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Research Article

A Note on Operator Sampling and Fractional Fourier Transform

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This paper presents that the kernel of the fractional Fourier transform (FRFT) satisfies the operator version of Kramer's Lemma (Hong and Pfander, 2010), which gives a new applicability of Kramer's Lemma. Moreover, we give a new sampling formulae for reconstructing the operators which are bandlimited in the FRFT sense.

1. Introduction and Notations

Sampling theory for operators motivated by the operator identification problem in communications engineering has been developed during the last few years [1–4]. In [4], Hong and Pfander gave an operator version of Kramer's Lemma (see [4, Theorem 25]). But they did not give any explicit kernel satisfying the hypotheses in [4, Theorem 25] other than the Fourier kernel. In this paper, we present that the kernel of the fractional Fourier transform satisfies the hypotheses in [4, Theorem 25]. Therefore, we give a new applicability of Kramer's method.

The FRFT—a generalization of the Fourier transform (FT)—has received much attention in recent years due to its numerous applications, including signal processing, quantum physics, communications, and optics [5–7]. Hong and Pfander studied the sampling theorem on the operators which are bandlimited in the FT sense (see [4]). In this paper, we generalize their results to bandlimited operators in the FRFT sense.

For $f \in L^2(\mathbb{R})$, its FRFT is defined by

$$F_\alpha(u) = F^\alpha\left[f(x)\right](u) = \int_{\mathbb{R}} f(x) K_\alpha(u,x)\,dx, \tag{1.1}$$

where $\alpha \in \mathbb{R}$, and the transform kernel is given by

$$K_\alpha(u,x) = \begin{cases} A_\alpha e^{(i/2)(x^2+u^2)\cot\alpha - iux\csc\alpha} & \text{if } \alpha \neq k\pi, \\ \delta(x-u) & \text{if } \alpha = 2k\pi, \\ \delta(x+u) & \text{if } \alpha = (2k-1)\pi, \end{cases} \tag{1.2}$$

where $\delta(\cdot)$ is Dirac distribution function over $\mathbb{R}$, $A_\alpha = \sqrt{(1-i\cot\alpha)/2\pi}$, and $k \in \mathbb{Z}$. The inverse FRFT is the FRFT at angle $-\alpha$, given by

$$f(x) = F^{-\alpha}[F_\alpha(u)] = \int_{\mathbb{R}} F_\alpha(u)\overline{K_\alpha(u,x)}du, \tag{1.3}$$

where the bar denotes the complex conjugation. Whenever $\alpha = \pi/2$, (1.2) reduces to the FT. Through this paper, we assume that $\alpha \neq k\pi$.

In FRFT domain, the function space with bandwidth Ω is defined by

$$\mathrm{FPW}_\Omega = \left\{ f \in L^2(\mathbb{R}) : \operatorname{supp} F^\alpha f \subseteq \left[-\frac{\Omega}{2}, \frac{\Omega}{2}\right] \right\}. \tag{1.4}$$

For the sake of simplicity, when $\alpha = \pi/2$, FPW_Ω is written as PW_Ω.

In the following, we use the notation

$$A(F) \asymp B(F), \quad F \in \mathcal{F} \tag{1.5}$$

if there exist positive constants c and C such that $cA(F) \leq B(F) \leq CA(F)$ for all objects F in the set $\mathcal{F}$.

