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Research Article

On Step Approximation for Roseau's Analytical Solution of Water Waves

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An indirect eigenfunction marching method (IEMM) is developed to provide step approximations for water wave problems. The bottom profile is in terms of successive flat shelves separated by abrupt steps. The marching conditions are represented by the horizontal velocities at the steps in the solution procedure. The approximated wave field can be obtained by solving a system of linear equations with unknown coefficients which represents the horizontal velocities under a proper basis. It is also demonstrated that this solution method can be exactly reduced to the transfer-matrix method (TM method) for a specific setting. The combined scattering effects of a series of steps can be described by a single two-by-two transfer matrix for connecting the far-field behaviors of both sides for this method. The solutions obtained by the IEMM are basically *exact* for water wave problems considering step-like bottoms. Numerical simulations were performed to validate the present and commonly used methods. Furthermore, it also shows that the solutions obtained by the IEMM converge very well to Roseau's analytical solutions for both mild and steep curved bottom profiles. The present method improves the converges of the TM method for solving water wave scattering over steep bathymetry.

1. Introduction

The scattering of linear water waves over arbitrary bed topography has been of interest for years. Analytic solutions are rare except for the cases of constant slope and Roseau's analytical solution [1]. Consequently, approximations are alternative for solving water waves. For example, Berkhoff [2] derived a mild-slope equation (MSE) by removing the vertical coordinate using the integration of depth function, and hence reducing the three-dimensional problem to a two-dimensional one. The mild-slope equation was later modified and extended with additional higher-order terms by several researches [3–6]. On the other hand, there are also other methods applicable for describing water wave problems, such as the integral equation method [7] and differential equation method in transformed domain [8] and so on.

For a two-dimensional problem, if the geometry of interest can be divided into separate regions with a constant fluid depth in each subdomain, the solution in each constant-depth subdomain is usually constructed in terms of eigenfunctions. The solutions are then matched at the vertical boundaries, resulting in a system of linear integral equations which must be truncated to a finite number of terms and solved numerically. Takano [9] used this direct eigenfunction marching method (DEMM) for solving the cases of waves travelling over an elevated sill and a fixed surface obstacle of normal wave incidence. Kirby and Dalrymple [10] extended the DEMM to the problems of waves passing a trench of oblique incidence. Moreover, Söylemez and Gören [11] applied the DEMM to problems of wave scattering over rectangle barriers.

On the other hand, Miles [12] innovated a variational formulation by approximating the horizontal velocity at each step using the corresponding propagating eigenfunction and incorporating with the wide-spacing approximation [13]. His method is formulated by a two-by-two scattering matrix for connecting the amplitudes of propagating modes in the two shelves near a step. Miles [12] used his variational formulation to study Newman's problem [14] of wave scattering by a step of infinite or arbitrarily finite depth. Mei and Black [15] applied the variational formulation to surface-wave scattering over rectangular obstacles. For waves propagating over a sequence of steps, Devillard et al. [13] generalized Miles' variational formulation by introducing a transfer matrix for representing the combined effects of all steps. For the case of an arbitrary bottom topography, O'Hare and Davies [16] approximated the smoothly varying bottom configuration using a series of shelves which are separated by abrupt steps and applied the prescribed method to find the desired transfer matrix of the problem. Furthermore, O'Hare and Davies [17] demonstrated that their solutions are comparable with those obtained using the MSE. However, their solutions converge to pseudosolutions for problems of steep bottom profiles.

In this study, an indirect eigenfunction marching method (IEMM) is developed to provide step approximations for describing wave propagating over an arbitrary bottom represented by several flat shelves separated by abrupt steps. In addition, it is demonstrated that the solution can be exactly reduced to the transfer-matrix (TM) method of Miles [12] and Devillard et al. [13] if the horizontal velocity at each step is approximated by the propagating eigenfunction and the wide-spacing approximation [13] is assumed. The accuracy of this new eigenfunction marching method is investigated to solve the problem of water wave scattering over steps. Furthermore, its applicability for wave propagating over an arbitrary bottom topography is examined by using Roseau's analytical solution [1]. Our results show that the solutions converge very well for both mild and steep curved beach profiles. This method circumvents the pseudoconverges for the TM method for solving problems of wave propagating over steep bottom profiles.

