehel [12], Cattani [13, 14], and references therein.

Denote the fractal dimension of $x(t)$ by D, which measures the local roughness or local irregularity or local self-similarity of $x(t)$; see Mandelbrot [15], Li [16]. Then, in the standard fractal time series, one has the positive dimension given by

$$D \in [1, 2). \tag{1.1}$$

The literature with respect to the time series with $D \in [1, 2)$ is rich; see, for example, [4–68], simply a drop in the bucket in the field. However, how to represent $D \in (0, 1)$ and $D < 0$ in particular for time series remains an open problem. This paper gives a solution to this problem in the case that $x(t)$ is a locally self-similar Gaussian random function the ACF of which follows the form of the GC process.

There are various definitions of dimensions (Mandelbrot [15]), such as the Minkowski dimension, the Rényi dimension, the Hausdorff dimension, the packing dimension, the box-counting dimension, the correlation dimension. Those dimensions may not be equal for a specific object but this does not matter. What an important thing is whether there are objects the dimensions of which are negative and how to represent negative dimensions of objects (Mandelbrot [69]). In this regard, the research conducted by Mandelbrot and his colleagues reveals a new outlook in negative dimensions in geometry, see Mandelbrot [69–75], applications of which were found to turbulence, see Molenaar et al. [76], Chhabra and Sreenivasan [77, 78].

Note that the negative dimensions described based on the Duplantier's function are from a view of geometry. Differing from the work by Mandelbrot [69–75], this paper addresses time series or random functions with negative dimensions. It is well known that commonly used models of fractal time series are fractional Brownian motion (fBm) and fractional Gaussian noise (fGn) that is the increment process of fBm (Mandelbrot [17]). For fBm as well as fGn, dimensions are restricted to be positive. As a matter of fact, denote by D_{fGn} and H_{fGn} the fractal dimension of fGn and its Hurst parameter, respectively. Denote fGn by $x_{\mathrm{fGn}}(t)$. Then, the autocorrelation function (ACF) of fGn follows

$$E[x_{\mathrm{fGn}}(t)x_{\mathrm{fGn}}(t+\tau)] \approx H_{\mathrm{fGn}}(2H_{\mathrm{fGn}}-1)\tau^{(2H_{\mathrm{fGn}}-2)}. \tag{1.2}$$

The right side of the above expression requires $0 < H_{\mathrm{fGn}} < 1$; see Mandelbrot [17]. Thus, $1 < D_{\mathrm{fGn}} < 2$ because $D_{\mathrm{fGn}} = 2 - H_{\mathrm{fGn}}$. In the LRD case, $0.5 < H_{\mathrm{fGn}} < 1$ and $1.5 < D_{\mathrm{fGn}} < 2$. As a result, I have the following note.

Note 1. The dimension of fGn is never negative. That is, $1 \le D_{\mathrm{fGn}} < 2$.

There are various types of fractal time series, such as fGn, alpha-stable processes, Levy processes. However, not all time series have negative dimensions. For example, fGn does not have negative dimensions. However, there may exist time series that have negative dimensions. This research of mine restricts my study to a specific class of Gaussian random functions. It is the extension of the GC process that were reported by Gneiting and Schlather [79], Lim and Li [80], Li and Lim [81], and Li et al. [82]. The extended GC (EGC) processes may have negative dimensions.

Let D_{GC} and H_{GC} be the fractal dimension and the Hurst parameter of the GC process, respectively. Then, D_{GC} and H_{GC} are independent of each other. In the previous research regarding the GC process, only the case of $1 < D_{\mathrm{GC}} < 2$ was discussed, see [79–81], Li [52], Li and Lim [83, 84]. In this paper, I extend the GC process such that $D_{\mathrm{EGC}} \in (0, 1)$ and $D_{\mathrm{EGC}} < 0$, where D_{EGC} is the fractal dimension of the EGC process in addition to the traditional case of $D_{\mathrm{EGC}} \in [1, 2)$. The term EGC process means that it is a class of processes that is based on the GC process but extended to the case of $D_{\mathrm{EGC}} < 1$, simply for the purpose of distinguishing it from the standard GC process.

The rest of paper is organized as follows. The class of negatively dimensional random functions, that is, EGC processes, is addressed in Section 2. Discussions are given in Section 3. Extending the presented class of negative dimensional random functions to the corresponding random fields is briefed in Section 4. Finally, Section 5 concludes the paper.

2. Extended Generalized Cauchy (EGC) Processes

Denote by (Ω, T, P) the probability space. Then, $x(t, \zeta)$ is said to be a stochastic process when the random variable x represents the value of the outcome of an experiment T for every time t, where Ω represents the sample space, T is the event space or sigma algebra, and P the probability measure.

As usual, $x(t, \zeta)$ is simplified to be written as $x(t)$, that is, the event space is usually omitted. Denote by $P(x)$ the probability function of x. Then, one can define the general nth order, time varying, joint distribution function $P(x_1, \ldots, x_n;\ t_1, \ldots, t_n)$ for the random variables $x(t_1), \ldots, x(t_n)$. The joint distribution density function is written by

$$p(x_1, \ldots, x_n;\ t_1, \ldots, t_n) = \frac{\partial^n P(x_1, \ldots, x_n;\ t_1, \ldots, t_n)}{\partial x_1 \cdots \partial x_n}. \tag{2.1}$$

In this paper, only the first- and second-order properties of processes are considered instead of the complete joint distribution function. Moreover, this research only considers Gaussian processes. Gaussian processes can be completely determined by the second-order properties, more precisely, mean and ACF, see Papoulis [85]. Without generality losing, this paper only considers processes with mean zero.

Note that $(1 + |\tau|^{\alpha})^{-\beta/\alpha}$ is a valid ACF for $\alpha > 0$ and $\beta > 0$. Denote $(1 + |\tau|^{\alpha})^{-\beta/\alpha}$ by $r_{\mathrm{EGC}}(\tau)$ that is the ACF of a Gaussian random function denoted by $x_{\mathrm{EGC}}(t)$. That is,

$$r_{\mathrm{EGC}}(\tau) = E[x_{\mathrm{EGC}}(t + \tau) x_{\mathrm{EGC}}(t)] = \left(1 + |\tau|^{\alpha}\right)^{-\beta/\alpha}, \quad \alpha > 0,\ \beta > 0. \tag{2.2}$$

Then, we call a random function $x(t)$ as an EGC process if it is a stationary Gaussian centred process with the ACF given by (2.2).

Usually, the norm of a random function $x(t)$ is expressed by $E[x(t)x(t)] = \|x(t)\|^2$, see, for example, Cramer [86, 87], Liu [88], Gelfand and Vilenkin [89], and Adler et al. [90]. However, for the EGC process, $E[x_{\mathrm{EGC}}(t) x_{\mathrm{EGC}}(t)] = r_{\mathrm{EGC}}(0) = 1$ regardless of the values of α and β. Therefore, it is inconvenient to use $\|x_{\mathrm{EGC}}(t)\|$. For this reason, I utilize $\|r_{\mathrm{EGC}}(\tau)\|$ rather than $\|x_{\mathrm{EGC}}(t)\|$ in this paper. The norm $\|r_{\mathrm{EGC}}(\tau)\|$ is suitable for our research purpose. In fact, since $x_{\mathrm{EGC}}(t)$ is Gaussian, it is uniquely determined by $r_{\mathrm{EGC}}(\tau)$ and vice versa. Note that

$$\int_{-\infty}^{\infty} |r_{\mathrm{EGC}}(\tau)|^2 d\tau = \infty \quad \text{for } 0 < \beta < 0.5. \tag{2.3}$$

Thus, I need to express $\|r_{\mathrm{EGC}}(\tau)\|$ in the domain of generalized functions.