Let $\mathbb{H}$ be a Hilbert space and $\{f_n : n \in \mathbb{Z}\}$ be a sequence in $\mathbb{H}$. The set $\{f_n : n \in \mathbb{Z}\}$ is said to be a frame [8, 9] for $\mathbb{H}$ if

$$\|f\|_{\mathbb{H}}^2 \asymp \sum_{n\in\mathbb{Z}} \left|\langle f, f_n\rangle_{\mathbb{H}}\right|^2, \quad f \in \mathbb{H}. \tag{1.6}$$

Let $\Lambda = \{\lambda_k : k \in \mathbb{Z}\} \subseteq \mathbb{R}$ with $(\lambda_k < \lambda_{k+1})$. Λ is a set sampling for FPW_Ω if

$$\|f\|_{L^2}^2 \asymp \sum_{k\in\mathbb{Z}} \left|f(\lambda_k)\right|^2, \quad f \in \mathrm{FPW}_\Omega. \tag{1.7}$$

2. The Properties of the Kernel of FRFT

In this section, we consider under what conditions $\{K_\alpha(\lambda_k + t, \cdot) : k \in \mathbb{Z}\}$ is a frame for $L^2[-\Omega/2, \Omega/2]$ for every $t \in \mathbb{R}$. The following theorem gives a necessary and sufficient condition for $\{K_\alpha(\lambda_k + t, \cdot) : k \in \mathbb{Z}\}$ to be a frame for $L^2[-\Omega/2, \Omega/2]$ for every $t \in \mathbb{R}$.

Theorem 2.1. *For any $t \in \mathbb{R}$ and $\sin\alpha > 0$. $\{K_\alpha(\lambda_k + t, \cdot) : k \in \mathbb{Z}\}$ is a frame for $L^2[-\Omega/2, \Omega/2]$ if and only if $\{e^{-i\lambda_k\omega} : k \in \mathbb{Z}\}$ is a frame for $L^2[-(\Omega/2)\csc\alpha, (\Omega/2)\csc\alpha]$.*

Remark 2.2. By Theorem 2.1, when taking appropriate λ_k, $\{K_\alpha(\lambda_k + t, \cdot) : k \in \mathbb{Z}\}$ is a frame for each $t \in \mathbb{R}$. Therefore, we give a kernel satisfying the hypotheses in [4, Theorem 25], which gives a new applicability of Kramer's Lemma.

To prove Theorem 2.1, we need to introduce the following results.

Lemma 2.3. *$\Lambda = \{\lambda_k : k \in \mathbb{Z}\}$ is a set of sampling for FPW_Ω if and only if $\{K_\alpha(\lambda_k + t, \cdot) : k \in \mathbb{Z}\}$ is a frame for $L^2[-\Omega/2, \Omega/2]$ for every $t \in \mathbb{R}$.*

Proof. Suppose that Λ is a set of sampling for FPW_Ω. Then, for any $F \in L^2[-\Omega/2, \Omega/2]$, there exists $f \in \mathrm{FPW}_\Omega$ such that $F = F^\alpha f$. Since

$$\begin{aligned}
\left\langle F(\cdot)e^{-i(\cdot)t\csc\alpha}, K_\alpha(\lambda_k + t, \cdot)\right\rangle_{L^2} &= \left\langle F(\cdot)e^{-i(\cdot)t\csc\alpha}, A_\alpha e^{(i/2)((\lambda_k+t)^2+(\cdot)^2)\cot\alpha - i(\cdot)(\lambda_k+t)\csc\alpha}\right\rangle_{L^2} \\
&= e^{-(i/2)(2\lambda_k t+t^2)}\left\langle F(\cdot), A_\alpha e^{(i/2)(\lambda_k^2+(\cdot)^2)\cot\alpha - i(\cdot)\lambda_k\csc\alpha}\right\rangle_{L^2} \\
&= e^{(-i/2)(2\lambda_k t+t^2)}\langle F(\cdot), K_\alpha(\cdot, \lambda_k)\rangle = e^{-(i/2)(2\lambda_k t+t^2)} f(\lambda_k),
\end{aligned} \tag{2.1}$$

we have

$$\begin{aligned}
\left\| F(\cdot)e^{-i(\cdot)t\csc\alpha}\right\|_{L^2}^2 = \|F\|_{L^2}^2 = \|f\|_{L^2}^2 &\asymp \sum_{k\in\mathbb{Z}} \left|f(\lambda_k)\right|^2 \\
&= \sum_{k\in\mathbb{Z}} \left|\left\langle F(\cdot)e^{-i(\cdot)t\csc\alpha}, K_\alpha(\lambda_k + t, \cdot)\right\rangle\right|^2.
\end{aligned} \tag{2.2}$$

Therewith, $\{K_\alpha(\lambda_k + t, \cdot) : k \in \mathbb{Z}\}$ is a frame for $L^2[-\Omega/2, \Omega/2]$ for any $t \in \mathbb{R}$.

