Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2011, Article ID 579645, 10 pages
doi:10.1155/2011/579645

Research Article

On the General Solution of the Ultrahyperbolic Bessel Operator

Rattapan Damkengpan[1] and Kamsing Nonlaopon[1,2]

[1] *Department of Mathematics, Khon Kaen University, Khon Kaen 40002, Thailand*
[2] *Centre of Excellence in Mathematics, Commission on Higher Education (CHE), Si Ayuthaya road, Bangkok 10400, Thailand*

Correspondence should be addressed to Kamsing Nonlaopon, nkamsi@kku.ac.th

Received 6 April 2011; Accepted 21 June 2011

Academic Editor: Alexei Mailybaev

We study the general solution of equation $\square_{B,c}^k u(x) = f(x)$, where $\square_{B,c}^k$ is the ultrahyperbolic Bessel operator iterated k-times and is defined by $\square_{B,c}^k = [(1/c^2)(B_{x_1} + B_{x_2} + \cdots + B_{x_p}) - (B_{x_{p+1}} + \cdots + B_{x_{p+q}})]^k$, $p + q = n$, n is the dimension of $\mathbb{R}_n^+ = \{x : x = (x_1, x_2, \ldots, x_n),\ _x > 0, \ldots, x_n > 0\}$, $B_{x_i} = \partial^2/\partial x_i^2 + (2v_i/x_i)(\partial/\partial x_i)$, $2v_i = 2\beta_i + 1$, $\beta_i > -1/2$, $x_i > 0\ (i = 1, 2, \ldots, n)$, $f(x)$ is a given generalized function, $u(x)$ is an unknown generalized function, k is a nonnegative integer, c is a positive constant, and $x \in \mathbb{R}_n^+$.

1. Introduction

The n-dimensional ultrahyperbolic operator $\square^k$ iterated k-times is defined by

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k, \tag{1.1}$$

where $p + q = n$, n is the dimension of space $\mathbb{R}^n$, and k is a nonnegative integer.

Consider the linear differential equation of the form

$$\square^k u(x) = f(x), \tag{1.2}$$

where $u(x)$ and $f(x)$ are generalized functions and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

Gel'fand and Shilov [1] first introduced the fundamental solution of (1.2), which is a complicated form. Later, Trione [2] has shown that the generalized function $R_{2k}(x)$, defined by (2.8) with $|v| = 0$, is a unique fundamental solution of (1.2) and Téllez [3] also proved that $R_{2k}(x)$ exists only in the case when p is odd with n odd or even and $p+q = n$. A wealth of some effective works on the fundamental solution of the n-dimensional classical ultrahyperbolic operator have, presented by Kananthai and Sritanratana [4–9].

In 2004, Yildirim et al. [10] have introduced the Bessel ultrahyperbolic operator iterated k-times with $x \in \mathbb{R}_n^+ = \{x : x = (x_1, x_2, \ldots, x_n),\ x_1 > 0, \ldots, x_n > 0\}$,

$$\Box_B^k = \left(B_{x_1} + B_{x_2} + \cdots + B_{x_p} - B_{x_{p+1}} - \cdots - B_{x_{p+q}}\right)^k, \tag{1.3}$$

where $p + q = n$, $B_{x_i} = \partial^2/\partial x_i^2 + (2v_i/x_i)(\partial/\partial x_i)$, $2v_i = 2\beta_i + 1$, $\beta_i > -1/2$ [11], k is a nonnegative integer, and n is the dimension of $\mathbb{R}_n^+$. They also have studied the fundamental solution of Bessel ultrahyperbolic operator.

In 2007, Sarikaya and Yildirim [12] have studied the weak solution of the compound Bessel ultrahyperbolic equation and also studied the Bessel ultrahyperbolic heat equation [13].

In 2009, Saglam et al. [14] have developed the operator of (1.3), defined by (1.6), and it is called the ultrahyperbolic Bessel operator iterated k-times. They have also studied the product of the ultrahyperbolic Bessel operator related to elastic waves.

Next, Srisombat and Nonlaopon [15] have studied the weak solution of

$$\Box_{B,c}^k u(x) = f(x), \tag{1.4}$$

where $u(x)$ and $f(x)$ are some generalized functions. They have developed (1.4) into the form

$$\sum_{k=0}^{m} C_k \Box_{B,c}^k u(x) = f(x), \tag{1.5}$$

which is called the compound ultrahyperbolic Bessel equation. In finding the solution of (1.5), they have used the properties of B-convolution for the generalized functions.