This paper is organized as follows: the wave problem is mathematically modeled in Section 2. Then, the indirect eigenfunction marching method is developed in Section 3. Its reduction to transfer-matrix method is given in Section 4. Some numerical experiments are carried out to validate the prescribed method in Section 5 and the conclusions are drawn in Section 6.

2. Wave Model

We consider the one-dimensional problem for a monochromatic wave propagating over an arbitrary bottom configuration with time dependence $e^{-i\omega t}$, where t is the time, ω the angular

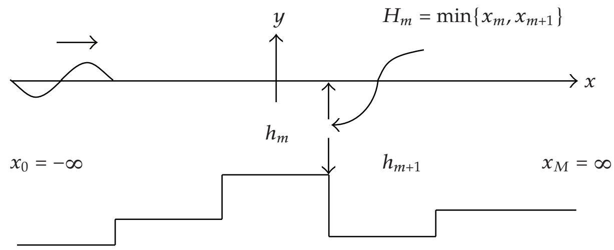


Figure 1: Schematic sketch of water wave scattering over a series of steps.

frequency and $\mathbf{i}$ the unit of complex number. The sea bottom is represented by a succession of flat shelves as depicted in Figure 1. In the figure, there are M shelves with depth h_m in the interval of $x_{m-1} < x < x_m$ for $m = 1, 2, 3, \ldots, M$ and $M - 1$ steps at $x = x_m$ for $m = 1, 2, 3, \ldots, M - 1$. In order to make the formulation easier, it is assumed that $x_0 = -\infty$ and $x_M = \infty$, respectively. The coordinate (x, y) in Figure 1 is defined such that x is the horizontal direction and y is the vertical direction upwards positively from the still water level on $y = 0$.

According to Airy's linear wave theory [18], the velocity potential is governed by Laplace equation given by

$$\nabla^2 \phi = 0, \tag{2.1}$$

which is subject to the free surface condition

$$\frac{\partial \phi}{\partial y} - \frac{\omega^2}{g}\phi = 0 \tag{2.2}$$

and the bottom boundary condition:

$$\frac{\partial \phi}{\partial y} = 0 \quad \text{on } y = -h_m, \qquad x_{m-1} < x < x_m \quad \text{for } m = 1, 2, 3, \ldots, M \tag{2.3}$$

and the condition for vertical walls

$$\frac{\partial \phi}{\partial x} = 0 \quad \text{on } x = x_m, \qquad -\max(h_m, h_{m+1}) < y < -\min(h_m, h_{m+1}) \quad \text{for } m = 1, 2, 3, \ldots, M - 1, \tag{2.4}$$

where $K = \omega^2/g$ with g being the acceleration of gravity. The following marching conditions are required at each step:

$$\phi_m = \phi_{m+1}, \quad \text{on } x = x_m, \qquad -\min(h_m, h_{m+1}) < y < 0 \quad \text{for } m = 1, 2, 3, \ldots, M - 1, \tag{2.5}$$

$$U_m(y) = \frac{\partial \phi_m}{\partial x} = \frac{\partial \phi_{m+1}}{\partial x} \quad \text{on } x = x_m, \qquad -\min(h_m, h_{m+1}) < y < 0 \quad \text{for } m = 1, 2, 3, \ldots, M - 1. \tag{2.6}$$

Furthermore, the following far-field conditions are required to make the solution unique:

$$\begin{aligned}\phi &= \frac{\cosh \kappa_1 (y + h_1)}{\cosh \kappa_1 h_1}\left(e^{\mathrm{i}\kappa_1 x} + Re^{-\mathrm{i}\kappa_1 x}\right) \quad \text{as } x \longrightarrow -\infty,\\ \phi &= \frac{\cosh \kappa_M (y + h_M)}{\cosh \kappa_M h_M}\left(Te^{\mathrm{i}\kappa_M x}\right) \quad \text{as } x \longrightarrow \infty,\end{aligned} \tag{2.7}$$

where R and T are the reflection and transmission coefficients, respectively, and κ_m is the wavenumber in the interval of $x_{m-1} < x < x_m$. The wavenumber κ_m is the positive root of the dispersion relation expressed by

$$\kappa_m \tanh \kappa_m h_m = K \quad \text{for } m = 1, 2, 3, \ldots, M. \tag{2.8}$$