On the other hand, suppose that $\{K_\alpha(\lambda_k + t, \cdot) : k \in \mathbb{Z}\}$ is a frame for $L^2[-\Omega/2, \Omega/2]$ for any $t \in \mathbb{R}$. Specifically, $\{K_\alpha(\lambda_k, \cdot) : k \in \mathbb{Z}\}$ is also a frame for $L^2[-\Omega/2, \Omega/2]$. Then, by (2.1),

$$\begin{aligned}
\|f\|_{L^2}^2 = \|F\|_{L^2}^2 &\asymp \sum_{k\in\mathbb{Z}} |\langle F(\cdot), K_\alpha(\lambda_k, \cdot)\rangle|^2 \\
&= \sum_{k\in\mathbb{Z}} \left|f(\lambda_k)\right|^2.
\end{aligned} \tag{2.3}$$

This completes our proof. □

The following proposition gives a necessary and sufficient condition about $\Lambda = \{\lambda_k : k \in \mathbb{Z}\}$ is set of sampling for PW_Ω.

Proposition 2.4 (see [10, Lemma 3.5]). *$\Lambda = \{\lambda_k : k \in \mathbb{Z}\}$ is set of sampling for PW_Ω if and only if $\{e^{-i\lambda_k\omega} : k \in \mathbb{Z}\}$ is a frame for $L^2[-\Omega/2, \Omega/2]$.*

Lemma 2.5. *$\Lambda = \{\lambda_k : k \in \mathbb{Z}\}$ is set of sampling for FPW_Ω if and only if $\{e^{-i\lambda_k\omega} : k \in \mathbb{Z}\}$ is a frame for $L^2[-(\Omega/2)\csc\alpha, (\Omega/2)\csc\alpha]$.*

Proof. Since

$$F^{\alpha}\left[f(t)\right](u) = \sqrt{2\pi}e^{(i/2)u^2\cot\alpha}\mathfrak{F}\left[f(t)e^{(i/2)t^2\cot\alpha}\right](u\csc\alpha), \tag{2.4}$$

where $\mathfrak{F}$ denotes the FT operator, we have

$$e^{(i/2)(\cdot)^2\cot\alpha}\mathrm{FPW}_{\Omega} = \mathrm{PW}_{\Omega\csc\alpha}. \tag{2.5}$$

By Proposition 2.4, this completes the proof. □

Proof of Theorem 2.1. By Lemmas 2.3 and 2.5, we immediately get the claim. □

3. Sampling of Operators Related to FRFT

In this section, a new sampling formulae for operator is proposed. First we introduce some definitions and notations about sampling of operator.

The class of Hilbert-Schmidt operators $\mathrm{HS}(L^2(\mathbb{R}))$ consists of bounded linear operators on $L^2(\mathbb{R})$ which can be represented as integral operators of the form

$$Hf(x) = \int_{\mathbb{R}} \kappa_H(x,t)f(t)dt \tag{3.1}$$

with kernel $\kappa_H \in L^2(\mathbb{R}^2)$. Let $h_H(t,x) = \kappa_H(x,x-t)$. We call $h_H(t,x)$ the time-varying impulse response of H. If $H \in \mathrm{HS}(L^2(\mathbb{R}))$, then the operator norm of H is defined by $\|H\|_{\mathrm{HS}} := \|\kappa_H\|_{L^2} = \|h_H\|_{L^2}$.