Definition 2.1 (see Griffel [91]). A function of rapid decay is a smooth function $\phi : \mathbb{R} \to \mathbb{C}$ such that $t^n \phi^{(r)}(t) \to 0$ as $t \to \pm\infty$ for all $n, r \geq 0$, where $\mathbb{C}$ is the space of complex numbers. The set of all functions of rapid decay is denoted by $\mathbf{S}$.

Lemma 2.2 (see Griffel [91]). *Every function belonging to* $\mathbf{S}$ *is absolutely integrable.*

Now, define the norm and inner product of $r \in H_{\mathrm{EGC}}$ by

$$\|r_{\mathrm{EGC}}\|^2 = (r_{\mathrm{EGC}}, r_{\mathrm{EGC}}) = \int_{-\infty}^{+\infty} [r_{\mathrm{EGC}}(u)]^2 g(u) du, \tag{2.4}$$

where $g \in \mathbf{S}$. Combining any $r \in H_{\mathrm{EGC}}$ with its limit yields that H_{EGC} is a Hilbert space. Denote it by

$$H_{\mathrm{EGC}} = \left\{ r_{\mathrm{EGC}};\ \|r_{\mathrm{EGC}}\|^2 < \infty;\ \alpha > 0,\ \beta > 0 \right\}. \tag{2.5}$$

For $0 < \alpha \leq 2$, I express the subspace of H_{EGC} by

$$H_{\mathrm{EGC1}} = \left\{ r_{\mathrm{EGC}};\ \|r_{\mathrm{EGC}}\|^2 < \infty;\ 0 < \alpha \leq 2,\ 0 < \beta \right\}. \tag{2.6}$$

Denote by H_{GC} the space of the standard GC process. Then, one immediately sees that H_{GC} is a subspace of H_{EGC}. More precisely,

$$H_{\mathrm{GC}} = H_{\mathrm{EGC1}}. \tag{2.7}$$

In the case of $2 < \alpha \leq 4$, I denote another subspace of H_{EGC} by

$$H_{\mathrm{EGC2}} = \left\{ r_{\mathrm{EGC}};\ \|r_{\mathrm{EGC}}\|^2 < \infty;\ 2 < \alpha \leq 4,\ 0 < \beta \right\}. \tag{2.8}$$

Further, I denote the subspace of H_{EGC} for $\alpha > 4$ by

$$H_{\mathrm{EGC3}} = \left\{ r_{\mathrm{EGC}};\ \|r_{\mathrm{EGC}}\|^2 < \infty;\ 4 < \alpha,\ 0 < \beta \right\}. \tag{2.9}$$

Then, I have the remark below.

Remark 2.3. One has $H_{\mathrm{EGC}} = H_{\mathrm{EGC1}} \cup H_{\mathrm{EGC2}} \cup H_{\mathrm{EGC3}}$.

Recall that each $r_{\mathrm{EGC}} \in H_{\mathrm{EGC}}$ corresponds to a Gaussian process, see Gelfand and Vilenkin [89, Chapter 4], Kanwal [92]. Its ACF is given by (2.2). However, the dimensions of processes in H_{EGC2} and H_{EGC3} are undefined, more precisely, unknown. The following theorems will describe the dimensions of processes in H_{EGC2} and H_{EGC3}.

Theorem 2.4. *Processes in H_{EGC2} have the dimensions less than one and greater than or equal to zero.*

Proof. Denote a process in H_{EGC2} by $x_{\mathrm{EGC2}}(t)$. Let an ACF in H_{EGC2} be r_{EGC2}. Then, taking into account the definition of the local self-similarity provided by Kent and Wood [93], Hall and Roy [94], Chan et al. [95], Adler [96], one says that a Gaussian stationary process is locally self-similar of order α if its ACF satisfies for $\tau \to 0$

$$r_{\mathrm{EGC2}}(\tau) = E[x_{\mathrm{EGC2}}(t+\tau)x_{\mathrm{EGC2}}(t)] = 1 - \frac{\beta}{\alpha}|\tau|^{\alpha}\left\{1 + O\left(|\tau|^{\alpha}\right)\right\}, \quad 2 < \alpha \le 4,\ 0 < \beta. \tag{2.10}$$

Therefore, according to [93–96], the fractal dimension denoted by D_{EGC2} is given by

$$0 \le D_{\mathrm{EGC2}} = \left(1 - \frac{\alpha}{2}\right) < 1. \tag{2.11}$$

This yields Theorem 2.4. □

Let $x_{\mathrm{EGC3}}(t)$ be a process in H_{EGC3}. Denote the ACF of $x_{\mathrm{EGC3}}(t)$ by r_{EGC3}. Then, the following theorem gives the negative dimensions of $x_{\mathrm{EGC3}}(t)$.

Theorem 2.5. *Processes in H_{EGC3} have negative dimensions.*

Proof. Similar to (2.10), one has

$$r_{\mathrm{EGC3}}(\tau) = E[x_{\mathrm{EGC3}}(t+\tau)x_{\mathrm{EGC3}}(t)] = 1 - \frac{\beta}{\alpha}|\tau|^{\alpha}\left\{1 + O\left(|\tau|^{\alpha}\right)\right\}, \quad 4 < \alpha,\ 0 < \beta. \tag{2.12}$$

Then, following (2.12), we have

$$D_{\mathrm{EGC3}} = \left(1 - \frac{\alpha}{2}\right) < 0. \tag{2.13}$$

This completes the proof of Theorem 2.5. □

Remark 2.6. For an EGC process $x_{\mathrm{EGC}}(t)$ with $0 < \alpha$ and $0 < \beta$, the fractal dimension of $x_{\mathrm{EGC}}(t)$ in general satisfies

$$D_{\mathrm{EGC}} < 2. \tag{2.14}$$

Theorem 2.7. *The power spectrum density (PSD) function of the EGC process is given*

$$
\begin{aligned}
S_{EGC}(\omega) = {} & \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\beta/\alpha + k)}{\pi \Gamma(\beta/\alpha)\Gamma(1+k)} I_1(\omega) * \mathrm{Sa}(\omega) \\
& + \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\beta/\alpha + k)}{\pi \Gamma(\beta/\alpha)\Gamma(1+k)} [\pi I_2(\omega) - I_2(\omega) * \mathrm{Sa}(\omega)],
\end{aligned}
\tag{2.15}
$$

where ω *is angular frequency,* $*$ *implies the convolution,* $\mathrm{Sa}(\omega) = \sin(\omega)/\omega$ *and*

$$
\begin{aligned}
I_1(\omega) &= -2\sin\left(\frac{\alpha k \pi}{2}\right)\Gamma(\alpha k + 1)|\omega|^{-\alpha k - 1}, \\
I_2(\omega) &= 2\sin\left[\frac{(\beta + \alpha k)\pi}{2}\right]\Gamma[1 - (\beta + \alpha k)]\,|\omega|^{(\beta + \alpha k) - 1}.
\end{aligned}
\tag{2.16}
$$

Proof. Note that $(1+x)^{\nu}$ can be expanded as a binomial series given by

$$
(1+x)^{\nu} = \sum_{k=0}^{\infty} \binom{\nu}{k} x^k = \sum_{k=0}^{\infty} \frac{\Gamma(\nu + k)}{\Gamma(\nu)\Gamma(1+k)} x^k \quad \text{for } |x| < 1,
\tag{2.17}
$$

where $\nu \in \mathbb{R}$. and $\binom{\nu}{k}$ is the binomial coefficient.