3. Indirect Eigenfunction Marching Method

According to the linear wave theory, a complete solution on the mth shelf can be constructed as follows:

$$\begin{aligned}\phi_m(x, y) = &\left(A_m e^{\mathrm{i}\kappa_m x} + B_m e^{-\mathrm{i}\kappa_m x}\right)\chi_m(y)\\ &+ \sum_{n=1}^{N}\left(C_{m,n} e^{k_{m,n}(x - x_m)} + D_{m,n} e^{-k_{m,n}(x - x_{m-1})}\right)\varphi_{m,n}(y).\end{aligned} \tag{3.1}$$

This representation is arbitrarily well if N is sufficiently large. In (3.1), the propagating eigenfunction $\chi_m(y)$ is written in the form:

$$\chi_m(y) = \frac{2\sqrt{\kappa_m}\cosh \kappa_m (y + h_m)}{\sqrt{2\kappa_m h_m + \sinh 2\kappa_m h_m}}, \tag{3.2}$$

and the evanescent eigenfunction $\varphi_{m,n}(y)$ is expressed as

$$\varphi_{m,n}(y) = \frac{2\sqrt{k_{m,n}}\cos k_{m,n}(y + h_m)}{\sqrt{2k_{m,n} h_m + \sin 2k_{m,n} h_m}}, \tag{3.3}$$

where $k_{m,n}$ is the nth smallest positive root of

$$k_{m,n} \tan k_{m,n} h_m = -K. \tag{3.4}$$

Clearly, the following orthonormal relation can be found based on the Sturm-Liouville theory, that is,

$$\begin{aligned}&\left\langle \chi_m \mid \chi_m \right\rangle = 1,\\&\left\langle \chi_m \mid \psi_{m,l} \right\rangle = 0,\\&\left\langle \psi_{m,n} \mid \psi_{m,l} \right\rangle = \delta_{nl},\end{aligned} \tag{3.5}$$

where δ_{nl} is the Kronecker delta function and $\langle F \mid G \rangle$ is defined by

$$\langle F \mid G \rangle = \int_{-\lambda}^{0} F(y)G(y)\,dy. \tag{3.6}$$

In (3.6), F and G are orthonormal eigenfunctions of $\chi_m(y)$ or $\psi_{m,n}(y)$, and λ is the corresponding water depth of the eigenfunction F.

Now, we are in a position to march the solutions on all shelves. Considering a specific step at $x = x_m$ with $m = 1, 2, 3, \ldots, M-1$, we apply the Galerkin method to (2.4) and (2.6) and use the orthonormal relation of (3.5) to obtain the following equations:

$$\int_{-H_m}^{0} U_m(y)\chi_m(y)\,dy = \mathbf{i}\kappa_m\left(A_m e^{\mathbf{i}\kappa_m x_m} - B_m e^{-\mathbf{i}\kappa_m x_m}\right), \tag{3.7}$$

$$\int_{-H_m}^{0} U_m(y)\psi_{m,n}(y)\,dy = k_{m,n}\left(C_{m,n} - D_{m,n}e^{-k_{m,n}(x_m - x_{m-1})}\right) \quad \text{for } n = 1, 2, 3, \ldots, N, \tag{3.8}$$

$$\int_{-H_m}^{0} U_m(y)\chi_{m+1}(y)\,dy = \mathbf{i}\kappa_{m+1}\left(A_{m+1}e^{\mathbf{i}\kappa_{m+1}x_m} - B_{m+1}e^{-\mathbf{i}\kappa_{m+1}x_m}\right), \tag{3.9}$$

$$\int_{-H_m}^{0} U_m(y)\psi_{m+1,n}(y)\,dy = k_{m+1,n}\left(C_{m+1,n}e^{-k_{m+1,n}(x_{m+1} - x_m)} - D_{m+1,n}\right) \quad \text{for } n = 1, 2, 3, \ldots, N, \tag{3.10}$$

where H_m is the minimum value between h_m and h_{m+1}, that is, $H_m = \min(h_m, h_{m+1})$. Equations (3.7)~(3.10) are exactly equivalent to (2.4) and (2.6) when $N \to \infty$ since $\chi_m(y)$ and $\psi_{m,n}(y)$ have formed a complete basis. This has been shown by Miles [12]. We make use of the orthonormal identity, (3.8) and (3.10), and are ready to obtain

