

*Research Article*

# Financial Applications of Bivariate Markov Processes

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This paper describes a methodology to approximate a bivariate Markov process by means of a proper Markov chain and presents possible financial applications in portfolio theory, option pricing and risk management. In particular, we first show how to model the joint distribution between market stochastic bounds and future wealth and propose an application to large-scale portfolio problems. Secondly, we examine an application to VaR estimation. Finally, we propose a methodology to price Asian options using a bivariate Markov process.

## 1. Introduction

In this paper, we propose an approach to some classical financial problems based on the analysis of bivariate Markov processes. In particular, we use a bivariate Markov process to examine three possible financial applications: portfolio selection, risk management, and option pricing.

Generally, portfolio selection, risk management, and option pricing problems are studied in financial literature assuming that the stock returns are Gaussian or elliptically distributed. As a matter of fact, the first analysis of the portfolio selection problem was given by Markowitz [1–3] and Tobin [4, 5] in terms of the mean and the variance of the portfolio returns. The portfolio selection based on investors' mean-variance preferences can be justified only assuming that the returns are elliptically distributed.

In risk-management theory, the risk measure mostly adopted by financial institutions to manage and evaluate the market risk exposition of the own portfolios is the value-at-risk (VaR). There exist many methodologies to compute VaR. The most used model

(proposed from RiskMetrics) hypothesizes Gaussian or conditionally elliptical distributed returns (see [6, 7]).

Finally, the benchmark model used in option pricing theory, the Black and Scholes model [8], is based on the assumption that the log-return evolves as a Brownian motion and thus log returns are Gaussian distributed.

However, it is well known that the stock returns present heavy tails and skewness and there exist a wide literature on the improvements performed on the Gaussian pioneer models (see, among others, [9] and the references therein). Many efforts have been destined to make the distributional hypothesis more realistic on the price process. Moreover, most of the alternative models are based on different Markovian processes. Effectiveness of Markov processes in describing the portfolios returns has been widely discussed in the literature (see, among others, [9, 10]). When the portfolio returns follow a Markov process, the estimation of the future wealth distribution can be a heavy computational task; nevertheless, the computational complexity can be controlled by means of a Markov chain, where the states are chosen in such a way that they induce a recombining effect on the future wealth (see [11] and the references therein).

Among Markovian models, we essentially distinguish two categories: parametric models (see, among others, [10, 12, 13]) and nonparametric models (see, among others, [11, 14]). In the first category, the Markovian hypothesis is used for diffusive models of the log returns. In the second category of models, the historical series are used to estimate the transition among the states. Nonparametric models have the main advantage in their capacity of adapting to the return distributions. In this paper, we propose a nonparametric Markovian model using an homogeneous Markov chain to describe the returns time evolution. The proposed approach extend the univariate approach proposed by Angelelli and Ortobelli Lozza [11], and for this reason, it is essentially different from other nonparametric models discussed in literature (see, among others, [15–18]).

The main contribution of this paper is twofold.

- (a) it extends the nonparametric univariate Markovian pricing valuation to the bivariate one to account joint behavior of the stock prices.
- (b) it shows the financial use and the impact of nonparametric bivariate Markov processes. In particular, we discuss the application of bivariate Markov processes in three financial problems: the large-scale portfolio selection problem, the valuation of the portfolio risk at a given future time, and the pricing of average strike Asian options.

In the first part of the paper, we approach a large-scale portfolio selection problem. The problem is attacked by means of a number of different techniques applied in steps. First, the randomness of the problem is reduced by applying *principal component analysis* (PCA) to the Pearson correlation of the forecasted wealth obtained with the approximating Markov process, which allows to approximate the returns using only few components deriving from the PCA. Secondly, we optimize a proper portfolio selection strategy that accounts for the joint behavior of the future portfolio wealth and of the predicted wealth obtained by the market stochastic bounds (see [19, 20]). The effectiveness of the approach is tested by an ex-post empirical analysis in which the results of this approach are compared to those obtained from a classical mean-variance strategy (see [21]).

In the second part of the paper, we propose to use the covariance matrix obtained by the estimated wealth at a given time to value the percentile of the future wealth. Then, we compare our estimates with the classical methodology used by Riskmetrics (see [7]). Finally,

we discuss the pricing of contingent claims that require the use of different random variables. In particular, we show how we can estimate the price of average strike Asian options.

The paper is organized as follows. In Section 2, we discuss how modeling bivariate Markov processes. Section 3 analyzes the large-scale portfolio problem and propose an ex-post empirical comparison. Section 4 discusses the use of bivariate Markov chains for the valuation of the portfolio risk and the pricing of average Asian options. The last section briefly summarizes the paper.