The purpose of this study is to find the general solution of equation $\Box_{B,c}^k u(x) = f(x)$, where $\Box_{B,c}^k$ is the ultrahyperbolic Bessel operator iterated k-times and is defined by

$$\Box_{B,c}^k = \left[\frac{1}{c^2}\left(B_{x_1} + B_{x_2} + \cdots + B_{x_p}\right) - \left(B_{x_{p+1}} + \cdots + B_{x_{p+q}}\right)\right]^k \tag{1.6}$$

$p + q = n$, n is the dimension of $\mathbb{R}_n^+ = \{x : x = (x_1, x_2, \ldots, x_n),\ x_1 > 0, \ldots, x_n > 0\}$, $B_{x_i} = \partial^2/\partial x_i^2 + (2v_i/x_i)(\partial/\partial x_i)$, $2v_i = 2\beta_i + 1$, $\beta_i > -1/2$, $x_i > 0$ $(i = 1, 2, \ldots, n)$, $f(x)$ is a given generalized function, $u(x)$ is an unknown generalized function, k is a nonnegative integer, c is a positive constant, and $x \in \mathbb{R}_n^+$.

2. Preliminaries

Let T_x^y be the generalized shift operator acting on the function φ, according to the law [11, 16]:

$$T_x^y \varphi(x) = C_v^* \int_0^\pi \cdots \int_0^\pi \varphi\left(\sqrt{x_1^2 + y_1^2 - 2x_1 y_1 \cos\theta_1}, \ldots, \sqrt{x_n^2 + y_n^2 - 2x_n y_n \cos\theta_n}\right) \times \left(\prod_{i=1}^{n} \sin^{2v_i - 1}\theta_i\right) d\theta_1 \cdots d\theta_n, \tag{2.1}$$

where $x, y \in \mathbb{R}_n^+$ and $C_v^* = \prod_{i=1}^{n}(\Gamma(v_i + 1)/\Gamma(1/2)\Gamma(v_i))$. We remark that this shift operator is closely connected to the Bessel differential operator [11]:

$$\frac{d^2U}{dx^2} + \frac{2v}{x}\frac{dU}{dx} = \frac{d^2U}{dy^2} + \frac{2v}{y}\frac{dU}{dy}, \quad U(x,0) = f(x), \quad U_y(x,0) = 0. \tag{2.2}$$

The convolution operator is determined by the T_x^y as follows:

$$(f * \varphi)(y) = \int_{\mathbb{R}_n^+} f(y) T_x^y \varphi(x) \left(\prod_{i=1}^{n} y_i^{2v_i}\right) dy. \tag{2.3}$$

The convolution (2.3) is known as a B-convolution. We note the following properties of the B-convolution and the generalized shift operator.

(a) $T_x^y \cdot 1 = 1$.

(b) $T_x^0 \cdot f(x) = f(x)$.

(c) If $f(x), g(x) \in C(\mathbb{R}_n^+)$, $g(x)$ is a bounded function all $x > 0$, and

$$\int_{\mathbb{R}_n^+} |f(x)| \left(\prod_{i=1}^{n} x_i^{2v_i}\right) dx < \infty, \tag{2.4}$$

then

$$\int_{\mathbb{R}_n^+} T_x^y f(x) g(y) \left(\prod_{i=1}^{n} y_i^{2v_i}\right) dy = \int_{\mathbb{R}_n^+} f(y) T_x^y g(x) \left(\prod_{i=1}^{n} y_i^{2v_i}\right) dy. \tag{2.5}$$

(d) From (c), we have the following equality for $g(x) = 1$:

$$\int_{\mathbb{R}_n^+} T_x^y f(x) \left(\prod_{i=1}^{n} y_i^{2v_i}\right) dy = \int_{\mathbb{R}_n^+} f(y) \left(\prod_{i=1}^{n} y_i^{2v_i}\right) dy. \tag{2.6}$$

(e) $(f * g)(x) = (g * f)(x)$.