$$\begin{aligned}C_{m,n} &= \frac{\int_{-H_m}^{0} U_m(y)\psi_{m,n}(y)\,dy - e^{-k_{m,n}(x_m - x_{m-1})}\int_{-H_{m-1}}^{0} U_{m-1}(y)\psi_{m,n}(y)\,dy}{k_{m,n}\left(1 - e^{-2k_{m,n}(x_m - x_{m-1})}\right)},\\D_{m,n} &= \frac{e^{-k_{m,n}(x_m - x_{m-1})}\int_{-H_m}^{0} U_m(y)\psi_{m,n}(y)\,dy - \int_{-H_{m-1}}^{0} U_{m-1}(y)\psi_{m,n}(y)\,dy}{k_{m,n}\left(1 - e^{-2k_{m,n}(x_m - x_{m-1})}\right)},\end{aligned} \tag{3.11}$$

for $m = 1, 2, 3, \ldots, M$.

On the other hand, the further application of the Galerkin method to the marching condition in (2.5) and using (3.11) can produce the following equations:

$$\begin{aligned}
&\left(A_m e^{\mathbf{i}\kappa_m x_m} + B_m e^{-\mathbf{i}\kappa_m x_m}\right)\langle \tilde{\chi}_m \mid \chi_m\rangle - \left(A_{m+1} e^{\mathbf{i}\kappa_{m+1} x_m} + B_{m+1} e^{-\mathbf{i}\kappa_{m+1} x_m}\right)\langle \tilde{\chi}_m \mid \chi_{m+1}\rangle \\
&\quad = \sum_{n=1}^{N}\left[\frac{\langle \tilde{\chi}_m \mid \psi_{m+1,n}\rangle \int_{-H_{m+1}}^{0} U_{m+1}(\eta)\psi_{m+1,n}(\eta)\,d\eta}{k_{m+1,n}\sinh k_{m+1,n}(x_{m+1}-x_m)}\right. \\
&\qquad - \frac{\langle \tilde{\chi}_m \mid \psi_{m+1,n}\rangle \int_{-H_m}^{0} U_m(\eta)\psi_{m+1,n}(\eta)\,d\eta}{k_{m+1,n}\tanh k_{m+1,n}(x_{m+1}-x_m)} - \frac{\langle \tilde{\chi}_m \mid \psi_{m,n}\rangle \int_{-H_m}^{0} U_m(\eta)\psi_{m,n}(\eta)\,d\eta}{k_{m+1,n}\tanh k_{m+1,n}(x_{m+1}-x_m)} \\
&\qquad \left. + \frac{\langle \tilde{\chi}_m \mid \psi_{m,n}\rangle \int_{-H_{m-1}}^{0} U_{m-1}(\eta)\psi_{m,n}(\eta)\,d\eta}{k_{m,n}\sinh k_{m,n}(x_m - x_{m-1})}\right],
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
&\left(A_m e^{\mathbf{i}\kappa_m x_m} + B_m e^{-\mathbf{i}\kappa_m x_m}\right)\langle \tilde{\psi}_{m,j} \mid \chi_m\rangle - \left(A_{m+1} e^{\mathbf{i}\kappa_{m+1} x_m} + B_{m+1} e^{-\mathbf{i}\kappa_{m+1} x_m}\right)\langle \tilde{\psi}_{m,j} \mid \chi_{m+1}\rangle \\
&\quad = \sum_{n=1}^{N}\left[\frac{\langle \tilde{\psi}_{m,j} \mid \psi_{m+1,n}\rangle \int_{-H_{m+1}}^{0} U_{m+1}(\eta)\psi_{m+1,n}(\eta)\,d\eta}{k_{m+1,n}\sinh k_{m+1,n}(x_{m+1}-x_m)}\right. \\
&\qquad - \frac{\langle \tilde{\psi}_{m,j} \mid \psi_{m+1,n}\rangle \int_{-H_m}^{0} U_m(\eta)\psi_{m+1,n}(\eta)\,d\eta}{k_{m+1,n}\tanh k_{m+1,n}(x_{m+1}-x_m)} - \frac{\langle \tilde{\psi}_{m,j} \mid \psi_{m,n}\rangle \int_{-H_m}^{0} U_m(\eta)\psi_{m,n}(\eta)\,d\eta}{k_{m+1,n}\tanh k_{m+1,n}(x_{m+1}-x_m)} \\
&\qquad \left. + \frac{\langle \tilde{\psi}_{m,j} \mid \psi_{m,n}\rangle \int_{-H_{m-1}}^{0} U_{m-1}(\eta)\psi_{m,n}(\eta)\,d\eta}{k_{m,n}\sinh k_{m,n}(x_m - x_{m-1})}\right]
\end{aligned} \tag{3.13}$$