## 2. Approximating Bivariate Markov Processes with a Markov Chain

Assume that an initial wealth  $W_0 = (W_{0x}, W_{0y})' = (1, 1)'$  is invested at time  $t = 0$  in two portfolios of weights  $x = [x_1, \dots, x_n]'$  and  $y = [y_1, \dots, y_m]'$  of  $n$  and  $m$  risky assets respectively. The vectors  $x$  and  $y$  represent the percentage of the initial wealths ( $W_{0x}$  and  $W_{0y}$ , resp.) invested in each asset. Denote the prices of these assets at time  $t$  by  $P_t^{(x)} = [P_{1,t}^{(x)}, \dots, P_{n,t}^{(x)}]'$  and  $P_t^{(y)} = [P_{1,t}^{(y)}, \dots, P_{m,t}^{(y)}]'$ . The portfolios returns during the period  $[t, t + 1]$  are given by the vector  $Z_{t+1} = (Z_{x,t+1}, Z_{y,t+1})'$  with components

$$Z_{x,t+1} = \sum_{i=1}^n x_i \frac{P_{i,t+1}^{(x)}}{P_{i,t}^{(x)}} \quad Z_{y,t+1} = \sum_{i=1}^m y_i \frac{P_{i,t+1}^{(y)}}{P_{i,t}^{(y)}}. \quad (2.1)$$

We assume that the portfolios returns  $Z_{x,t}$  and  $Z_{y,t}$  follow two homogeneous Markov processes. In this section, we introduce an approximation of the bivariate process  $Z_t = (Z_{x,t}, Z_{y,t})'$  by a bivariate homogeneous Markov chain. We introduce the multi-index  $i = (i_x, i_y)$  and denote by  $z^{(i)} = (z_x^{(i_x)}, z_y^{(i_y)})'$ ,  $i \in I := \{(i_x, i_y) : 1 \leq i_x \leq N, 1 \leq i_y \leq M\}$  the states of the Markov chain. First, we discretize the support of the Markov process  $\{Z_t\}$ . Given a set of past observations  $\{z_{-K}, \dots, z_0\}$ , we consider the range of the portfolios returns

$$\left( \min_{k=-K, \dots, 0} z_{x,k}, \max_{k=-K, \dots, 0} z_{x,k} \right) \times \left( \min_{k=-K, \dots, 0} z_{y,k}, \max_{k=-K, \dots, 0} z_{y,k} \right), \quad (2.2)$$

and divide it into  $N \cdot M$  bidimensional intervals  $(a_i, a_{i-1}) \times (b_j, b_{j-1})$ , where  $\{a_i\}$  and  $\{b_j\}$  are two decreasing sequences given by

$$a_i := \left( \frac{\min_k z_{x,k}}{\max_k z_{x,k}} \right)^{i/N} \max_k z_{x,k}, \quad i = 0, \dots, N, \\ b_j := \left( \frac{\min_k z_{y,k}}{\max_k z_{y,k}} \right)^{j/M} \max_k z_{y,k}, \quad j = 0, \dots, M. \quad (2.3)$$

The idea is to approximate the returns associated to values of the Markov process in  $(a_{i_x}, a_{i_x-1}) \times (b_{i_y}, b_{i_y-1})$  by the state  $(z_x^{(i_x)}, z_y^{(i_y)})$  of the Markov chain defined by

$$\begin{aligned} z_x^{(i_x)} &= \sqrt{a_{i_x} a_{i_x-1}} = \max_k z_{x,k} \left( \frac{\max_k z_{x,k}}{\min_k z_{x,k}} \right)^{(1-2i_x)/2N}, \quad i_x = 1, \dots, N, \\ z_y^{(i_y)} &= \sqrt{b_{i_y} b_{i_y-1}} = \max_k z_{y,k} \left( \frac{\max_k z_{y,k}}{\min_k z_{y,k}} \right)^{(1-2i_y)/2M}, \quad i_y = 1, \dots, M. \end{aligned} \quad (2.4)$$

Introducing

$$u_x := \left( \frac{\max_k z_{x,k}}{\min_k z_{x,k}} \right)^{1/N} \quad u_y := \left( \frac{\max_k z_{y,k}}{\min_k z_{y,k}} \right)^{1/M}, \quad (2.5)$$

we may write  $z_x^{(i_x)} = z_x^{(1)} u_x^{1-i_x}$  and  $z_y^{(i_y)} = z_y^{(1)} u_y^{1-i_y}$ . Assuming the Markov chain  $\{Z_t\}$  homogeneous, we denote its transition matrix by  $Q = \{q(i, j)\}_{i, j \in I}$ , where

$$q(i, j) = P(Z_{t+1} = z^{(j)} \mid Z_t = z^{(i)}), \quad i, j \in I \quad (2.6)$$

represents the probability of observing the returns  $z^{(j)}$  in  $t + 1$  being in  $z^{(i)}$  at time  $t$ . These probabilities are estimated by the maximum likelihood estimates

$$\hat{q}(i, j) = \frac{\pi_{ij}}{\pi_i}, \quad (2.7)$$

where  $\pi_{ij}$  is the number of observations that transit from  $z^{(i)}$  to  $z^{(j)}$  and  $\pi_i$  is the number of observations in  $z^{(i)}$ . Let us now consider the bivariate wealth process generated by the gross returns. The wealth  $W_t = (W_{tx}, W_{ty})'$  at time  $t$  is a bivariate random variable with  $N \cdot M$  possible values