Definition 2.1. Let $x = (x_1, x_2, \dots, x_n)$ be a point of the n-dimensional space $\mathbb{R}_n^+$. Denote the nondegenerated quadratic form by

$$V = c^2\left(x_1^2 + x_2^2 + \cdots + x_p^2\right) - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2, \tag{2.7}$$

where $p + q = n$. The interior of the forward cone is defined by $\Gamma_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}_n^+ : x_i > 0,\ i = 1, \dots, n \text{ and } V > 0\}$, where $\overline{\Gamma}_+$ designates its closure. For any complex number α, we define

$$R_{\alpha,c}^H(x) = \begin{cases} \dfrac{V^{(\alpha-n-2|v|)/2}}{K_n(\alpha)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.8}$$

where

$$K_n(\alpha) = \frac{\pi^{(n+2|v|-1)/2}\Gamma((2+\alpha-n-2|v|)/2)\Gamma((1-\alpha)/2)\Gamma(\alpha)}{\Gamma((2+\alpha-p-2|v|)/2)\Gamma((p+2|v|-\alpha)/2)}. \tag{2.9}$$

The function $R_{\alpha,c}^H(x)$ is introduced by [10, 12, 17, 18]. It is well known that $R_{\alpha,c}^H(x)$ is an ordinary function if $\mathrm{Re}(\alpha) \geq n$ and is the distribution of α if $\mathrm{Re}(\alpha) < n$. Let supp $R_{\alpha,c}^H(x) \subset \overline{\Gamma}_+$, where supp $R_{\alpha,c}^H(x)$ denotes the support of $R_{\alpha,c}^H(x)$.

By putting $p = c = 1$ into (2.7), (2.8), and (2.9), and using the Legendre's duplication of $\Gamma(z)$,

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right), \tag{2.10}$$

the formula (2.8) is reduced to

$$M_\alpha^H(x) = \begin{cases} \dfrac{V^{((\alpha-n-2|v|)/2)}}{H_n(\alpha)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.11}$$

where $V = x_1^2 - x_2^2 - \cdots - x_n^2$ and

$$H_n(\alpha) = \pi^{(n+2|v|-1)/2} 2^{\alpha-1}\Gamma\left(\frac{2+\alpha-n-2|v|}{2}\right)\Gamma\left(\frac{\alpha}{2}\right). \tag{2.12}$$

Note that the function $M_\alpha^H(x)$ is precisely the Bessel hyperbolic kernel of Marcel Riesz.

Lemma 2.2. *Given the equation*

$$\Box_{B,c}^k u(x) = \delta(x), \tag{2.13}$$

where $\square_{B,c}^k$ *is defined by* (1.6) *and* $x \in \mathbb{R}_n^+$*, then we obtain* $u(x) = R_{2k,c}^H(x)$ *as a fundamental solution of* (2.13)*, where* $R_{2k,c}^H(x)$ *is defined by* (2.8)*.*

The proof of this Lemma is given in [14].

Lemma 2.3. *The B-convolutions of tempered distributions.*

(a) $(\square_{B,c}^k \delta) * u(x) = \square_{B,c}^k u(x)$*, where* $u(x)$ *is any tempered distribution.*

(b) *Let* $R_{2k,c}^H(x)$ *and* $R_{2m,c}^H(x)$ *be defined by* (2.8)*; then* $R_{2k,c}^H(x) * R_{2m,c}^H(x)$ *exists and is a tempered distribution.*

(c) *Let* $R_{2k,c}^H(x)$ *and* $R_{2m,c}^H(x)$ *be defined by* (2.8)*; then* $R_{2k,c}^H(x) * R_{2m,c}^H(x) = R_{2k+2m,c}^H(x)$*, where* k *and* m *are nonnegative integers.*

The proof of this Lemma is given in [15].

Lemma 2.4. *Given that* P *is a hypersurface*

$$P\delta^{(m)}(P) + mP\delta^{(m-1)}(P) = 0, \tag{2.14}$$

where $\delta^{(m)}$ *is the Dirac-delta distribution with* m *derivatives.*

The proof of this Lemma is given in [1].