for $j = 1, 2, 3, \ldots, J$. In (3.12) and (3.13), $\tilde{\chi}_m$ and $\tilde{\psi}_{m,l}$ are defined, respectively, by

$$\begin{aligned}
\tilde{\chi}_m(y) &= \begin{cases} \chi_m(y) & \text{if } h_m < h_{m+1}, \\ \chi_{m+1}(y) & \text{if } h_{m+1} < h_m, \end{cases} \\
\tilde{\psi}_{m,l}(y) &= \begin{cases} \psi_{m,l}(y) & \text{if } h_m < h_{m+1}, \\ \psi_{m+1,l}(y) & \text{if } h_{m+1} < h_m. \end{cases}
\end{aligned} \tag{3.14}$$

Note that (3.7), (3.9), (3.12), and (3.13) form a functional problem of $U_m(y)$. In order to solve this functional problem, the horizontal velocities in above steps should be represented by the complete basis, $\tilde{\chi}_m$ and $\tilde{\psi}_{m,l}$, as follows:

$$U_m(y) = u_{m,0}\tilde{\chi}_m(y) + \sum_{l=1}^{L} u_{m,l}\tilde{\psi}_{m,l}(y), \tag{3.15}$$

where $u_{m,l}$ are unknown coefficients to be determined. Then, substitution of (3.15) into (3.7), (3.9), (3.12), and (3.13) respectively, yield