$$W_t = z^{(i)} \otimes W_{t-1} = \left( z_x^{(i_x)} W_{(t-1)x}, z_y^{(i_y)} W_{(t-1)y} \right)', \quad i \in I, \quad (2.8)$$

where  $W_{t-1}$  is the wealth at time  $t - 1$ . Denoting  $i_s = (i_{x,s}, i_{y,s})$  the realized state of Markov chain at time  $s$ , the value of  $W_t$  is given by

$$W_t = \begin{pmatrix} W_{0x} z_x^{(i_{x,1})} z_x^{(i_{x,2})} \dots z_x^{(i_{x,t})} \\ W_{0y} z_y^{(i_{y,1})} z_y^{(i_{y,2})} \dots z_y^{(i_{y,t})} \end{pmatrix}. \quad (2.9)$$

It is clear that the sequence  $\langle i_0, i_1, \dots, i_t \rangle$  identifies uniquely the path followed by the bivariate wealth process up to time  $t$ . Thus, using formulas (2.5), the wealth obtained along the path  $\langle i_0, i_1, \dots, i_t \rangle$  is given by

$$W_t = \begin{pmatrix} W_{0x} \left( z_x^{(1)} u_x \right)^t u_x^{-(i_{x,1} + i_{x,2} + \dots + i_{x,t})} \\ W_{0y} \left( z_y^{(1)} u_y \right)^t u_y^{-(i_{y,1} + i_{y,2} + \dots + i_{y,t})} \end{pmatrix}. \quad (2.10)$$

Notice that vector  $x$  and  $y$  represent the percentages of the initial wealths. Thus, if we want to evaluate the sample path of the ex-post wealths, we have to recalibrate each portfolio in order to maintain these percentages constant over time.

Moreover, describing the gross returns by a general bivariate Markov chain with  $N \cdot M$  possible states implies that the number of possible values for  $W_t$  grows exponentially with the time. However,  $W_t$  can take only  $[1 + t(N - 1)] \cdot [1 + t(M - 1)]$  values. In particular, in this way, the final wealth  $W_t$  does not depend on the specific path followed by the process, but only on the sums of the indices of the states traversed by the Markov chain in the first  $t$  steps. This property is called *recombining effect* of the Markov chain on the wealth process  $W$ .

Let us denote the  $[1 + t(N - 1)] \times [1 + t(M - 1)]$  possible values of  $W_t$  at time  $t$  by

$$w^{(l,t)} = \begin{pmatrix} w_x^{(l_x,t)} \\ w_y^{(l_y,t)} \end{pmatrix} = \begin{pmatrix} \left( z_x^{(1)} \right)^t u_x^{1-l_x} \\ \left( z_y^{(1)} \right)^t u_y^{1-l_y} \end{pmatrix}, \quad (2.11)$$

where  $l = (l_x, l_y) \in L_t := \{(l_x, l_y) : 1 \leq l_x \leq 1 + t(N - 1), 1 \leq l_y \leq 1 + t(M - 1)\}$ . The possible values of  $W_t$  up to time  $T$  can be stored in  $T$  matrices of dimension  $[1 + (N - 1)T] \times [1 + (M - 1)T]$  or in a monodimensional vector of size  $\sum_{t=1}^T [1 + (N - 1)t][1 + (M - 1)t] = O(NMT^3)$ .

The wealth  $W_t$  can be represented by a three-dimensional Markovian tree, starting with a single node  $w^{((1,1),0)} = (1, 1)'$  and presenting at each time instant  $t$  the  $[1 + t(N - 1)] \times [1 + t(M - 1)]$  nodes given by  $w^{(l,t)}$ ,  $l \in L_t$ .

We are interested in the evolution of such a process  $\{W_t\}$ , which is clearly connected to the evolution of  $\{Z_t\}$ . Consider the matrix

$$P_{(w_t, z_t)} = \{p(w_t, z_t)(l, i)\}_{l \in L_t, i \in I'} \quad (2.12)$$

with components

$$p(w_t, z_t)(l, i) = P(W_t = w^{(l,t)} \cap Z_t = z^{(i)}), \quad (2.13)$$

which represents the probability of obtaining the wealth  $w^{(l,t)}$  and to be in state  $z^{(i)}$  at time  $t$  and the vector  $P_{W_t} = \{p_{W_t}(l)\}_{l \in L_t}$  with components

$$p_{W_t}(l) = P(W_t = w^{(l,t)}), \quad l \in L_t. \quad (2.14)$$

The probabilities  $p_{(W_t, Z_t)}(l, i)$  and  $p_{W_t}(l)$  can be computed recursively by

$$p_{(W_t, Z_t)}(l, i) = \begin{cases} p_i & t = 0, l = 1, \\ \sum_{h \in I} p_{(W_{t-1}, Z_{t-1})}(l - (i - 1), h) q(h, i) & t > 0, l_x - (i_x - 1) > 0, \\ & l_y - (i_y - 1) > 0, \\ 0 & \text{otherwise,} \end{cases} \quad (2.15)$$

$$p_{W_t}(l) = \begin{cases} 1 & t = 0, l = 1, \\ \sum_{h \in I} p_{(W_t, Z_t)}(l, h) & t > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where  $p_i = P(Z_0 = z^{(i)})$  is the probability that the return at time zero is  $z^{(i)}$ . We assume these probabilities to be known from past observations.