The Feichtinger algebra is defined by

$$S_0(\mathbb{R}) = \left\{f \in L^2(\mathbb{R}) : V_{\mathfrak{g}}f(t,\nu) \in L^1\left(\mathbb{R}^2\right)\right\}, \tag{3.2}$$

where $V_{\mathfrak{g}}f(t,\nu) = \langle f, M_{\nu}T_t\mathfrak{g}\rangle$ is the short-time Fourier transform of f with respect to the Gaussian $\mathfrak{g}(x) = e^{-\pi x^2}$. An operator class $O \subseteq \mathrm{HS}(L^2(\mathbb{R}))$ is identifiable if all $H \in O$ extend to a domain containing a so-called identifier $f \in S_0'(\mathbb{R})$ and

$$\|H\|_{\mathrm{HS}} \asymp \|Hf\|_{L^2}, \quad H \in O. \tag{3.3}$$

The operator class $O \subseteq \mathrm{HS}(L^2(\mathbb{R}))$ permits operator sampling if one can choose f in (3.3) with discrete support in $\mathbb{R}$ in the distributional sense. In that case, $\mathrm{supp}\, f$ is called sampling set for O.

For $T,\Omega > 0$, let

$$\begin{gathered}\mathrm{OPW}_{T,\Omega} = \left\{H \in \mathrm{HS}\left(L^2(\mathbb{R})\right) : \mathrm{supp}\,\mathfrak{F}(h_H(t,\cdot)) \subseteq [0,T] \times \left[-\frac{\Omega}{2},\frac{\Omega}{2}\right]\right\},\\ \mathrm{OFPW}_{T,\Omega} = \left\{H \in \mathrm{HS}\left(L^2(\mathbb{R})\right) : \mathrm{supp}\,F^{\alpha}(h_H(t,\cdot)) \subseteq [0,T] \times \left[-\frac{\Omega}{2},\frac{\Omega}{2}\right]\right\}.\end{gathered} \tag{3.4}$$

The following theorem states that a bandlimited operator in FRFT sense permits operator sampling.

Theorem 3.1. *For* $\Omega, T, T > 0$ *and* $0 < T'\Omega \leq T\Omega \leq 2\pi \sin\alpha$, *choose* $\varphi \in PW([-(2\pi/T - (\Omega\csc\alpha/2)), (2\pi/T - (\Omega\csc\alpha/2))])$ *with* $\mathfrak{F}(\varphi) = 1$ *on* $[-\Omega\csc\alpha/2,\ \Omega\csc\alpha/2]$ *and* $r \in L^\infty(\mathbb{R})$ *with* $\operatorname{supp} r \subseteq [-T+T', T]$ *and* $r = 1$ *on* $[0, T]$. *Then* $OFPW_{T',\Omega}$ *permits operator sampling as*

$$\|H\|_{HS} = \sqrt{T}\left\| H\sum_{k\in\mathbb{Z}}\delta_{kT}\right\|_{L^2}, \quad H \in OFPW_{T',\Omega}, \tag{3.5}$$

and operator reconstruction is possible by means of the L^2*-convergent series*

$$\begin{aligned} h_H(t,x) = {}& e^{-(i/2)x^2\cot\alpha} r(t) T \sum_{n\in\mathbb{Z}} e^{-(i/2)(t+nT)^2\cot\alpha} \\ & \times \left(H\sum_{k\in\mathbb{Z}}\delta_{kT}\right)(t+nT)\varphi(x-t-nT). \end{aligned} \tag{3.6}$$

Before we give the proof of Theorem 3.1, the following two propositions are needed.