$$
\begin{aligned}
&u_{m,0}\langle \tilde{\chi}_m \mid \chi_m\rangle + \sum_{l=1}^{L} u_{m,l}\langle \tilde{\varphi}_{m,l} \mid \chi_m\rangle = \mathbf{i}\kappa_m\left(A_m e^{\mathbf{i}\kappa_m x_m} - B_m e^{-\mathbf{i}\kappa_m x_m}\right),\\
&u_{m,0}\langle \tilde{\chi}_m \mid \chi_{m+1}\rangle + \sum_{l=1}^{L} u_{m,l}\langle \tilde{\varphi}_{m,l} \mid \chi_{m+1}\rangle = \mathbf{i}\kappa_{m+1}\left(A_{m+1} e^{\mathbf{i}\kappa_{m+1} x_m} - B_{m+1} e^{-\mathbf{i}\kappa_{m+1} x_m}\right),\\
&\left(A_m e^{\mathbf{i}\kappa_m x_m} + B_m e^{-\mathbf{i}\kappa_m x_m}\right)\langle \tilde{\chi}_m \mid \chi_m\rangle - \left(A_{m+1} e^{\mathbf{i}\kappa_{m+1} x_m} + B_{m+1} e^{-\mathbf{i}\kappa_{m+1} x_m}\right)\langle \tilde{\chi}_m \mid \chi_{m+1}\rangle\\
&\quad = u_{m+1,0}\sum_{n=1}^{N}\left\{\frac{\langle \tilde{\chi}_{m+1} \mid \psi_{m+1,n}\rangle\langle \tilde{\chi}_m \mid \psi_{m+1,n}\rangle}{k_{m+1,n}\sinh k_{m+1,n}(x_{m+1}-x_m)}\right\}\\
&\qquad + \sum_{l=1}^{L} u_{m+1,l}\sum_{n=1}^{N}\left\{\frac{\langle \tilde{\varphi}_{m+1,l} \mid \psi_{m+1,n}\rangle\langle \tilde{\chi}_m \mid \psi_{m+1,n}\rangle}{k_{m+1,n}\sinh k_{m+1,n}(x_{m+1}-x_m)}\right\}\\
&\qquad - u_{m,0}\sum_{n=1}^{N}\left\{\frac{\langle \tilde{\chi}_m \mid \psi_{m+1,n}\rangle\langle \tilde{\chi}_m \mid \psi_{m+1,n}\rangle + \langle \tilde{\chi}_m \mid \psi_{m,n}\rangle\langle \tilde{\chi}_m \mid \psi_{m,n}\rangle}{k_{m+1,n}\tanh k_{m+1,n}(x_{m+1}-x_m)}\right\}\\
&\qquad - \sum_{l=1}^{L} u_{m,l}\sum_{n=1}^{N}\left\{\frac{\langle \tilde{\varphi}_{m,l} \mid \psi_{m+1,n}\rangle\langle \tilde{\chi}_m \mid \psi_{m+1,n}\rangle + \langle \tilde{\varphi}_{m,l} \mid \psi_{m,n}\rangle\langle \tilde{\chi}_m \mid \psi_{m,n}\rangle}{k_{m+1,n}\tanh k_{m+1,n}(x_{m+1}-x_m)}\right\}\\
&\qquad + u_{m-1,0}\sum_{n=1}^{N}\left\{\frac{\langle \tilde{\chi}_{m-1} \mid \psi_{m,n}\rangle\langle \tilde{\chi}_m \mid \psi_{m,n}\rangle}{k_{m,n}\sinh k_{m,n}(x_m-x_{m-1})}\right\} + \sum_{l=1}^{L} u_{m-1,l}\sum_{n=1}^{N}\left\{\frac{\langle \tilde{\varphi}_{m-1,l} \mid \psi_{m,n}\rangle\langle \tilde{\chi}_m \mid \psi_{m,n}\rangle}{k_{m,n}\sinh k_{m,n}(x_m-x_{m-1})}\right\},\\
&\left(A_m e^{\mathbf{i}\kappa_m x_m} + B_m e^{-\mathbf{i}\kappa_m x_m}\right)\langle \tilde{\varphi}_{m,j} \mid \chi_m\rangle - \left(A_{m+1} e^{\mathbf{i}\kappa_{m+1} x_m} + B_{m+1} e^{-\mathbf{i}\kappa_{m+1} x_m}\right)\langle \tilde{\varphi}_{m,j} \mid \chi_{m+1}\rangle\\
&\quad = u_{m+1,0}\sum_{n=1}^{N}\left\{\frac{\langle \tilde{\chi}_{m+1} \mid \psi_{m+1,n}\rangle\langle \tilde{\varphi}_{m,j} \mid \psi_{m+1,n}\rangle}{k_{m+1,n}\sinh k_{m+1,n}(x_{m+1}-x_m)}\right\}\\
&\qquad + \sum_{l=1}^{L} u_{m+1,l}\sum_{n=1}^{N}\left\{\frac{\langle \tilde{\varphi}_{m+1,l} \mid \psi_{m+1,n}\rangle\langle \tilde{\varphi}_{m,j} \mid \psi_{m+1,n}\rangle}{k_{m+1,n}\sinh k_{m+1,n}(x_{m+1}-x_m)}\right\}\\
&\qquad - u_{m,0}\sum_{n=1}^{N}\left\{\frac{\langle \tilde{\chi}_m \mid \psi_{m+1,n}\rangle\langle \tilde{\varphi}_{m,j} \mid \psi_{m+1,n}\rangle + \langle \tilde{\chi}_m \mid \psi_{m,n}\rangle\langle \tilde{\varphi}_{m,j} \mid \psi_{m,n}\rangle}{k_{m+1,n}\tanh k_{m+1,n}(x_{m+1}-x_m)}\right\}\\
&\qquad - \sum_{l=1}^{L} u_{m,l}\sum_{n=1}^{N}\left\{\frac{\langle \tilde{\varphi}_{m,l} \mid \psi_{m+1,n}\rangle\langle \tilde{\varphi}_{m,j} \mid \psi_{m+1,n}\rangle + \langle \tilde{\varphi}_{m,l} \mid \psi_{m,n}\rangle\langle \tilde{\varphi}_{m,j} \mid \psi_{m,n}\rangle}{k_{m+1,n}\tanh k_{m+1,n}(x_{m+1}-x_m)}\right\}\\
&\qquad + u_{m-1,0}\sum_{n=1}^{N}\left\{\frac{\langle \tilde{\chi}_{m-1} \mid \psi_{m,n}\rangle\langle \tilde{\varphi}_{m,j} \mid \psi_{m,n}\rangle}{k_{m,n}\sinh k_{m,n}(x_m-x_{m-1})}\right\} + \sum_{l=1}^{L} u_{m-1,l}\sum_{n=1}^{N}\left\{\frac{\langle \tilde{\varphi}_{m-1,l} \mid \psi_{m,n}\rangle\langle \tilde{\varphi}_{m,j} \mid \psi_{m,n}\rangle}{k_{m,n}\sinh k_{m,n}(x_m-x_{m-1})}\right\}
\end{aligned}
\tag{3.16}
$$

for $j = 1, 2, 3, \ldots, J$.