### 3. The Portfolio Selection Problem

In this section, we provide two applications of bivariate Markov processes to the portfolio selection problem

- (1) to account the joint behavior of the portfolio with the market stochastic bounds,
- (2) to reduce the dimensionality of large scale portfolio problems.

Finally, we compare a classical (static) portfolio selection strategy and a dynamic one based on the forecasted wealth obtained with Markov processes.

The static portfolio selection problem when no short sales are allowed consists of the maximization of a functional  $f$  (performance measure or utility functional) defined on the space of possible returns  $Z_{x,t}$  with respect to the portfolio  $x$ , which is assumed to belong to the  $(n - 1)$ -dimensional simplex  $S = \{x \in \mathbb{R}^n | x_i \geq 0, \sum_{i=1}^n x_i = 1\}$ . In other words, the investors compute the portfolio  $x \in S$  solution of

$$\max_{x \in S} f(Z_{x,t}). \quad (3.1)$$

Among the various static strategies that have been proposed in the literature, in our empirical application, we consider the *Sharpe ratio (SR) strategy* (see [21]) which evaluates the expected excess return for unit of risk (standard deviation); that is,

$$\text{SR}(X) = \frac{E(X - r_b)}{\sigma_{X-r_b}}, \quad (3.2)$$

where  $r_b$  is a given benchmark and  $\sigma_{X-r_b}$  is the standard deviation of the random variable  $X - r_b$ . When the benchmark  $r_b$  is the risk-free rate and  $X$  is the portfolio return, the Sharpe ratio is isotonic with nonsatiable risk averse preferences. For a discussion on the choice of  $f$  see [11].

Consider now the dynamic framework. Assume the initial wealth  $W_0 = 1$  and denote by  $W_x = \{W_{t,x}\}_{t \geq 0}$  all the admissible wealth processes depending on an initial portfolio  $x \in S$ . The dynamic portfolio selection problem consists of the maximization over  $S$  of a functional  $f$  depending on the wealth process. In our application, we consider a portfolio selection strategy where investors optimize their portfolio every  $T$  periods maximizing a functional  $f(\cdot)$  applied to the forecasted wealth  $W_T$  at time  $T$ . Since the weights  $x \in S$  represent the percentages of wealth invested in each asset and the value of the assets change every day, we should recalibrate daily the wealth maintaining constant the percentage every day during each period  $[t_k, t_k + T]$ , (where  $t_k$  is the time in which we compute the new portfolio composition). Thus, investors periodically compute the portfolio  $x_M \in S$  solution of

$$\max_{x \in S} f(W_{T,x}), \quad (3.3)$$

and then they recalibrate their portfolio every  $t = 1, \dots, T - 1$  in order to maintain constant the percentages  $x_M$  invested in each asset. Moreover, we will make use of a nonmyopic functional, that is, a functional depending on the entire stochastic process  $W_x$ . Since we want to value the impact of bivariate processes, we propose to optimize a distance between the wealth of the portfolio and wealth obtained from the market stochastic bounds. Upper and lower market stochastic bounds are, respectively, defined by  $y^{(M)} = \max_{x \in S} Z_x$  and  $y^{(m)} = \min_{x \in S} Z_x$  and satisfy the relation  $y^{(M)} \geq Z_x \geq y^{(m)}$  for all vectors of portfolio weights  $x$  belonging to the simplex  $S = \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = 1; x_i \geq 0\}$  (see [19, 20]). Thus, the returns during the period  $[t, t + 1]$  of stochastic bounds are given by  $y_t^{(M)} = \max_{x \in S} Z_{x,t}$  and  $y_t^{(m)} = \min_{x \in S} Z_{x,t}$ . Generally, investors would like to minimize a distance measure between the portfolio and the upper market bound  $y^{(M)}$  and to maximize a distance measure with the lower market bound  $y^{(m)}$ . To account these investors' preferences, we consider the following *OA-Stochastic Bound Ratio* (OA-SBR) performance functional defined by

$$\text{OA-SBR}(W_T(x)) = \frac{E\left(\sum_{t=1}^T \left(W_{t,x} - W_{t,y^{(m)}}\right)_+\right)}{E\left(\sum_{t=1}^T \left(W_{t,y^{(M)}} - W_{t,x}\right)_+\right)}, \quad (3.4)$$

where  $(X)_+ = X \cdot I_{(X \geq 0)}$  denotes the positive part of a function  $W_{t,y^{(M)}}$ ,  $W_{t,y^{(m)}}$  are the wealth processes at time  $t$ , deriving, respectively, from the upper and lower market stochastic bounds. Assuming that the returns follow the Markov chain introduced in Section 2, we can compute the previous expectations for the bivariate processes  $(W_{t,x}, W_{t,y^{(m)}})$  and  $(W_{t,x}, W_{t,y^{(M)}})$  exploiting the results of the previous section. In general, for a bivariate wealth process  $W_t = (W_{t,x}, W_{t,y})$ , we have

$$E(f(W_{t,x} - W_{t,y})) = \sum_{l \in L_t} f\left(w_x^{(l_x, t)} - w_y^{(l_y, t)}\right) p_{W_t}(l), \quad (3.5)$$

with  $p_{W_t}(l) = P(W_{tx} = w_x^{(l_x, t)}, W_{ty} = w_y^{(l_y, t)})$ .