Now, I expand the ACF of the EGC process by

$$
\begin{aligned}
C(\tau) = {} & \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma[(\beta/\alpha) + k]}{\Gamma(\beta/\alpha)\Gamma(1+k)} |\tau|^{\alpha k} \right\} [u(\tau + 1) - u(\tau - 1)] \\
& + \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma[(\beta/\alpha) + k]}{\Gamma(\beta/\alpha)\Gamma(1+k)} |\tau|^{-(\beta + \alpha k)} \right\} [u(\tau - 1) + u(-\tau - 1)],
\end{aligned}
\tag{2.18}
$$

where $u(\tau)$ is the Heaviside unit step function (Li and Lim [84]).

Because the Fourier transform (FT) of $|t|^{\lambda}$ is expressed by

$$
F\left(|t|^{\lambda}\right) = -2\sin\left(\frac{\lambda\pi}{2}\right)\Gamma(\lambda + 1)|\omega|^{-\lambda - 1},
\tag{2.19}
$$

where $\lambda \neq -1, -3, \ldots$, I have

$$
F\left[|\tau|^{\alpha k}\right] = -2\sin\left(\frac{\alpha k \pi}{2}\right)\Gamma(\alpha k + 1)|\omega|^{-\alpha k - 1} = I_1(\omega).
\tag{2.20}
$$

Similarly,

$$F\left[|\tau|^{-(\beta+\alpha k)}\right] = 2\sin\left[\frac{(\beta+\alpha k)\pi}{2}\right]\Gamma[1-(\beta+\alpha k)]|\omega|^{(\beta+\alpha k)-1} = I_2(\omega). \tag{2.21}$$

Note that $F[u(\tau+1)-u(\tau-1)] = 2\,\mathrm{Sa}(\omega)$. Doing the FT of the first term on the right side of (2.18) term-by-term yields the following:

$$\begin{aligned} &F\left\{\left[\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma[(\beta/\alpha)+k]}{\Gamma(\beta/\alpha)\Gamma(1+k)}|\tau|^{\alpha k}\right][u(\tau+1)-u(\tau-1)]\right\} \\ &\quad= \frac{1}{2\pi}\left\{\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma[(\beta/\alpha)+k]}{\Gamma(\beta/\alpha)\Gamma(1+k)}F\left(|\tau|^{\alpha k}\right)\right\} * F[u(\tau+1)-u(\tau-1)] \\ &\quad= \left\{\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma[(\beta/\alpha)+k]}{\pi\Gamma(\beta/\alpha)\Gamma(1+k)}I_1(\omega)\right\} * \mathrm{Sa}(\omega) \\ &\quad= \sum_{k=0}^{\infty}\frac{(-1)^k\Gamma[(\beta/\alpha)+k]}{\pi\Gamma(\beta/\alpha)\Gamma(1+k)}I_1(\omega) * \mathrm{Sa}(\omega). \end{aligned} \tag{2.22}$$

In addition, computing the FT of the second term on the right side of (2.18) term-by-term yields

$$\begin{aligned} &F\left\{\left\{\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma[(\beta/\alpha)+k]}{\Gamma(\beta/\alpha)\Gamma(1+k)}|\tau|^{-(\beta+\alpha k)}\right\}[u(\tau-1)+u(-\tau-1)]\right\} \\ &\quad= \frac{1}{2\pi}\left\{\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma[(\beta/\alpha)+k]}{\Gamma(\beta/\alpha)\Gamma(1+k)}F\left[|\tau|^{-(\beta+\alpha k)}\right]\right\} * F[u(\tau-1)+u(-\tau-1)] \\ &\quad= \left\{\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma[(\beta/\alpha)+k]}{2\pi\Gamma(\beta/\alpha)\Gamma(1+k)}I_2(\omega)\right\} * [2\pi\delta(\omega)-2\mathrm{Sa}(\omega)] \\ &\quad= \sum_{k=0}^{\infty}\frac{(-1)^k\Gamma[(\beta/\alpha)+k]}{\pi\Gamma(\beta/\alpha)\Gamma(1+k)}[\pi I_2(\omega)-I_2(\omega) * \mathrm{Sa}(\omega)]. \end{aligned} \tag{2.23}$$

Adding the right sides of (2.22) and (2.23) yields the result of this theorem. □

Remark 2.8. The EGC processes are non-Markovian since $r_{\mathrm{EGC}}(t_1, t_2)$ does not satisfy the triangular relation given by

$$r_{\mathrm{EGC}}(t_1, t_3) = \frac{r_{\mathrm{EGC}}(t_1, t_2)r_{\mathrm{EGC}}(t_2, t_3)}{r_{\mathrm{EGC}}(t_2, t_2)}, \qquad t_1 < t_2 < t_3, \tag{2.24}$$

which is a necessary condition for a Gaussian process to be Markovian; see Todorovic [97]. In fact, up to a multiplicative constant, the Ornstein-Uhlenbeck process is the only stationary Gaussian Markov process; see Lim and Muniandy [98], Wolperta and Taqqu [99].

Note 1. Since $r_{\mathrm{EGC}}(\tau) \sim |\tau|^{-\beta}$ for $\tau \to \infty$, one has the Hurst parameter of the EGC processes given by

$$H_{\mathrm{EGC}} = 1 - \frac{\beta}{2}. \tag{2.25}$$

An EGC process is LRD if $0 < \beta < 1$. It is short-range dependent (SRD) if $\beta > 1$.

Note 2. Parameter α is independent of β and vice versa in the ACF of an EGC process.

Note 3. Dimensions of an EGC process relies on the value of α, irrelevant with β. That is, dimensions of an EGC process is irrelevant with its statistic dependenc.

I will discuss the meaning of D_{EGC2} and D_{EGC3} in the next section.

3. Discussions

The emphasized point I will explain is that the ACF, (2.2), of the EGC process differs significantly from that of the GC process because I relax the restriction of α to be $\alpha > 0$ instead of $0 < \alpha \le 2$ as that in the GC process, though two ACFs appear the similar, referring [80, 81] for the details of the GC process. Relaxing the range of α from $0 < \alpha \le 2$ in the GC model to $\alpha > 0$ in this paper makes a considerable step further in the aspect of dimensions of random functions. To exhibit this step, I should explain the meaning of dimensions less than one and negative for a random function.

The fractal index α of a random function $x(t)$ is considered for $\tau \to 0$ in (2.10) or in the following expression if the ACF of $x(t)$ is sufficiently smooth on $(0, \infty)$:

$$r_{xx}(0) - r_{xx}(\tau) \sim c|\tau|^{\alpha} \quad \text{for } |\tau| \longrightarrow 0, \tag{3.1}$$

where α relates to D by $D = 1 - \alpha/2$. Obviously, r_{EGC} is sufficiently smooth on $(0, \infty)$. As implied by (3.1), one sees that the larger the value of α, the smoother the sample path of a random function. The following notes become apparent, accordingly.

Note 1. The present fractal dimensions, say D_{EGC2} and D_{EGC3}, imply that conventionally random functions are not the locally smoothest because there are random functions with dimensions less than one, for example, D_{EGC2}, or even negative, for example, D_{EGC3}.

Note 2. The zero dimension occurs when $\alpha = 4$. That is,

$$D_{\mathrm{EGC}} = D_{\mathrm{EGC2}} = 0 \quad \text{for } \alpha = 4. \tag{3.2}$$

Note 3. Random functions with $D_{\mathrm{EGC}} = 0$ are not the locally smoothest. They are locally rougher than those with $D_{\mathrm{EGC3}} < 0$.

In the extreme case of $\alpha \to \infty$, I have

$$\lim_{\alpha \to \infty} D_{\mathrm{EGC}} = -\infty. \tag{3.3}$$

Because

$$\lim_{\alpha \to \infty} |\tau|^{\alpha} = 0 \quad \text{for } |\tau| \longrightarrow 0, \tag{3.4}$$

we say that the locally smoothest random function is that with $D_{\mathrm{EGC}} \to -\infty$.