Lemma 2.5. *Given the equation*

$$\square_{B,c}^k u(x) = 0, \tag{2.15}$$

where $\square_{B,c}^k$ *is the ultrahyperbolic Bessel operator iterated* k*-times, as defined by* (1.6)*, and* $x \in \mathbb{R}_n^+$*, then*

$$u(x) = \left[R_{2(k-1),c}^H(x)\right]^{(m)}, \tag{2.16}$$

defined by (2.8) *with* m *derivatives, as a solution of* (2.15) *with* $m = ((n+2|v|-4)/2)$*,* $n+2|v| \geq 4$ *and* n *is an even dimension.*

Proof. We first show that the generalized function $\delta^{(m)}(c^2r^2 - s^2)$, where $r^2 = x_1^2 + x_2^2 + \cdots + x_p^2$, $s^2 = x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}^2$, $p + q = n$, is a solution of

$$\square_{B,c} u(x) = 0, \tag{2.17}$$

and $\square_{B,c}$ is defined by (1.6) with $k = 1$ and $x \in \mathbb{R}_n^+$. Now for $1 \leq i \leq p$, we have

$$\begin{aligned} \frac{\partial}{\partial x_i}\delta^{(m)}\left(c^2r^2 - s^2\right) &= 2c^2 x_i \delta^{(m+1)}\left(c^2r^2 - s^2\right), \\ \frac{\partial^2}{\partial x_i^2}\delta^{(m)}\left(c^2r^2 - s^2\right) &= 2c^2\delta^{(m+1)}\left(c^2r^2 - s^2\right) + 4c^4x_i^2\delta^{(m+2)}\left(c^2r^2 - s^2\right). \end{aligned} \tag{2.18}$$

Thus, we have

$$\begin{aligned}
&\frac{1}{c^2}\sum_{i=1}^{p}\left[\frac{\partial^2}{\partial x_i^2}\delta^{(m)}\left(c^2r^2-s^2\right)+\frac{2v_i}{x_i}\frac{\partial}{\partial x_i}\delta^{(m)}\left(c^2r^2-s^2\right)\right]\\
&\quad= 2p\delta^{(m+1)}\left(c^2r^2-s^2\right)+4c^2r^2\delta^{(m+2)}\left(c^2r^2-s^2\right)+4\left|v'\right|\delta^{(m+1)}\left(c^2r^2-s^2\right)\\
&\quad= \left(2p+4\left|v'\right|\right)\delta^{(m+1)}\left(c^2r^2-s^2\right)+4\left(c^2r^2-s^2\right)\delta^{(m+2)}\left(c^2r^2-s^2\right)+4s^2\delta^{(m+2)}\left(c^2r^2-s^2\right)\\
&\quad= \left(2p+4\left|v'\right|\right)\delta^{(m+1)}\left(c^2r^2-s^2\right)-4(m+2)\delta^{(m+1)}\left(c^2r^2-s^2\right)+4s^2\delta^{(m+2)}\left(c^2r^2-s^2\right)\\
&\quad= \left[2p+4\left|v'\right|-4(m+2)\right]\delta^{(m+1)}\left(c^2r^2-s^2\right)+4s^2\delta^{(m+2)}\left(c^2r^2-s^2\right)
\end{aligned} \tag{2.19}$$

by applying Lemma 2.4 with $P = c^2r^2 - s^2$, where $|v'| = v_1 + v_2 + \cdots + v_p$.

Similarly, we have

$$\begin{aligned}
&\sum_{i=p+1}^{p+q}\left[\frac{\partial^2}{\partial x_i^2}\delta^{(m)}\left(c^2r^2-s^2\right)+\frac{2v_i}{x_i}\frac{\partial}{\partial x_i}\delta^{(m)}\left(c^2r^2-s^2\right)\right]\\
&\quad= \left[-\left(2q+4\left|v''\right|\right)+4(m+2)\right]\delta^{(m+1)}\left(c^2r^2-s^2\right)+4c^2r^2\delta^{(m+2)}\left(c^2r^2-s^2\right)
\end{aligned} \tag{2.20}$$

by applying Lemma 2.4 with $P = c^2r^2 - s^2$, where $|v''| = v_{p+1} + v_{p+2} + \cdots + v_{p+q}$.