As we show in the next subsection, bivariate Markov processes are useful even to reduce the dimensionality of large scale portfolio selection problems.

### 3.1. Large-Scale Portfolios

The number of observations necessary in the optimization process increases proportionally with the dimension of the portfolios considered. Since the number of observations available on the market is relatively small compared to the number of assets, it is clear that a procedure to reduce the dimensionality of large-scale problems is needed. To this purpose, we apply a principal component analysis (PCA). The idea of PCA is to reduce the dimensionality of a data set made of a large number of possible correlated variables (assets) while preserving the largest possible variability in the data. This is done by transforming the initial variables (assets) into a new set of variables (called the principal components) which are uncorrelated and ordered in decreasing order of importance. Consider the assets returns at time  $t + 1$

$$Z_{i,t+1} = \frac{P_{i,t+1}}{P_{i,t}}. \quad (3.6)$$

Applying the PCA methodology to the Pearson correlation matrix of the historical series, we replace the original  $n$  correlated time series  $\{Z_{i,t}\}_{i=1}^n$  with  $n$  uncorrelated time-series  $\{R_{i,t}\}_{i=1}^n$ . The dimensionality reduction is obtained by choosing only those components (principal components) whose variability is significantly different from zero. We call these principal components *factors* and denote them by  $f_j$ ,  $j = 1, \dots, s$ .

Thus, each series  $Z_i$  can be written as the linear combination of the identified factors plus a small (uncorrelated) noise

$$Z_{i,t} = \sum_{j=1}^s a_{i,j} f_{j,t} + \sum_{j=s+1}^n a_{i,j} R_{j,t} = \sum_{j=1}^s a_{i,j} f_{j,t} + \varepsilon_{i,t}. \quad (3.7)$$

We can further reduce the variability of the error by performing a PCA of the Pearson correlation matrix of the *forecasted wealth* obtained by the single returns. Notice that in order to compute the correlation matrix of the forecasted wealth, it is necessary to use bivariate Markov processes to account the joint behavior of the future wealth (as suggested in Section 2).

Once identified, the  $s$  factors  $f_j$  ( $j = 1, \dots, s$ ) accounting for most of the variability of the returns and the  $r$  factors  $\tilde{f}_l$  ( $l = 1, \dots, r$ ) accounting for most of the variability of the forecasted wealth, we regress the return of each asset on the factors as follows:

$$Z_{i,t} = b_{i,0} + \sum_{j=1}^s b_{i,j} f_{j,t-1} + \sum_{l=1}^r b_{i,l} \tilde{f}_{l,t} + \varepsilon_{i,t}. \quad (3.8)$$

Then, we can use the approximated returns

$$\tilde{Z}_{i,t} = b_{i,0} + \sum_{j=1}^s b_{i,j} f_{j,t-1} + \sum_{l=1}^r b_{i,l} \tilde{f}_{l,t}, \quad (3.9)$$

for selecting the optimal portfolio.



### 3.2. An Empirical Comparison between Portfolio Strategies

In order to value the impact of the bivariate Markovian approximation on portfolio selection strategies, we compare the performance of strategies based either on the Sharpe ratio or on the OA-stochastic bound ratio. The comparison consists of the ex-post evaluation of the wealth produced by the strategies. In particular, we assume that the riskless asset is not allowed; that is, the Sharpe ratio is given by  $SR(Z_x) = E(Z_x - 1)/\sigma_{Z_x}$ . We approximate the Markovianity assuming  $N = 9$  states for each asset and a temporal horizon  $T = 20$  working days.

As dataset, we consider 3805 assets from the main US markets (NYSE and NASDAQ) available in DataStream during the period 05-Aug-2009, 17-Oct-2010. For each optimization, we consider a 6-month time window (about 125 market days) of historical data. Thus, we need a strong dimensionality reduction in order to keep statistical significance of historical data.

For any portfolio optimization, we first preselect the “best” 30 assets following the eight preselection criteria suggested by Ortobelli et al. [20]. The preselection is a methodology to reduce the dimensionality of the portfolio problem. It consists of selecting some assets for their appealing characteristics. In particular, with the proposed preselection criteria, we account the consistency with investors’ preferences, the timing of the choices, the association with market stochastic bounds, and the Markovian and asymptotic behavior of wealth (see [20]). On these preselected assets, we apply the principal component analysis as suggested in Section 3.1. In particular, we consider 14 factors: 7 obtained with the PCA applied to the forecasted Pearson correlation matrix of the future wealth and the other 7 obtained with the PCA applied to the Pearson correlation matrix of the historical series. For any estimation, every 20 working days starting from 05 August 2009, we compute the optimal portfolio composition that maximize each performance ratio (SR or OA-SBR) considering the following constrains on the weights  $0 \leq x_i \leq 0.2$ . Since portfolio selection problems based on the Markovian hypothesis presents more local optima, we solve the optimization problem using the heuristic for global optimization proposed by [11]. Then, we value the ex-post wealth.