Note 4. For $D_{\mathrm{EGC}} \to -\infty$, $x_{\mathrm{EGC}}(t)$ is locally uncorrelated at any point of t. However, it may not be a white noise because it may globally be LRD if $0 < \beta < 1$. As a matter of fact, there are the Hurst effects on $x_{\mathrm{EGC}}(t)$ regardless of the value of D_{EGC}.

Note 5. The standard GC process is a special case of the EGC process for $0 < \alpha \le 2$ and $\beta > 0$, which has applications to relaxation description in physics (Lim and Li [80]), the Internet traffic (Li and Lim [81], Li and Zhao [100]), chromatin morphologies in breast cancer cells (Muniandy and Stanslas [101]).

Note 6. The usual Cauchy process is a special case of the EGC process. In fact, when $\alpha = \beta = 2$, one gets the ACF of the usual Cauchy process. Denote by $r_{\mathrm{C}}(\tau)$ the ACF of the usual Cauchy process. Then,

$$r_{\mathrm{C}}(\tau) = \left(1 + |\tau|^2\right)^{-1}. \tag{3.5}$$

It is easily seen that the fractal dimension of the usual Cauchy process is one. It is SRD since $r_{\mathrm{C}}(\tau) \sim |\tau|^{-2}$ for $\tau \to \infty$.

In what follows, I denote $r_{\mathrm{EGC}}(\tau)$ by $r_{\mathrm{EGC}}(\tau; \alpha, \beta)$ for facilitating the explanation. When $\alpha = 2$ and $\beta = 1$, one has

$$r_{\mathrm{EGC}}(\tau; 2, 1) = \left(1 + |\tau|^2\right)^{-1/2}. \tag{3.6}$$

The above $r_{\mathrm{EGC}}(\tau; 2, 1)$ is the ACF proposed by Spector and Grant [102] for interpreting areomagnetic data, which is a special case of the EGC process. When $\alpha = 2$ and $\beta = 3$, $r_{\mathrm{EGC}}(\tau)$ reduces to

$$r_{\mathrm{EGC}}(\tau; 2, 3) = \left(1 + |\tau|^2\right)^{-3/2}, \tag{3.7}$$

which has applications to magnetic fields; see Chilés and Delfiner [103]. The ACF of the Cauchy type stated by Chilés and Delfiner [103, page 86] is a reduced case of $r_{\mathrm{EGC}}(\tau)$ in the cases of $\alpha = 2$ and $\beta > 2b$ for $b > 0$. That is,

$$r_{\mathrm{EGC}}(\tau; 2, 2b) = \left(1 + |\tau|^2\right)^{-b} \quad \text{for } b > 0. \tag{3.8}$$

At moment, I am unaware what practical data may have the fractal dimensions less than one or negative but we are able to synthesize such data following the simulation method by Li [104]. Denote by $w(t)$ the standard white noise. Let F^{-1} be the operator of the inverse Fourier transform. Denote by $y(t)$ the synthesized random function that follows the ACF of the EGC process. Then,

$$y(t) = w(t) * F^{-1}\left\{ \left[F\left(1 + |t|^{\alpha}\right)^{-\beta/\alpha} \right]^{0.5} \right\}. \tag{3.9}$$

Replacing α and β by D_{EGC} and H_{EGC}, respectively, yields

$$y(t) = w(t) * F^{-1}\left\{ \left[F\left(1 + |t|^{4-2D_{\text{EGC}}}\right)^{-(1-H_{\text{EGC}})/(2-D_{\text{EGC}})} \right]^{0.5} \right\}. \tag{3.10}$$

In the discrete case, we have

$$\begin{aligned} y(i) &= w(i) * \text{IFFT}\left\{ \left[\text{FFT}\left(1 + |i|^{\alpha}\right)^{-\beta/\alpha} \right]^{0.5} \right\} \\ &= w(i) * \text{IFFT}\left\{ \left[\text{FFT}\left(1 + |i|^{4-2D_{\text{EGC}}}\right)^{-(1-H_{\text{EGC}})/(2-D_{\text{EGC}})} \right]^{0.5} \right\}, \end{aligned} \tag{3.11}$$

where FFT represents the fast Fourier transform and IFFT stands for its inverse. Figure 1 indicates the realizations of the EGC process with various values of α for $\beta = 0.8$ (the LRD case).

4. Extension to Corresponding Random Field with Negative Dimension

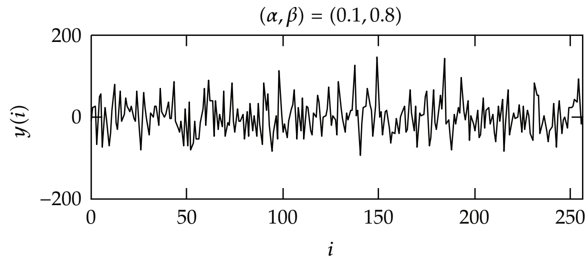
Denote by $\mathbb{R}^n$ the n-dimensional Euclidean space. The bold letters $\mathbf{t}$ and $\boldsymbol{\tau}$ represent vectors belonging to $\mathbb{R}^n$, respectively denoting $\mathbf{t} = (t_1, \ldots, t_n)$ and $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)$. The symbols $\|\mathbf{t}\|$ and $\|\boldsymbol{\tau}\|$ represent their Euclidean norms.

In the work by Lim and Teo [105], a random field $\mathbf{X}(\mathbf{t})$ is called a Gaussian field with the GC's covariance function if its covariance is given by

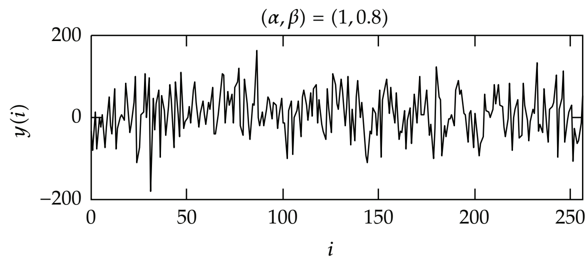
$$\mathbf{R}(\boldsymbol{\tau}) = E[\mathbf{X}(\mathbf{t})\mathbf{X}(\mathbf{t} + \boldsymbol{\tau})] = \left(1 + \|\boldsymbol{\tau}\|^{\alpha}\right)^{-\beta/\alpha}, \tag{4.1}$$

where $\beta > 0$ and α is restricted by $0 < \alpha \leq 2$. Lim and Teo termed $\mathbf{X}(\mathbf{t})$ as the GFGCC (Gaussian field with the generalized Cauchy covariance) in short. We denote $\mathbf{X}(\mathbf{t})$ by $\mathbf{X}_{\text{GFGCC}}(\mathbf{t})$ for simplicity. Over the hyperrectangle $C = \prod_{i=1}^{n}[a_i, b_i]$, they obtained the positive fractal dimension of $\mathbf{X}_{\text{GFGCC}}(\mathbf{t})$ by the following expression:

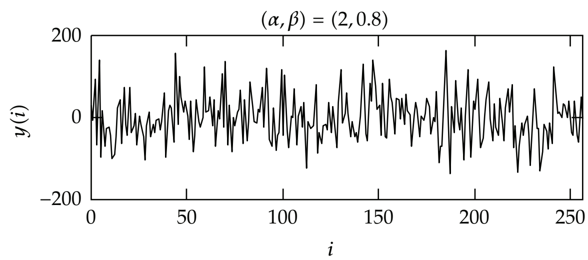
$$D_{\text{GFGCC}} = n + 1 - \frac{\alpha}{2}, \quad 0 < \alpha \leq 2. \tag{4.2}$$