Thus, we have

$$\begin{aligned}
\Box_{B,c}\delta^{(m)}\left(c^2r^2-s^2\right) &= \frac{1}{c^2}\sum_{i=1}^{p}\left[\frac{\partial^2}{\partial x_i^2}+\frac{2v_i}{x_i}\frac{\partial}{\partial x_i}\right]\delta^{(m)}\left(c^2r^2-s^2\right)\\
&\quad-\sum_{i=p+1}^{p+q}\left[\frac{\partial^2}{\partial x_i^2}+\frac{2v_i}{x_i}\frac{\partial}{\partial x_i}\right]\delta^{(m)}\left(c^2r^2-s^2\right)\\
&= \left[2\left(p+q+2|v|\right)-8(m+2)\right]\delta^{(m+1)}\left(c^2r^2-s^2\right)\\
&\quad-4\left(c^2r^2-s^2\right)\delta^{(m+2)}\left(c^2r^2-s^2\right)\\
&= [2(n+2|v|)-8(m+2)]\delta^{(m+1)}\left(c^2r^2-s^2\right)+4(m+2)\delta^{(m+1)}\left(c^2r^2-s^2\right)\\
&= [2(n+2|v|)-4(m+2)]\delta^{(m+1)}\left(c^2r^2-s^2\right)
\end{aligned} \tag{2.21}$$

by applying Lemma 2.4 with $P = c^2r^2 - s^2$, where $|v| = |v'| + |v''|$.

If $[2(n+2|v|) - 4(m+2)] = 0$, we obtain

$$\Box_{B,c}\delta^{(m)}\left(c^2r^2-s^2\right) = 0. \tag{2.22}$$

That is, $u(x) = \delta^{(m)}(c^2r^2 - s^2)$ is a solution of (2.15) with $m = (n+2|v|-4)/2$, $n+2|v| \geq 4$, and n is an even dimension. Now $\square_{B,c}^{k} u(x) = 0$ can be written in the form

$$\square_{B,c}\left(\square_{B,c}^{k-1} u(x)\right) = 0. \tag{2.23}$$

From (2.17), we have

$$\square_{B,c}^{k-1} u(x) = \delta^{(m)}\left(c^2r^2 - s^2\right) \tag{2.24}$$

with $m = (n + 2|v| - 4)/2, n + 2|v| \geq 4$, and n being an even dimension. By Lemma 2.3(a), we can write (2.24) in the from

$$\square_{B,c}^{k-1}\delta * u(x) = \delta^{(m)}\left(c^2r^2 - s^2\right). \tag{2.25}$$

B-convolving both sides of the above equation with the function $R^{H}_{2(k-1),c}(x)$, we obtain

$$\begin{gathered}
R^{H}_{2(k-1),c}(x) * \square_{B,c}^{k-1}\delta * u(x) = R^{H}_{2(k-1),c}(x) * \delta^{(m)}(c^2r^2 - s^2), \\
\square_{B,c}^{k-1}\left[R^{H}_{2(k-1),c}(x)\right] * u(x) = \left[R^{H}_{2(k-1),c}(x)\right]^{(m)}, \\
\delta * u(x) = u(x) = \left[R^{H}_{2(k-1),c}(x)\right]^{(m)},
\end{gathered} \tag{2.26}$$

by Lemma 2.2.

It follows that $u(x) = [R^{H}_{2(k-1),c}(x)]^{(m)}$ is a solution of (2.15) with $m = (n+2|v|-4)/2$, $n+2|v| \geq 4$ and n is an even dimension.

The generalized function $\delta^{(m)}(c^2r^2-s^2)$ mentioned in Lemma 2.5 has been also studied on the aspect of multiplicative product, distributional product and applications, for more details, see [19–23]. □

3. Main Result

Theorem 3.1. *Given the equation*

$$\square_{B,c}^{k} u(x) = f(x), \tag{3.1}$$

where $\square_{B,c}^{k}$ *is the ultrahyperbolic Bessel operator iterated* k*-times and is defined by* (1.6), $f(x)$ *is a generalized function,* $u(x)$ *is an unknown generalized function,* $x \in \mathbb{R}_n^+$, *and* n *is an even, then* (3.1) *has the general solution*

$$u(x) = \left[R^{H}_{2(k-1),c}(x)\right]^{(m)} + R^{H}_{2k,c}(x) * f(x), \tag{3.2}$$

where $[R^{H}_{2k,c}(x)]^{(m)}$ *is a function defined by* (2.8) *with* m *derivatives.*

Proof. B-convolving both sides of (3.1) with $R^H_{2k,c}(x)$, we obtain

$$R^H_{2k,c}(x) * \left(\square^k_{B,c} u(x)\right) = R^H_{2k,c}(x) * f(x). \quad (3.3)$$

By Lemma 2.2, we have

$$\square^k_{B,c}\left(R^H_{2k,c}(x)\right) * u(x) = \delta * u(x) = R^H_{2k,c}(x) * f(x). \quad (3.4)$$

So, we obtain that

$$u(x) = R^H_{2k,c}(x) * f(x) \quad (3.5)$$

is the solution of (3.1).