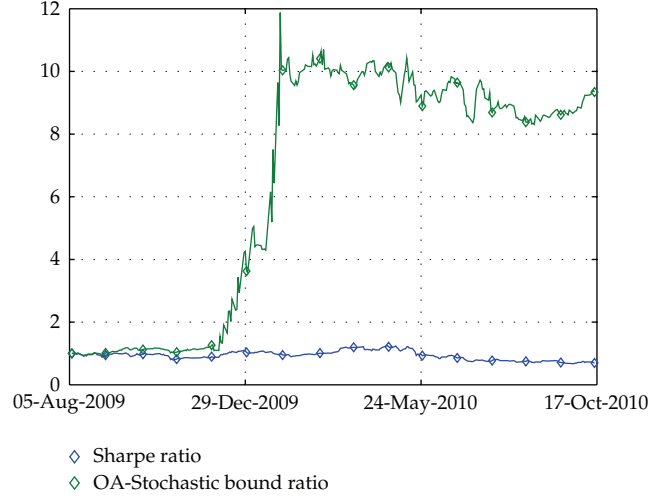
For each strategy, we consider an initial wealth  $W_0 = 1$ , and we use the last 6 months of daily observations. Thus, starting from 05 August 2009 at the  $k$ -th recalibration ( $k = 0, 1, 2, \dots$ ), three main steps are performed to compute the ex-post final wealth.

*Step 1.* Preselect the “best” 30 assets among 3805 assets (as suggested by [11]). On these assets apply the principal component analysis and approximate the returns, as suggested in Section 3.1.

*Step 2.* Determine the market portfolio  $x_M^{(k)}$  that maximizes the performance ratio  $\rho(W(x))$  (SR or OA-SBR) associated to the strategy, that is, the solution of the following optimization problem:

$$\max_{x^{(k)}} \rho\left(W_T\left(x^{(k)}\right)\right) \quad \text{s.t.} \quad \sum_{i=1}^n x_i^{(k)} = 1, \quad 0 \leq x_i^{(k)} \leq 0.2; \quad i = 1, \dots, n. \quad (3.10)$$

*Step 3.* During the period  $[t_k, t_{k+1}]$  (where  $t_{k+1} = t_k + T$ ), we have to recalibrate daily the portfolio maintaining the percentages invested in each asset equal to those of the market



**Figure 1:** *Ex-post* sample paths of wealth obtained maximizing either the Sharpe ratio or the OA-stochastic bound ratio.

portfolio  $x^{(k)}$ . Thus, the ex-post final wealth is given by

$$W_{t_{k+1}} = W_{t_k} \left( \prod_{i=1}^T (x_M^{(k)})' z_{(t_k+i)}^{(\text{ex post})} \right), \quad (3.11)$$

where  $z_{(t_k+i)}^{(\text{ex post})}$  is the vector of observed daily gross returns between  $(t_k + i - 1)$  and  $(t_k + i)$ . The optimal portfolio  $x_M^{(k)}$  is the new starting point for the  $(k + 1)$ th optimization problem.

Steps 1, 2 and 3 are repeated until the observations are available.

Figure 1 reports the ex-post sample paths of the wealth obtained maximizing the Sharpe ratio and the OA-stochastic bound ratio. In particular, we observe that the ex-post wealth of the OA-stochastic bound strategy multiplies of about six times in two months and half during the last week of November 2009 and the first week of February 2010. Instead, the strategy based on the maximization of the Sharpe ratio is not able to produce wealth during the same period. While during the European countries crisis (period from May till September, 2010) the loss of each strategy is no more than the 15% of the wealth. Therefore, this first comparison shows a very high impact (more than 900% in one year) on the ex-post final wealth obtained using the bivariate Markov process.

#### 4. Value at Risk at a Given Time and Applications in Option Pricing Theory

In this section, we consider other two possible applications of the proposed approximation of a bivariate Markov process: the valuation of VaR at a given time  $T$  and the pricing of average strike Asian options.

#### 4.1. VAR at a Given Time $T$

In the classical risk-management problem, a financial institution has to evaluate the market risk exposition of the owned portfolio. The classical tool proposed and used by practitioners is the value at risk (VaR) that synthesizes in a single value the possible losses which could be realized with a given probability, for a fixed temporal horizon. Namely, indicating with  $t$  the current time, with  $\tau$  the investor's temporal horizon, with  $R_t(\tau)$  the profit/loss realized in the interval  $[t, t + \tau]$  and with  $\theta$  a level of confidence, the value at risk  $VaR_{t+\tau, (1-\theta)}(R_t(\tau))$  is the possible loss at time  $t + \tau$  implicitly defined by

$$P(R_t(\tau) \leq -VaR_{t+\tau, (1-\theta)}(R_t(\tau))) = 1 - \theta, \quad (4.1)$$

note that  $VaR_{t+\tau, (1-\theta)}(R_t(\tau))$  is the opposite of the  $(1 - \theta)$ -percentile of the profit/loss distribution in the interval  $[t, t + \tau]$ .