To obtain a complete solution in the present model, it is desirable to compare the far-field behaviors (2.7) with (3.1) for obtaining the following relations:

$$A_1 = \frac{\sqrt{2\kappa_1 h_1 + \sinh 2\kappa_1 h_1}}{2\sqrt{\kappa_1}\cosh \kappa_1 h_1}, \qquad B_M = 0, \tag{3.17}$$

$$B_1 = \frac{R\sqrt{2\kappa_1 h_1 + \sinh 2\kappa_1 h_1}}{2\sqrt{\kappa_1}\cosh \kappa_1 h_1}, \qquad A_M = \frac{T\sqrt{2\kappa_M h_M + \sinh 2\kappa_M h_M}}{2\sqrt{\kappa_M}\cosh \kappa_M h_M}. \tag{3.18}$$

It should be noted that the model defined in Section 2 is analytically converted to the problem of finding $u_{m,l}$ and A_m and B_m such that (3.16) are satisfied. This conversion is arbitrarily well if all of N, L, and J are sufficiently large.

To obtain a numerical solution to the prescribed problem, the numbers N, L, and J should be truncated to finite values. In the model, there are $(L+3)(M-1)$ unknowns of $u_{m,l}$ and A_m and B_m, except that A_1 and B_M have been defined in (3.17). Therefore, $J = L$ should be set to obtain $(L+3)(M-1)$ equations as given in (3.16). After the coefficients A_m and B_m are solved, (3.18) can thus be used to obtain the reflection and transmission coefficients of the water wave problem. These complete the procedure of numerical solutions in the IEMM. This method has the advantage to solve the velocity U_m in a direct manner instead of solving the velocity potential. If it is required to solve the velocity potential, (3.1), (3.11) can be implemented to get the solution ϕ_m.

4. Transfer-Matrix Method

Equation (3.16) is readily reduced to the TM method of Miles [12] and Devillard et al. [13] using the wide-spacing approximation [13] subject to $J = L = 0$, defined as

$$\sinh k_{m+1,n}(x_{m+1} - x_m) \longrightarrow 0, \qquad \tanh k_{m+1,n}(x_{m+1} - x_m) \longrightarrow 1. \tag{4.1}$$

In (4.1), we assumed that the evanescent eigenfunctions originating at one step are negligible when they reach the next step. Based on these assumption, (3.16) can be simplified in the following expressions:

$$u_{m,0}\langle \tilde{\chi}_m \mid \chi_m \rangle = \mathbf{i}\kappa_m \left(A_m e^{\mathbf{i}\kappa_m x_m} - B_m e^{-\mathbf{i}\kappa_m x_m} \right),$$
$$u_{m,0}\langle \tilde{\chi}_m \mid \chi_{m+1} \rangle = \mathbf{i}\kappa_{m+1} \left(A_{m+1} e^{\mathbf{i}\kappa_{m+1} x_m} - B_{m+1} e^{-\mathbf{i}\kappa_{m+1} x_m} \right),$$
$$\left(A_m e^{\mathbf{i}\kappa_m x_m} + B_m e^{-\mathbf{i}\kappa_m x_m} \right)\langle \tilde{\chi}_m \mid \chi_m \rangle - \left(A_{m+1} e^{\mathbf{i}\kappa_{m+1} x_m} + B_{m+1} e^{-\mathbf{i}\kappa_{m+1} x_m} \right)\langle \tilde{\chi}_m \mid \chi_{m+1} \rangle = -u_{m,0} X_{m,N}, \tag{4.2}$$

where

$$X_{m,N} = \sum_{n=1}^{N} \left\{ \frac{\langle \tilde{\chi}_m \mid \varphi_{m+1,n} \rangle \langle \tilde{\chi}_m \mid \varphi_{m+1,n} \rangle + \langle \tilde{\chi}_m \mid \varphi_{m,n} \rangle \langle \tilde{\chi}_m \mid \varphi_{m,n} \rangle}{k_{m+1,n}} \right\}. \tag{4.3}$$