For a homogeneous equation $\square^k_{B,c} u(x) = 0$, we have a solution

$$u(x) = \left[R^H_{2(k-1),c}(x)\right]^{(m)} \quad (3.6)$$

by Lemma 2.5. Thus the general solution of (3.1) is

$$u(x) = \left[R^H_{2(k-1),c}(x)\right]^{(m)} + R^H_{2k,c}(x) * f(x). \quad (3.7)$$

This completes the proof. □

By putting $c = 1$, (3.1) becomes the Bessel ultrahyperbolic equation

$$\square^k_B w(x) = f(x), \quad (3.8)$$

where $\square^k_B$ is the Bessel ultrahyperbolic operator iterated k-times, and is defined by (1.3), $f(x)$ is a generalized function and $w(x)$ is an unknown generalized function. From (3.5) we have that

$$w(x) = R^H_{2k}(x) * f(x) \quad (3.9)$$

is a solution of (3.8), where $R^H_{2k}(x) = R^H_{2k,1}(x)$ defined by (2.8).

From (3.2), we obtain that the general solution of the Bessel ultrahyperbolic equation is

$$w(x) = \left[R^H_{2(k-1)}(x)\right]^{(m)} + R^H_{2k}(x) * f(x). \quad (3.10)$$

Moreover, if we put $k = 1$, $p = 1$ and $x_1 = t$(times), then (3.8) is reduced to the Bessel wave equation

$$\square_B w(x) = \left(B_t - \sum_{i=2}^{n} B_{x_i}\right) w(x) = f(x), \tag{3.11}$$

where

$$\square_B = B_t - \sum_{i=2}^{n} B_{x_i} \tag{3.12}$$

is the Bessel wave operator and $B_{x_i} = \partial^2/\partial x_i^2 + (2v_i/x_i)(\partial/\partial x_i)$.

Thus, we obtain $w(x) = M_2(x) * f(x)$ as a solution of the Bessel wave equation, since $R_2^H(x)$ becomes $M_2^H(x)$, where $M_2^H(x)$ is the Bessel ultrahyperbolic kernel of Marcel Riesz, and is defined by (2.11) with $\alpha = 2$. And from (3.2), we obtain the general solution of Bessel wave equation as

$$w(x) = \delta^{(m)}(x) + M_2^H(x) * f(x), \tag{3.13}$$

where $\delta^{(m)}(x)$ is a solution of

$$\left(B_t - \sum_{i=2}^{n} B_{x_i}\right) w(x) = 0. \tag{3.14}$$

Now we put $V = t^2 - x_2^2 - x_3^2 - \cdots - x_n^2$ and $s^2 = x_2^2 + x_3^2 + \cdots + x_n^2$. By [24], we obtain that

$$w(x,t) = \delta^{(m)}\left(t^2 - s^2\right) \tag{3.15}$$

is the solution of (3.14) with the initial conditions $w(x,0) = 0$ and $\partial w(x,0)/\partial t = (-1)^m 2\pi^{m+1}\delta(x)$ at $t = 0$ and $x = (x_2, x_3, \ldots, x_n) \in \mathbb{R}_{n-1}^+$.

Acknowledgments

The authors would like to thank an anonymous referee who provided very useful comments and suggestions. This work is supported by the Commission on Higher Education, the Thailand Research Fund, and Khon Kaen University (contract number MRG5380118), and the Centre of Excellence in Mathematics, Thailand.

References

[1] I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, vol. 1, Academic Press, New York, NY, USA, 1964.

[2] S. E. Trione, "On Marcel Riesz's ultra-hyperbolic kernel," *Trabajos de Mathematica*, vol. 116, pp. 1–12, 1987.