The well-known RiskMetrics model, also called *exponential weighted moving average* (EWMA) model, assumes a Gaussian distribution for the conditional distribution of  $R_t(\tau)$ . Such an hypothesis dramatically simplifies the VaR calculation, in particular for portfolios with many assets whose returns are assumed conditional jointly normal distributed. Thus, if we point out with  $x = [x_1, x_2, \dots, x_n]'$  the composition vector of a portfolio, then the portfolio profit/loss at time  $t + 1$  is given by

$$R_{p,t}(1) = \sum_{i=1}^n x_i R_{i,t+1}, \quad (4.2)$$

where  $R_{i,t+1} = Z_{i,t+1} - E(Z_{i,t+1})$ . We use centered returns to simplify the computation, but clearly, these results can be easily extended to real returns at less of an additive shift. When the conditional joint distribution of centered return vector  $R = [R_{1,t+1}, R_{2,t+1}, \dots, R_{n,t+1}]'$  is Gaussian, every linear combination of the primary components is also normally distributed. Since the expected centered return is null, the 1-day VaR of a portfolio  $p$  with profit/loss  $R_p$  is completely determined from the portfolio standard deviation

$$VaR_{t+1, (1-\theta)}(R_{p,t}(1)) = k_\theta \sigma_{p,t}, \quad (4.3)$$

where  $k_\theta$  is the  $\theta$  percentile of a standard normal distribution,  $\sigma_{p,t} = \sqrt{x' \cdot Q_t \cdot x}$  and  $Q_t = [\sigma_{ij,t}^2]$  is the covariance matrix whose evolution over time is described by

$$\sigma_{ij,t}^2 = \lambda \sigma_{ij,t-1}^2 + (1 - \lambda) R_{i,t-1} R_{j,t-1}. \quad (4.4)$$

where  $\lambda$  is the so called *decay factor* (see [7]).

Moreover, RiskMetrics proposes to approximate the VaR at a given time  $T$  by using the time rule

$$VaR_{t+T, (1-\theta)}(R_{p,t}(T)) \simeq \sqrt{T} VaR_{t+1, (1-\theta)}(R_{p,t}(1)) = \sqrt{T} k_\theta \sigma_{p,t} = \sqrt{T} k_\theta \sqrt{x' \cdot Q_t \cdot x}. \quad (4.5)$$

However, this approximation can produce very big errors (see, among others, [22, 23]). Straightforward extensions of the RiskMetrics model can be obtained by using any other

elliptical distribution (see, among others, [6] and the references therein). That is, if the conditional joint distribution of return vector  $R$  is elliptically distributed, every linear combination of the primary components follows the same elliptical law. For example, if the joint conditional distribution of  $R$  is a  $t$ -Student with  $\nu$  ( $\nu > 2$ ) degrees of freedom, then

- (1) formula (4.3) is still valid provided that we substitute  $k_\theta$  with the  $\theta$ -percentile of a  $t$ -Student with  $\nu$  degrees of freedom and;
- (2) formula (4.5) changes by substituting the  $\theta$ -percentile of a sum of  $T$  standard normal distributions with the  $\theta$ -percentile  $\hat{k}_\theta$  of the sum of  $T$  random variables distributed as  $t$ -Student with  $\nu$  degrees of freedom, (see [6]); that is,

$$VaR_{t+T, (1-\theta)}(R_{p,t}(1)) \simeq \hat{k}_\theta \sigma_{p,t} = \hat{k}_\theta \sqrt{x' \cdot Q_t \cdot x}. \quad (4.6)$$

In order to overcome the approximating error of formulas (4.5) and (4.6), we suggest to value the risk at a given time  $T$  by using the covariance matrix  $Q_{t+T}$  obtained considering the joint distribution of the forecasted wealth at time  $T$  for each couple of risky assets. Therefore, if we assume that the vector of the centered returns of wealth at time  $T$  is conditionally elliptical distributed with null mean and covariance matrix  $Q_{t+T}$ , we get

$$VaR_{t+T, (1-\theta)}(R_p(1)) \simeq \tilde{k}_\theta \sqrt{w' Q_{t+T} w}. \quad (4.7)$$

where  $\tilde{k}_\theta$  is either  $k_\theta$  or  $\hat{k}_\theta / \sqrt{T}$  according to the above definitions of  $k_\theta$  and  $\hat{k}_\theta$ .