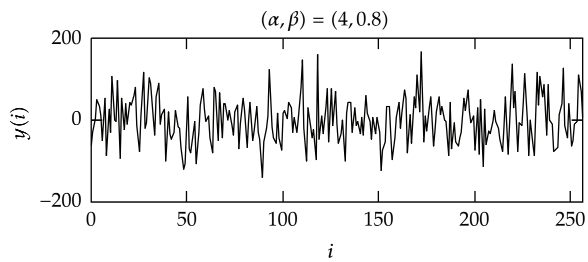
(a)



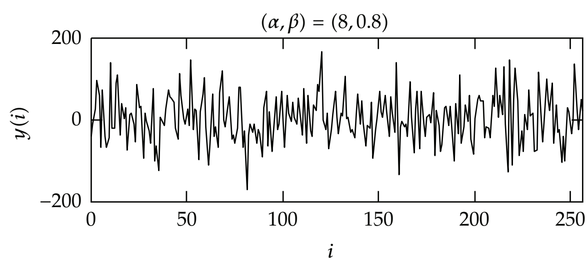
(b)



(c)

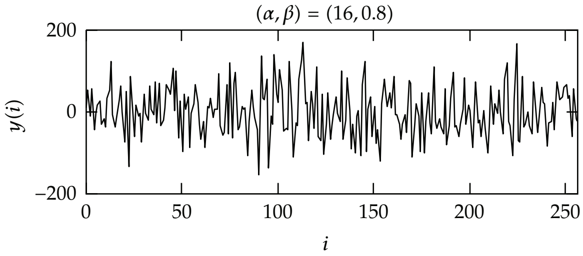


(d)



(e)

Figure 1: Continued.



(f)

Figure 1: Realizations of the EGC process for $\beta = 0.8$, that is, $H_{\mathrm{EGC}} = 0.6$. (a) Realization for $\alpha = 0.1$, that is, $D_{\mathrm{EGC}} = 1.95$. (b) Realization for $\alpha = 1$, that is, $D_{\mathrm{EGC}} = 1.5$. (c) Realization for $\alpha = 2$, that is, $D_{\mathrm{EGC}} = 1$. (d) Realization for $\alpha = 4$, that is, $D_{\mathrm{EGC}} = 0$. (e) Realization for $\alpha = 8$, that is, $D_{\mathrm{EGC}} = -2$. (f) Realization for $\alpha = 16$, that is, $D_{\mathrm{EGC}} = -6$.

Clearly,

$$n \le D_{\mathrm{GFGCC}} < n + 1. \tag{4.3}$$

As previously discussed in Sections 2 and 3, the restriction of α can be relaxed to $\alpha > 0$. Therefore, GFGCC can be extended to the case of $\alpha > 0$. We call such an extension by EGFGCC (extended Gaussian field with the generalized Cauchy covariance) and denote it by $\mathbf{X}_{\mathrm{EGFGCC}}(\mathbf{t})$. The fractal dimension of $\mathbf{X}_{\mathrm{EGFGCC}}(\mathbf{t})$ is expressed by

$$D_{\mathrm{EGFGCC}} = n + 1 - \frac{\alpha}{2}, \qquad \alpha > 0. \tag{4.4}$$

The difference between $\mathbf{X}_{\mathrm{EGFGCC}}(\mathbf{t})$ and $\mathbf{X}_{\mathrm{GFGCC}}(\mathbf{t})$ is considerable because D_{EGFGCC} may be negative while D_{GFGCC} is always positive. As a matter of fact, we have

$$D_{\mathrm{EGFGCC}} < 0 \quad \text{if } \alpha > 2(n + 1). \tag{4.5}$$

The meaning of negative dimension for $\mathbf{X}_{\mathrm{EGFGCC}}(\mathbf{t})$ is similar to that explained in Section 3. That is, locally, $\mathbf{X}_{\mathrm{EGFGCC}}(\mathbf{t})$ is more regular for smaller D_{EGFGCC}.

5. Conclusions

I have explained that the GC process can be extended to the EGC process with dimensions less than 1 and negative. The EGC process is rich such that it takes the standard GC process as its special case. I have explained that the EGC process with smaller fractal dimensions is smoother than that with larger ones. The present results are theoretic but data with negative dimensions have been synthesized in this paper. One interesting thing, as a consequence of this paper, is to explore such a class of data in various fields, for example, either in time series as those in [13, 14, 106–127], or random fields [103].

Acknowledgment

This paper was supported in part by the National Natural Science Foundation of China under the project Grant nos. 60573125, 60873264, 61070214, and the 973 plan under the project no. 2011CB302800/2011CB302802.

References

[1] S. R. Massel, *Ocean Surface Waves: Their Physics and Prediction*, World Scientific, River Edge, NJ, USA, 1997.

[2] ITTC, "The Specialist Committee on Waves—final report and recommendations to the 23rd ITTC," in *Proceedings of the 23rd ITTC*, vol. 2, pp. 497–543, 2002.

[3] M. Li, "A method for requiring block size for spectrum measurement of ocean surface waves," *IEEE Transactions on Instrumentation and Measurement*, vol. 55, no. 6, pp. 2207–2215, 2006.

[4] J. Alvarez-Ramirez, J. Alvarez, L. Dagdug, E. Rodriguez, and J. Carlos Echeverria, "Long-term memory dynamics of continental and oceanic monthly temperatures in the recent 125 years," *Physica A*, vol. 387, no. 14, pp. 3629–3640, 2008.

[5] J. Beran, *Statistics for Long-Memory Processes*, vol. 61 of *Monographs on Statistics and Applied Probability*, Chapman & Hall, New York, NY, USA, 1994.

[6] B. B. Mandelbrot, *The Fractal Geometry of Nature*, W. H. Freeman, San Francisco, Calif, USA, 1982.

[7] G. Korvin, "Fractal Models in the Earth Sciences," Elsevier, New York, NY, USA, 1992.

[8] B. J. West and W. Deering, "Fractal physiology for physicists: Lévy statistics," *Physics Report*, vol. 246, no. 1-2, pp. 1–100, 1994.

[9] T. Schreiber, "Interdisciplinary application of nonlinear time series methods," *Physics Reports*, vol. 308, no. 1, pp. 1–64, 1999.

[10] P. Abry, P. Borgnat, F. Ricciato, A. Scherrer, and D. Veitch, "Revisiting an old friend: on the observability of the relation between long range dependence and heavy tail," *Telecommunication Systems*, vol. 43, no. 3-4, pp. 147–165, 2010.

[11] G. Werner, "Fractals in the nervous system: conceptual implications for theoretical neuroscience," *Frontiers in Fractal Physiology*, vol. 1, p. 12, 2010.

[12] J. Levy-Vehel, E. Lutton, and C. Tricot, Eds., *Fractals in Engineering*, Springer, New York, NY, USA, 1997.

[13] C. Cattani, "Harmonic wavelet approximation of random, fractal and high frequency signals," *Telecommunication Systems*, vol. 43, no. 3-4, pp. 207–217, 2010.

[14] C. Cattani, "Fractals and hidden symmetries in DNA," *Mathematical Problems in Engineering*, vol. 2010, Article ID 507056, 31 pages, 2010.

[15] B. B. Mandelbrot, *Gaussian Self-Affinity and Fractals*, Springer, New York, NY, USA, 2002.

[16] M. Li, "Fractal time series—a tutorial review," *Mathematical Problems in Engineering*, vol. 2010, Article ID 2570932, 26 pages, 2010.

[17] B. B. Mandelbrot and J. W. van Ness, "Fractional Brownian motions, fractional noises and applications," *SIAM Review*, vol. 10, pp. 422–437, 1968.