[3] M. A. Téllez, "The distributional Hankel transform of Marcel Riesz's ultrahyperbolic kernel," *Studies in Applied Mathematics*, vol. 93, no. 2, pp. 133–162, 1994.
[4] A. Kananthai, "On the convolution equation related to the diamond kernel of Marcel Riesz," *Journal of Computational and Applied Mathematics*, vol. 100, no. 1, pp. 33–39, 1998.
[5] A. Kananthai, "On the convolution equation related to the N-dimensional ultra-hyperbolic operator," *Journal of Computational and Applied Mathematics*, vol. 115, no. 1-2, pp. 301–308, 2000.
[6] A. Kananthai, "On the diamond operator related to the wave equation," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 47, no. 2, pp. 1373–1382, 2001.
[7] A. Kananthai and G. Sritanratana, "On the nonlinear diamond operator related to the wave equation," *Nonlinear Analysis: Real World Applications*, vol. 3, no. 4, pp. 465–470, 2002.
[8] A. Kananthai, "On the product of the ultra-hyperbolic operator related to the elastic waves," *Computational Technologies*, vol. 4, no. 6, pp. 88–91, 1999.
[9] A. Kananthai, "On the solutions of the n-dimensional diamond operator," *Applied Mathematics and Computation*, vol. 88, no. 1, pp. 27–37, 1997.
[10] H. Yildirim, M. Z. Sarikaya, and S. Öztürk, "The solutions of the n-dimensional Bessel diamond operator and the Fourier-Bessel transform of their convolution," *Proceedings Indian Academy of Sciences*, vol. 114, no. 4, pp. 375–387, 2004.
[11] B. M. Levitan, "Expansion in Fourier series and integrals with Bessel functions," *Uspekhi Matematicheskikh Nauk*, vol. 62, no. 42, pp. 102–143, 1951.
[12] M. Z. Sarikaya and H. Yildirim, "On the weak solutions of the compound Bessel ultra-hyperbolic equation," *Applied Mathematics and Computation*, vol. 189, no. 1, pp. 910–917, 2007.
[13] A. Saglam, H. Yildirim, and M. Z. Sarikaya, "On the Bessel ultra-hyperbolic heat equation," *Thai Journal of Mathematics*, vol. 8, no. 1, pp. 149–159, 2010.
[14] A. Saglam, H. Yıldırım, and M. Z. Sarıkaya, "On the product of the ultra-hyperbolic Bessel operator related to the elastic waves," *Selcuk Journal of Applied Mathematics*, vol. 10, no. 1, pp. 85–93, 2009.
[15] P. Srisombat and K. Nonlaopon, "On the weak solutions of the compound ultra-hyperbolic Bessel equation," *Selcuk Journal of Applied Mathematics*, vol. 11, no. 1, pp. 127–136, 2010.
[16] H. Yıldırım, *Riesz potentials generated by a generalized shift operator*, Ph.D. thesis, Ankara University, Ankara, Turkey, 1995.
[17] M. Z. Sarıkaya and H. Yıldırım, "On the Bessel diamond and the nonlinear Bessel diamond operator related to the Bessel wave equation," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 68, no. 2, pp. 430–442, 2008.
[18] M. Z. Sarıkaya, *On the elementary solution of the Bessel diamond operator*, Ph.D. thesis, Afyon Kocatepe University, Afyon Kocatepe, Turkey, 2007.
[19] M. Aguirre Téllez, "The distributional product of Dirac's delta in a hypercone," *Journal of Computational and Applied Mathematics*, vol. 115, no. 1-2, pp. 13–21, 2000.
[20] M. A. Aguirre Téllez, "The product of convolution $P\pm^{\lambda} * P\mp^{\mu}$ and the multiplicative product $P\pm^{\lambda} * \delta^{(k)}(P\pm)$," *Mathematical and Computer Modelling*, vol. 23, no. 10, pp. 135–144, 1996.
[21] M. A. Aguirre and K. Nonlaopan, "Generalization of distributional product of Diracs delta in hypercone," *Integral Transforms and Special Functions*, vol. 18, no. 3-4, pp. 155–164, 2007.
[22] S. E. Trione, "Product between $\delta^{(k)}(m^2+P)$ and the distribution (m^2+P)," *Trabajos de Matematica*, vol. 55, 1983.
[23] M. Oberguggenberger, *Multiplication of Distributions and Applications to Partial Differential Equations*, vol. 259 of *Pitman Research Notes in Mathematics*, Longman, Harlow, UK, 1992.
[24] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Interscience, New York, NY, USA, 1966.