Proposition 3.2 (see [4, Theorem 8]). *For* $\Omega, T, T > 0$ *and* $0 < T'\Omega \leq T\Omega \leq 2\pi$, *choose* $\varphi \in PW([-(2\pi/T - \Omega/2), (2\pi/T - \Omega/2)])$ *with* $\mathfrak{F}(\varphi) = 1$ *on* $[-\Omega/2, \Omega/2]$ *and* $r \in L^\infty(\mathbb{R})$ *with* $\operatorname{supp} r \subseteq [-T+T', T]$ *and* $r = 1$ *on* $[0, T]$. *Then* $OPW_{T',\Omega}$ *permits operator sampling as*

$$\|H\|_{HS} = \sqrt{T}\left\| H\sum_{k\in\mathbb{Z}}\delta_{kT}\right\|_{L^2}, \quad H \in OPW_{T',\Omega}, \tag{3.7}$$

and operator reconstruction is possible by means of the L^2*-convergent series*

$$h_H(t,x) = r(t) T \sum_{n\in\mathbb{Z}} \left(H\sum_{k\in\mathbb{Z}}\delta_{kT}\right)(t+nT)\varphi(x-t-nT). \tag{3.8}$$

Proposition 3.3 (see [11, Lemma 1]). *Assume a signal* $f(t) \in FPW_\Omega$. *Let*

$$g(t) = \int_{-\Omega/2}^{\Omega/2} F^\alpha[f(t)](u) e^{-(i/2)u^2\cot\alpha + iut\csc\alpha} du. \tag{3.9}$$

Then $g(t) \in PW_{\Omega\csc\alpha}$.

Proof of Theorem 3.1. Due to (2.4), we have

$$f(t,x) \in \mathrm{OFPW}_{T',\Omega} \Longleftrightarrow e^{(i/2)x^2\cot\alpha} f(t,x) \in OPW_{T',\Omega\csc\alpha}. \tag{3.10}$$

By the proof of Proposition 3.2,

$$\|H\|_{\mathrm{HS}} = \sqrt{T}\left\| H\sum_{k\in\mathbb{Z}}\delta_{kT}\right\|_{L^2}, \quad H \in \mathrm{OFPW}_{T',\Omega}. \tag{3.11}$$

Applying Proposition 3.3, we obtain

$$h_H(t,x) = A_\alpha e^{-(i/2)x^2\cot\alpha}\overline{h}_H(t,x), \tag{3.12}$$

where $\overline{h}_H(t,x) = \int_{-\Omega/2}^{\Omega/2} F^\alpha[h_H(t,\cdot)](u)e^{-(i/2)u^2\cot\alpha+iux\csc\alpha}du$ and $\overline{h}_H(t,x) \in \mathrm{OPW}_{T',\Omega\csc\alpha}$. Let

$$\overline{H}f(x) = \int_{\mathbb{R}} \overline{h}_H(t,x)f(x-t). \tag{3.13}$$

Using Proposition 3.2 again,

$$\overline{h}_H(t,x) = r(t)T\sum_{n\in\mathbb{Z}}\left(\overline{H}\sum_{k\in\mathbb{Z}}\delta_{kT}\right)(t+nT)\varphi(x-t-nT). \tag{3.14}$$

Therewith,

$$\begin{aligned}
h_H(t,x) &= A_\alpha e^{-(i/2)x^2\cot\alpha}r(t)T\sum_{n\in\mathbb{Z}}\left(\overline{H}\sum_{k\in\mathbb{Z}}\delta_{kT}\right)(t+nT)\varphi(x-t-nT)\\
&= A_\alpha e^{-(i/2)x^2\cot\alpha}r(t)T\sum_{n\in\mathbb{Z}}\overline{h}_H(t,t+nT)\varphi(x-t-nT)\\
&= e^{-(i/2)x^2\cot\alpha}r(t)T\sum_{n\in\mathbb{Z}}e^{-(i/2)(t+nT)^2\cot\alpha}h_H(t,t+nT)\varphi(x-t-nT)\\
&= e^{-(i/2)x^2\cot\alpha}r(t)T\sum_{n\in\mathbb{Z}}e^{-(i/2)(t+nT)^2\cot\alpha}\\
&\quad\times\left(H\sum_{k\in\mathbb{Z}}\delta_{kT}\right)(t+nT)\varphi(x-t-nT).
\end{aligned} \tag{3.15}$$

□

Next, we give an important multichannel operator sampling theorem, namely, derivative operator sampling. Checking the proof of [4, Theorem 32], we have following lemma.