[18] V. M. Sithi and S. C. Lim, "On the spectra of Riemann-Liouville fractional Brownian motion," *Journal of Physics A*, vol. 28, no. 11, pp. 2995–3003, 1995.

[19] S. V. Muniandy and S. C. Lim, "Modeling of locally self-similar processes using multifractional Brownian motion of Riemann-Liouville type," *Physical Review E*, vol. 63, no. 4, Article ID 046104, 7 pages, 2001.

[20] A. Razdan, "Multifractal nature of extensive air showers," *Chaos, Solitons & Fractals*, vol. 42, no. 5, pp. 2735–2740, 2009.

[21] B. J. West, "Fractal physiology and the fractional calculus: a perspective," *Frontiers in Fractal Physiology*, vol. 1, p. 5, 2010.

[22] A. Neuenkirch, S. Tindel, and J. Unterberger, "Discretizing the fractional Lévy area," *Stochastic Processes and Their Applications*, vol. 120, no. 2, pp. 223–254, 2010.

[23] G. W. Wornell, "Wavelet-based representations for the $1/f$ family of fractal processes," *Proceedings of the IEEE*, vol. 81, no. 10, pp. 1428–1450, 1993.

[24] H. E. Schepers, J. H. G. M. van Beek, and J. B. Bassingthwaighte, "Four methods to estimate the fractal dimension from self-affine signals [medical application]," *IEEE Engineering in Medicine and Biology Magazine*, vol. 11, no. 2, pp. 57–64, 1992.

[25] C. Fortin, R. Kumaresan, W. Ohley, and S. Hoefer, “Fractal dimension in the analysis of medical images,” *IEEE Engineering in Medicine and Biology Magazine*, vol. 11, no. 2, pp. 65–71, 1992.

[26] T. Bedford, “Hölder exponents and box dimension for self-affine fractal functions,” *Constructive Approximation*, vol. 5, no. 1, pp. 33–48, 1989.

[27] H. E. Stanley, L. A. N. Amaral, A. L. Goldberger, S. Havlin, P. Ch. Ivanov, and C. K. Peng, “Statistical physics and physiology: monofractal and multifractal approaches,” *Physica A*, vol. 270, no. 1, pp. 309–324, 1999.

[28] P. Flandrin, “Wavelet analysis and synthesis of fractional Brownian motion,” *IEEE Transactions on Information Theory*, vol. 38, no. 2, part 2, pp. 910–917, 1992.

[29] P. Christie, “Fractal analysis of interconnection complexity,” *Proceedings of the IEEE*, vol. 81, no. 10, pp. 1492–1499, 1993.

[30] K. Chandra and C. Thompson, “Ultrasonic characterization of fractal media,” *Proceedings of the IEEE*, vol. 81, no. 10, pp. 1523–1533, 1993.

[31] A. A. Suleymanov, A. A. Abbasov, and A. J. Ismaylov, “Fractal analysis of time series in oil and gas production,” *Chaos, Solitons & Fractals*, vol. 41, no. 5, pp. 2474–2483, 2009.

[32] E. Conte, A. Federici, and J. P. Zbilut, “A new method based on fractal variance function for analysis and quantification of sympathetic and vagal activity in variability of R-R time series in ECG signals,” *Chaos, Solitons & Fractals*, vol. 41, no. 3, pp. 1416–1426, 2009.

[33] W. Deering and B. J. West, “Fractal physiology,” *IEEE Engineering in Medicine and Biology Magazine*, vol. 11, no. 2, pp. 40–46, 1992.

[34] A. K. Mishra and S. Raghav, “Local fractal dimension based ECG arrhythmia classification,” *Biomedical Signal Processing and Control*, vol. 5, no. 2, pp. 114–123, 2010.

[35] D. Khoshnevisan and Y. Xiao, “Packing-dimension profiles and fractional Brownian motion,” *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 145, no. 1, pp. 205–213, 2008.

[36] A. Das and P. Das, “Fractal analysis of songs: performer’s preference,” *Nonlinear Analysis: Real World Applications*, vol. 11, no. 3, pp. 1790–1794, 2010.

[37] J. Dávila and A. C. Ponce, “Hausdorff dimension of ruptures sets and removable singularities,” *Comptes Rendus Mathématique*, vol. 346, no. 1-2, pp. 27–32, 2008.

[38] G. Kekovic, M. Culic, L. Martac et al., “Fractal dimension values of cerebral and cerebellar activity in rats loaded with aluminium,” *Medical & Biological Engineering & Computing*, vol. 48, no. 7, pp. 671–679, 2010.

[39] S. Rehman and A. H. Siddiqi, “Wavelet based Hurst exponent and fractal dimensional analysis of Saudi climatic dynamics,” *Chaos, Solitons & Fractals*, vol. 40, no. 3, pp. 1081–1090, 2009.

[40] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, John Wiley & Sons, Hoboken, NJ, USA, 2nd edition, 2003.

[41] P. Ch. Ivanov, L. A. Nunes Amaral, A. L. Goldberger et al., “Multifractality in human heartbeat dynamics,” *Nature*, vol. 399, no. 6735, pp. 461–465, 1999.

[42] K. Daoudi and J. L. Véhel, “Signal representation and segmentation based on multifractal stationarity,” *Signal Processing*, vol. 82, no. 12, pp. 2015–2024, 2002.

[43] M. Tanaka, R. Kato, Y. Kimura, and A. Kayama, “Automated image processing and analysis of fracture surface patterns formed during creep crack growth in austenitic heat-resisting steels with different microstructures,” *ISIJ International*, vol. 42, no. 12, pp. 1412–1418, 2002.

[44] R. Zuo, Q. Cheng, Q. Xia, and F. P. Agterberg, “Application of fractal models to distinguish between different mineral phases,” *Mathematical Geosciences*, vol. 41, no. 1, pp. 71–80, 2009.

[45] B. Kaulakys and M. Alaburda, “Modeling scaled processes and $1/f^{\beta}$ noise using nonlinear stochastic differential equations,” *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2009, no. 2, Article ID P02051, 2009.

[46] C. Song, L. K. Gallos, S. Havlin, and H. A. Makse, “How to calculate the fractal dimension of a complex network: the box covering algorithm,” *Journal of Statistical Mechanics: Theory and Experiment*, no. 3, Article ID P03006, 2007.

[47] R. C. García, A. S. Galán, J. R. Castrejón Pita, and A. A. Castrejón Pita, “The fractal dimension of an oil spray,” *Fractals*, vol. 11, no. 2, pp. 155–161, 2003.

[48] M. Radziejewski and Z. W. Kundzewicz, “Fractal analysis of flow of the river Warta,” *Journal of Hydrology*, vol. 200, no. 1-4, pp. 280–294, 1997.

[49] J. Shinmoto and F. Takeo, “The Hausdorff dimension of sub-self-similar sets,” *Fractals*, vol. 11, no. 1, pp. 9–18, 2003.

[50] M. Li and W. Zhao, "Detection of variations of local irregularity of traffic under DDOS flood attack," *Mathematical Problems in Engineering*, vol. 2008, Article ID 475878, 11 pages, 2008.

[51] B. R. Hunt, "The Hausdorff dimension of graphs of Weierstrass functions," *Proceedings of the American Mathematical Society*, vol. 126, no. 3, pp. 791–800, 1998.

[52] M. Li, "Self-similarity and long-range dependence in teletraffic," in *Proceedings of the 9th WSEAS International Conference on Multimedia Systems and Signal Processing*, Hangzhou, China, May 2009.

[53] G. A. Hirchoren and C. E. D'Attellis, "Estimation of fractal signals using wavelets and filter banks," *IEEE Transactions on Signal Processing*, vol. 46, no. 6, pp. 1624–1630, 1998.