Next, we test and compare the performance of the two alternative models. We compute the VaR with  $\theta = 1\%$ ,  $3\%$ ,  $5\%$  by using a Markov model and the classical EWMA model. Both models are implemented with Gaussian and Student assumption. Tests are executed on 440 NASDAQ assets from January 1997 till July 2010. For the elliptical distributions, the average Student degrees of freedom estimated on 01/01/1997 among the 440 assets is 4.732, and we use this value for all the ex-post analyses. For the Markov processes, we consider 9 states and  $T = 20$ . For the ex-post computations, we use a time window of 500 working days, and we assume that the historical observations present an exponential probability with  $\lambda = 0.995$ . We estimated this value as the average (during the period 1997–2000) of optimal decay factor computed as suggested by Lamantia et al. [6]. Moreover, Kondor et al. [24] suggests to use a large value of the decay factor  $\lambda$  (near to 1) to compute the covariance matrix for large portfolio in contrast to “the rule of thumb” ( $\lambda = 0.94$ ) proposed by RiskMetrics (see [7]). We consider 22500 random portfolios of the NASDAQ assets. The average of the number of portfolio observations that violate the VaR limits under the two distributional assumptions are shown in Table 1.

The percentages of violations should be, respectively, equal to the VaR limits 1%, 3%, 5%. From this first analysis, we observe that the Markov valuations respect well enough the percentage of violations, while EWMA models generally overestimate the losses. In particular, when we assume that the conditional distribution of the returns follows a Student distribution, both models seem to give better performance than the Gaussian model. Moreover, we test how this valuation is accurate using the conditional (LRc) and unconditional (LRu) likelihood ratio tests proposed by Christoffersen [25] with 95%

**Table 1**

VaR	1%	3%	5%
Markov Gaussian	0.0100	0.0295	0.0485
EWMA Gaussian	0.0066	0.0206	0.0344
Markov Student	0.0099	0.0302	0.0497
EWMA Student	0.0092	0.0276	0.0398

**Table 2**

VaR	1%	3%	5%
LRu Markov Gaussian	85.1%	85.6%	85.3%
LRc Markov Gaussian	60.3%	60.5%	60.6%
LRu EWMA Gaussian	58.6%	57.2%	56.3%
LRc EWMA Gaussian	41.2%	41.1%	40.8%
LRu Markov Student	92.6%	93.2%	92.8%
LRc Markov Student	75.4%	77.1%	77.7%
LRu EWMA Student	63.3%	64.6%	65.2%
LRc EWMA Student	55.1%	54.8%	55.5%

confidence interval. The percentages of acceptably accurate valuation of VaR are given in Table 2.

Thus, Christoffersen's tests show clearly the best performance of the Markovian approximation even if further analysis are probably still necessary to confirm these studies. In particular, we believe that using other different distributional assumptions that consider also the skewness effects, which are generally observed in the portfolio returns, we should get better results.

#### **4.2. Average Strike Asian Options**

In this last subsection, we deal with the problem of pricing average strike price options by using a bivariate Markov process. With average strike Asian options, the final payoff at a maturity  $T$  is given by

(i)  $\max(S_{\text{Ave}} - S_T, 0)$  for a put option,

(ii)  $\max(S_T - S_{\text{Ave}}, 0)$  for a call option,

where  $S_T$  is the stock price at a given time  $T$  and  $S_{\text{Ave}}$  is the average price during the period  $[0, T]$ . It is well known that when the average is the arithmetic mean, we have not a close form solution for option pricing even when we assume that prices evolve as a geometric Brownian motion. Generally, to price continuous arithmetic average strike Asian options analysts calculate the first and second moments and then fits the approximating lognormal distribution—for the average—to the moments. Further approximations are needed if the average is done on daily prices.

Since by means of the bivariate Markov process we can easily value the joint distribution of two random variables, we can describe the joint Markovian behavior of the random vector  $(Z_t, U_t)$ , where  $Z_t = S_t/S_{t-1}$ ,  $S_t$  is the stock price at time  $t$ , and  $U_t = \exp(S_t/S_0)$ . Considering a joint Markovian evolution of the vector  $(Z_t, U_t)$ , we get

a pyramidal tree that after  $T$  steps describes the "wealth process"  $(W_{1,T}, W_{2,T})$ , where  $S_T = S_0 W_{1,T}$  and  $S_{Ave} = S_0 \ln(W_{2,T})/T$ .

Thus, at less of an increasing transformation, we have the joint distribution of  $(S_T, S_{Ave})$ . Therefore, we can price Bermudan and European average strike Asian options using the Iaquina and Ortobelli's algorithm [26] to compute the risk neutral matrix and the prices. An empirical analysis of this option pricing model requires the use of data from the over-the-counter (OTC) market (market where are priced these derivatives), and it should be object of future discussions and studies.

## 5. Conclusions

This paper proposes a simple way to value bivariate Markov processes in portfolio, risk management, and option pricing problems. In particular, we have observed that the Markovian previsions of the future present a very big impact on the portfolio choices. Moreover, the bivariate Markov process can be used to estimate the covariance matrix at a given future time. Thus, using the forecasted variability, we can value the risk of a given portfolio at a future time  $T$ . The comparison of the Markovian prevision with the classical EWMA model shows the highest performance of the first. Finally, we have discussed how to deal with average strike options by using the proposed approximation of a bivariate Markov process.

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