Lemma 3.4. *Let $M,N\in\mathbb{N}$ and $f^{(r)}$ denotes the rth derivative of f in the distributional sense. Then,*

$$\|H\|_{HS}^2 \asymp \sum_{j=0}^{MN-1}\left\|\sum_{r=0}^{j}\binom{j}{r}(-1)^r\left(H\sum_{k\in\mathbb{Z}}\delta_{kN}^{(r)}\right)^{(j-r)}\right\|^2, \quad H\in OPW_{N,M2\pi}. \tag{3.16}$$

and operator reconstruction is possible by means of the L^2-convergent series

$$h_H(t,x) = \sum_{j=0}^{MN-1} \sum_{n\in\mathbb{Z}} \left(H_j \sum_{k\in\mathbb{Z}} \delta_{kN} \right)(t+nN)\varphi_j(x-t-nN), \tag{3.17}$$

where $\{\varphi_j(x-t-nN) :\ 0 \le j \le MN,\ n \in \mathbb{N}\}$ *is a Riesz basis for* $PW_{2\pi M}$ *for each fixed* $t \in [0,N]$, *and* $H_j f(x) = (\sum_{r=0}^{j} \binom{j}{r} (-1)^r (Hf^{(r)})^{j-r})(x)$.

For the operators which are bandlimited in FRFT sense. We have the following theorem.

Theorem 3.5. *Let* $M,N \in \mathbb{N}$ *and* $f^{(r)}$ *denotes the* r*th derivative of* f *in the distributional sense. Then,*

$$\|H\|_{HS}^2 \asymp \sum_{j=0}^{MN-1} \left\| \sum_{r=0}^{j} \binom{j}{r} \left(e^{(i/2)x^2\cot\alpha}\right)^{(j-r)} \sum_{p=0}^{r} \binom{r}{p} (-1)^p \left(H \sum_{k\in\mathbb{Z}} \delta_{kN}^{(r)} \right)^{r-p} \right\|^2, \quad H \in OFPW_{N,M2\pi\sin\alpha}. \tag{3.18}$$

and operator reconstruction is possible by means of the L^2-convergent series

$$\begin{aligned} h_H(t,x) = e^{-(i/2)x^2\cot\alpha} &\sum_{j=0}^{MN-1} \sum_{n\in\mathbb{Z}} \sum_{r=0}^{j} \binom{j}{r} \\ &\times \left(\left(e^{(i/2)x^2\cot\alpha}\right)^{(j-r)} H_r \sum_{k\in\mathbb{Z}} \delta_{kN} \right)(t+nN)\varphi_j(x-t-nN), \end{aligned} \tag{3.19}$$

where $\{\varphi_j(x-t-nN) : 0 \le j \le MN, n \in \mathbb{N}\}$ *is a Riesz basis for* $PW_{M2\pi}$ *for each fixed* $t \in [0,N]$ *and* $H_j f(x) = (\sum_{r=0}^{j} \binom{j}{r} (-1)^r (Hf^{(r)})^{j-r})(x)$.

Proof. Due to (2.4), we have

$$f(t,x) \in \mathrm{OFPW}_{N,M2\pi\sin\alpha} \Longleftrightarrow e^{(i/2)x^2\cot\alpha} f(t,x) \in \mathrm{OPW}_{N,M2\pi}. \tag{3.20}$$