[54] H. E. Stanley and P. Meakin, "Multifractal phenomena in physics and chemistry," *Nature*, vol. 335, no. 6189, pp. 405–409, 1988.

[55] R. D. Mauldin and S. C. Williams, "On the Hausdorff dimension of some graphs," *Transactions of the American Mathematical Society*, vol. 298, no. 2, pp. 793–803, 1986.

[56] R. R. Prasad, C. Meneveau, and K. R. Sreenivasan, "Multifractal nature of the dissipation field of passive scalars in fully turbulent flows," *Physical Review Letters*, vol. 61, no. 1, pp. 74–77, 1988.

[57] T. Higuchi, "Approach to an irregular time series on the basis of the fractal theory," *Physica D*, vol. 31, no. 2, pp. 277–283, 1988.

[58] B. Dubuc, J. F. Quiniou, C. Roques-Carmes, C. Tricot, and S. W. Zucker, "Evaluating the fractal dimension of profiles," *Physical Review A*, vol. 39, no. 3, pp. 1500–1512, 1989.

[59] N. Patzschke and M. Zähle, "Self-similar random measures are locally scale invariant," *Probability Theory and Related Fields*, vol. 97, no. 4, pp. 559–574, 1993.

[60] B. Ninness, "Estimation of $1/f$ noise," *IEEE Transactions on Information Theory*, vol. 44, no. 1, pp. 32–46, 1998.

[61] J. M. Girault, D. Kouamé, and A. Ouahabi, "Analytical formulation of the fractal dimension of filtered stochastic signals," *Signal Processing*, vol. 90, no. 9, pp. 2690–2697, 2010.

[62] P. Paramanathan and R. Uthayakumar, "An algorithm for computing the fractal dimension of waveforms," *Applied Mathematics and Computation*, vol. 195, no. 2, pp. 598–603, 2008.

[63] B. B. Mandelbrot, "Self-affine fractals and fractal dimension," *Physica Scripta*, vol. 32, no. 4, pp. 257–260, 1985.

[64] A. Das and P. Das, "Fractal analysis of different eastern and western musical instruments," *Fractals*, vol. 14, no. 3, pp. 165–170, 2006.

[65] S. C. Lim, L. Ming, and L. P. Teo, "Locally self-similar fractional oscillator processes," *Fluctuation and Noise Letters*, vol. 7, no. 2, pp. L169–L179, 2007.

[66] C. R. Tolle, T. R. McJunkin, and D. J. Gorsich, "Suboptimal minimum cluster volume cover-based method for measuring fractal dimension," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 25, no. 1, pp. 32–41, 2003.

[67] P. Maragos and A. Potamianos, "Fractal dimensions of speech sounds: computation and application to automatic speech recognition," *Journal of the Acoustical Society of America*, vol. 105, no. 3, pp. 1925–1932, 1999.

[68] J. M. Halley, S. Hartley, A. S. Kallimanis, W. E. Kunin, J. J. Lennon, and S. P. Sgardelis, "Uses and abuses of fractal methodology in ecology," *Ecology Letters*, vol. 7, no. 3, pp. 254–271, 2004.

[69] B. B. Mandelbrot, "Negative fractal dimensions and multifractals," *Physica A*, vol. 163, no. 1, pp. 306–315, 1990.

[70] B. B. Mandelbrot, "Multifractal power law distributions: negative and critical dimensions and other "anomalies," explained by a simple example," *Journal of Statistical Physics*, vol. 110, no. 3–6, pp. 739–774, 2003.

[71] B. B. Mandelbrot, "Random multifractals: negative dimensions and the resulting limitations of the thermodynamic formalism," *Proceedings of the Royal Society. Series A*, vol. 434, no. 1890, pp. 79–88, 1991.

[72] B. B. Mandelbrot, "Negative dimensions and Hölders, multifractals and their Hölder spectra, and the role of lateral preasymptotics in science," *The Journal of Fourier Analysis and Applications*, vol. 2, pp. 409–432, 1995.

[73] B. B. Mandelbrot and R. H. Riedi, "Inverse measures, the inversion formula, and discontinuous multifractals," *Advances in Applied Mathematics*, vol. 18, no. 1, pp. 50–58, 1997.

[74] B. B. Mandelbrot and M. Frame, "A primer of negative test dimensions and degrees of emptiness for latent sets," *Fractals*, vol. 17, no. 1, pp. 1–14, 2009.

[75] B. B. Mandelbrot, "Heavy tails in finance for independent or multifractal price increments," in *Handbook on Heavy Tailed Distributions in Finance*, S. T. Rachev, Ed., vol. 1 of *Handbooks in Finance 30*, pp. 1–34, Elsevier, New York, NY, USA, 2003.

[76] J. Molenaar, J. Herweijer, and W. van de Water, "Negative dimensions of the turbulent dissipation field," *Physical Review E*, vol. 52, no. 1, pp. 496–509, 1995.

[77] A. B. Chhabra and K. R. Sreenivasan, "Negative dimensions: theory, computation, and experiment," *Physical Review A*, vol. 43, no. 2, pp. 1114–1117, 1991.

[78] A. B. Chhabra and K. R. Sreenivasan, "Scale-invariant multiplier distributions in turbulence," *Physical Review Letters*, vol. 68, no. 18, pp. 2762–2765, 1992.

[79] T. Gneiting and M. Schlather, "Stochastic models that separate fractal dimension and the Hurst effect," *SIAM Review*, vol. 46, no. 2, pp. 269–282, 2004.

[80] S. C. Lim and M. Li, "A generalized Cauchy process and its application to relaxation phenomena," *Journal of Physics A*, vol. 39, no. 12, pp. 2935–2951, 2006.

[81] M. Li and S. C. Lim, "Modeling network traffic using generalized Cauchy process," *Physica A*, vol. 387, no. 11, pp. 2584–2594, 2008.

[82] M. Li, W. Jia, and W. Zhao, "A whole correlation structure of asymptotically self-similar traffic in communication networks," in *Proceedings of the 1st International Conference on Web Information Systems Engineering (WISE '00)*, pp. 461–466, Hong Kong, June 2000.

[83] M. Li and S. C. Lim, "Modeling network traffic using cauchy correlation model with long-range dependence," *Modern Physics Letters B*, vol. 19, no. 17, pp. 829–840, 2005.

[84] M. Li and S. C. Lim, "Power spectrum of generalized Cauchy process," *Telecommunication Systems*, vol. 43, no. 3-4, pp. 219–222, 2010.

[85] A. Papoulis, *Probability, Random Variables and Stochastic Processes*, McGraw-Hill, New York, NY, USA, 1997.

[86] H. Cramer, *Random Variable and Probability Distributions*, Cambridge Tracts in Mathematics, no. 36, Cambridge University, Cambridge, UK, 1937.

[87] H. Cramer, "On the theory of stationary random processes," *The Annals of Mathematics*, vol. 43, no. 2, pp. 351–369, 1942.

[88] C. K. Liu, *Applied Functional Analysis*, Defense Press, Beijing, China, 1986.

[89] I. M. Gelfand and K. Vilenkin, *Generalized Functions*, vol. 4, Academic Press, New York, NY, USA, 1964.

[90] R. J. Adler, R. E. Feldman, and M. S. Taqqu, Eds., *A Practical Guide to Heavy Tails: Statistical Techniques and Applications*, Birkhäuser, Boston, Mass, USA, 1998.

[91] D. H. Griffel, *Applied Functional Analysis*, Ellis Horwood Series in Mathematics and Its Application, Ellis Horwood, Chichester, UK, 1981.

[92] R. P. Kanwal, *Generalized Functions: Theory and Applications*, Birkhäuser, Boston, Mass, USA, 3rd edition, 2004.