By Proposition 3.3, we obtain

$$h_H(t,x) = A_\alpha e^{-(i/2)x^2\cot\alpha} \overline{h}_H(t,x), \tag{3.21}$$

where $\overline{h}_H(t,x) = \int_{-\Omega/2}^{\Omega/2} F^\alpha[h_H(t,\cdot)](u) e^{-(i/2)u^2\cot\alpha + iux\csc\alpha} du$. Put

$$\overline{H} f(x) = \int_{\mathbb{R}} \overline{h}_H(t,x) f(x-t) dt. \tag{3.22}$$

By Lemma 3.4, we have

$$h_{\overline{H}}(t,x) = \sum_{j=0}^{MN-1}\sum_{n\in\mathbb{Z}}\left(\overline{H}_j\sum_{k\in\mathbb{Z}}\delta_{kN}\right)(t+nN)\varphi_j(x-t-nN), \tag{3.23}$$

where

$$\begin{aligned}
\overline{H}_j f(x) &= \left(\sum_{r=0}^{j}\binom{j}{r}(-1)^r\left(\overline{H}f^{(r)}\right)^{j-r}\right)(x) = \int_{\mathbb{R}}\frac{\partial^j}{\partial x^j}\overline{h}_H(t,x)f(x-t)dt \\
&= \frac{1}{A_\alpha}\int_{\mathbb{R}}\frac{\partial^j}{\partial x^j}\left(e^{(i/2)x^2\cot\alpha}h_H(t,x)\right)f(x-t)dt \\
&= \frac{1}{A_\alpha}\sum_{r=0}^{j}\binom{j}{r}\left(e^{(i/2)x^2\cot\alpha}\right)^{(j-r)}(x)\int_{\mathbb{R}}\frac{\partial^r}{\partial x^r}h_H(t,x)f(x-t)dt \\
&= \frac{1}{A_\alpha}\sum_{r=0}^{j}\binom{j}{r}\left(e^{(i/2)x^2\cot\alpha}\right)^{(j-r)}(x)H_r f(x) \\
&= \frac{1}{A_\alpha}\sum_{r=0}^{j}\binom{j}{r}\left(e^{(i/2)x^2\cot\alpha}\right)^{(j-r)}(x) \\
&\quad\times\left(\sum_{p=0}^{r}\binom{r}{p}(-1)^p\left(Hf^{(p)}\right)^{r-p}\right)(x).
\end{aligned} \tag{3.24}$$

By the proof of Lemma 3.4, (3.18) holds. Moreover, by (3.21) we obtain

$$\begin{aligned}
h_H(t,x) &= A_\alpha e^{-(i/2)x^2\cot\alpha}\sum_{j=0}^{MN-1}\sum_{n\in\mathbb{Z}}\left(\overline{H}_j\sum_{k\in\mathbb{Z}}\delta_{kN}\right)(t+nN)\varphi_j(x-t-nN) \\
&= e^{-(i/2)x^2\cot\alpha}\sum_{j=0}^{MN-1}\sum_{n\in\mathbb{Z}}\sum_{r=0}^{j}\binom{j}{r} \\
&\quad\times\left(\left(e^{(i/2)x^2\cot\alpha}\right)^{(j-r)}H_r\sum_{k\in\mathbb{Z}}\delta_{kN}\right)(t+nN)\varphi_j(x-t-nN).
\end{aligned} \tag{3.25}$$

□

Remark 3.6. After careful development of pertinent tools, one can formulate extensions of results in this paper to the linear canonical transform case (see [11–13]). We have presented the **FRFT** case because of its simplicity and applicability.

4. Conclusion

Kramer's Lemma is very important in the proofs of a number of sampling theorems. In [4, Theorem 25], Hong and Pfander proved an operator sampling version of Kramer's

Lemma. But they did not give any explicit kernel satisfying the hypotheses in [4, Theorem 25] other than the Fourier kernel. In this paper, we find that the kernel of the fractional Fourier transform satisfies the hypotheses in [4, Theorem 25]. This observation gives a new applicability of Kramer's method. Moreover, we give a new sampling formulae for reconstructing the operators which are bandlimited in the FRFT sense. This is an extension of some results in [4].

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