[93] J. T. Kent and A. T. A. Wood, "Estimating the fractal dimension of a locally self-similar Gaussian process by using increments," *Journal of the Royal Statistical Society. Series B*, vol. 59, no. 3, pp. 679–699, 1997.

[94] P. Hall and R. Roy, "On the relationship between fractal dimension and fractal index for stationary stochastic processes," *The Annals of Applied Probability*, vol. 4, no. 1, pp. 241–253, 1994.

[95] G. Chan, P. Hall, and D. S. Poskitt, "Periodogram-based estimators of fractal properties," *The Annals of Statistics*, vol. 23, no. 5, pp. 1684–1711, 1995.

[96] R. J. Adler, *The Geometry of Random Fields*, Wiley Series in Probability and Mathematical Statistics, John Wiley & Sons, Chichester, UK, 1981.

[97] P. Todorovic, *An Introduction to Stochastic Processes and Their Applications*, Springer Series in Statistics: Probability and Its Applications, Springer, New York, NY, USA, 1992.

[98] S. C. Lim and S. V. Muniandy, "Generalized Ornstein-Uhlenbeck processes and associated self-similar processes," *Journal of Physics A*, vol. 36, no. 14, pp. 3961–3982, 2003.

[99] R. L. Wolpert and M. S. Taqqu, "Fractional Ornstein-Uhlenbeck Lévy processes and the Telecom process: upstairs and downstairs," *Signal Processing*, vol. 85, no. 8, pp. 1523–1545, 2005.

[100] M. Li and W. Zhao, "Representation of a Stochastic Traffic Bound," *IEEE Transactions on Parallel and Distributed Systems*, vol. 21, no. 9, pp. 1368–1372, 2010.

[101] S. V. Muniandy and J. Stanslas, "Modelling of chromatin morphologies in breast cancer cells undergoing apoptosis using generalized Cauchy field," *Computerized Medical Imaging and Graphics*, vol. 32, no. 7, pp. 631–637, 2008.

[102] A. Spector and F. S. Grant, "Statistical methods for interpreting aeromagnetic data," *Geophysics*, vol. 35, no. 2, pp. 293–302, 1970.

[103] J.-P. Chilès and P. Delfiner, *Geostatistics, Modeling Spatial Uncertainty*, Wiley Series in Probability and Statistics: Applied Probability and Statistics, John Wiley & Sons, New York, NY, USA, 1999.

[104] M. Li, "Generation of teletraffic of generalized Cauchy type," *Physica Scripta*, vol. 81, no. 2, Article ID 025007, 2010.

[105] S. C. Lim and L. P. Teo, "Gaussian fields and Gaussian sheets with generalized Cauchy covariance structure," *Stochastic Processes and Their Applications*, vol. 119, no. 4, pp. 1325–1356, 2009.

[106] M. Scalia, G. Mattioli, and C. Cattani, "Analysis of large-amplitude pulses in short time intervals: application to neuron interactions," *Mathematical Problems in Engineering*, vol. 2010, Article ID 895785, 15 pages, 2010.

[107] E. G. Bakhoum and C. Toma, "Dynamical aspects of macroscopic and quantum transitions due to coherence function and time series events," *Mathematical Problems in Engineering*, vol. 2010, Article ID 428903, 13 pages, 2010.

[108] E. G. Bakhoum and C. Toma, "Mathematical transform of traveling-wave equations and phase aspects of quantum interaction," *Mathematical Problems in Engineering*, vol. 2010, Article ID 695208, 15 pages, 2010.

[109] X. Yang and D. She, "A new adaptive local linear prediction method and its application in hydrological time series," *Mathematical Problems in Engineering*, vol. 2010, Article ID 205438, 15 pages, 2010.

[110] T. Y. Sung, Y. S. Shieh, and H. C. Hsin, "An efficient VLSI linear array for DCT/IDCT using subband decomposition algorithm," *Mathematical Problems in Engineering*, vol. 2010, Article ID 185398, 21 pages, 2010.

[111] M. Humi, "Assessing local turbulence strength from a time series," *Mathematical Problems in Engineering*, vol. 2010, Article ID 316841, 13 pages, 2010.

[112] A. R. Messina, P. Esquivel, and F. Lezama, "Time-dependent statistical analysis of wide-area time-synchronized data," *Mathematical Problems in Engineering*, vol. 2010, Article ID 751659, 17 pages, 2010.

[113] K. Friston, K. Stephan, B. Li, and J. Daunizeau, "Generalised filtering," *Mathematical Problems in Engineering*, vol. 2010, Article ID 621670, 34 pages, 2010.

[114] M. Dong, "A tutorial on nonlinear time-series data mining in engineering asset health and reliability prediction: concepts, models, and algorithms," *Mathematical Problems in Engineering*, vol. 2010, Article ID 175936, 22 pages, 2010.

[115] M. Li and J.-Y. Li, "On the predictability of long-range dependent series," *Mathematical Problems in Engineering*, vol. 2010, Article ID 397454, 9 pages, 2010.

[116] W. Qiu, Y. Zheng, and K. Chen, "Building representative-based data aggregation tree in wireless sensor networks," *Mathematical Problems in Engineering*, vol. 2010, Article ID 732892, 11 pages, 2010.

[117] Z. Liu, "Chaotic time series analysis," *Mathematical Problems in Engineering*, vol. 2010, Article ID 720190, 31 pages, 2010.

[118] G. Toma, "Specific differential equations for generating pulse sequences," *Mathematical Problems in Engineering*, vol. 2010, Article ID 324818, 11 pages, 2010.

[119] A. N. Al-Rabadi, O. M. Abuzeid, and H. S. Alkhaldi, "Fractal geometry-based hypergeometric time series solution to the hereditary thermal creep model for the contact of rough surfaces using the Kelvin-Voigt medium," *Mathematical Problems in Engineering*, vol. 2010, Article ID 652306, 22 pages, 2010.

[120] X. Zhao, J. Yue, and P. Shang, "Effect of trends on detrended fluctuation analysis of precipitation series," *Mathematical Problems in Engineering*, vol. 2010, Article ID 749894, 15 pages, 2010.

[121] L. Li, H. Peng, Y. Shao, and Y. Yang, "Cryptanalysis of a chaotic communication scheme using parameter observer," *Mathematical Problems in Engineering*, vol. 2010, Article ID 361860, 18 pages, 2010.

[122] M. Li and M. Li, "An adaptive approach for defending against DDoS attacks," *Mathematical Problems in Engineering*, vol. 2010, Article ID 570940, 15 pages, 2010.

[123] M. Li, "Change trend of averaged Hurst parameter of traffic under DDOS flood attacks," *Computers and Security*, vol. 25, no. 3, pp. 213–220, 2006.

[124] M. Li, S. C. Lim, and S.-Y. Chen, "Exact solution of impulse response to a class of fractional oscillators and its stability," *Mathematical Problems in Engineering*, vol. 2011, 9 pages, 2011.

[125] M. Li, C. Cattani, and S.-Y. Chen, "Viewing sea level by a one-dimensional random function with long memory," *Mathematical Problems in Engineering*, vol. 2011, Article ID 654284, 13 pages, 2011.

[126] W. B. Mikhael and T. Yang, "A gradient-based optimum block adaptation ICA technique for interference suppression in highly dynamic communication channels," *EURASIP Journal on Applied Signal Processing*, vol. 2006, 10 pages, 2006.

[127] X. Fu, W. Yu, S. Jiang, S. Graham, and Y. Guan, "TCP performance in flow-based mix networks: modeling and analysis," *IEEE Transactions on Parallel and Distributed Systems*, vol. 20, no. 5, pp. 695–709, 2009.