Following Devillard et al. [13], (4.2) can solved independently for every single step. Considering a specific step at $x = x_m$, suppose that the following two quantities on the left shelf are known as follows:

$$\begin{aligned} \Psi_m &= A_m e^{\mathbf{i}\kappa_m x_m} + B_m e^{-\mathbf{i}\kappa_m x_m}, \\ \Omega_m &= -\frac{1}{\kappa_m} \frac{d\Psi_m}{dx_m} = -\mathbf{i} A_m e^{\mathbf{i}\kappa_m x_m} + \mathbf{i} B_m e^{-\mathbf{i}\kappa_m x_m}. \end{aligned} \tag{4.4}$$

Details of these definitions are referred to [19]. Using (4.4), we can replace A_m and B_m by Ψ_m and Ω_m in (4.2). The resultant equations read

$$\begin{aligned} &u_{m,0} \langle \tilde{\chi}_m \mid \chi_m \rangle + \kappa_m \Omega_m = 0, \\ &u_{m,0} \langle \tilde{\chi}_m \mid \chi_{m+1} \rangle - \kappa_{m+1} (\Psi_{m+1} \sin \kappa_{m+1}(x_{m+1} - x_m) - \Omega_{m+1} \cos \kappa_{m+1}(x_{m+1} - x_m)) = 0, \\ &\Psi_m \langle \tilde{\chi}_m \mid \chi_m \rangle - (\Psi_{m+1} \cos \kappa_{m+1}(x_{m+1} - x_m) + \Omega_{m+1} \sin \kappa_{m+1}(x_{m+1} - x)) \langle \tilde{\chi}_m \mid \chi_{m+1} \rangle \\ &\quad + u_{m,0} X_{m,N} = 0. \end{aligned} \tag{4.5}$$

Equations (4.5) are sufficient to obtain the unknowns Ψ_{m+1} and Ω_{m+1} on the right shelf in which $u_{m,0}$ have been cancelled as follows:

$$\begin{aligned} \begin{pmatrix} \Psi_{m+1} \\ \Omega_{m+1} \end{pmatrix} &= \begin{pmatrix} \cos \kappa_{m+1}(x_{m+1} - x_m) & -\sin \kappa_{m+1}(x_{m+1} - x_m) \\ \sin \kappa_{m+1}(x_{m+1} - x_m) & \cos \kappa_{m+1}(x_{m+1} - x_m) \end{pmatrix} \\ &\quad \times \begin{pmatrix} \dfrac{\langle \tilde{\chi}_m \mid \chi_m \rangle}{\langle \tilde{\chi}_m \mid \chi_{m+1} \rangle} & \dfrac{-\kappa_m X_{m,N}}{\langle \tilde{\chi}_m \mid \chi_m \rangle \langle \tilde{\chi}_m \mid \chi_{m+1} \rangle} \\ 0 & \dfrac{\kappa_m \langle \tilde{\chi}_m \mid \chi_{m+1} \rangle}{\kappa_{m+1} \langle \tilde{\chi}_m \mid \chi_m \rangle} \end{pmatrix} \begin{pmatrix} \Psi_m \\ \Omega_m \end{pmatrix}. \end{aligned} \tag{4.6}$$

Equation (4.6) is the transfer matrix method of Devillard et al. [13] based on the variational formulation of Miles [12]. Clearly, the combined effect of the series of steps can be achieved by simply applying matrix multiplications to the transfer matrices of all the steps. We also notice that (4.6) can be further reduced to the plane-wave approximation of Lamb [20] by assuming $X_{m,N} = 0$. This reduction has been addressed by Miles [12].

5. Numerical Results

The validity of the present approximation is examined by three cases of waves propagating over a step, a rectangle obstacle, and a trench. All the IEMM, DEMM, TM method, and plane-wave approximations are applied to solve the prescribed wave-scattering problems. Here,

Table 1: Reflection by a single step and comparison with the solution of Miles [12].

K, ε	1, 0.5	1, 0.1	0.1, 0.1	1, 5	0.1, 5
IEMM ($N = L = 10$)	0.122322	0.416221	0.510080	0.064880	0.340511
IEMM ($N = L = 20$)	0.122484	0.417915	0.510243	0.066488	0.340920
IEMM ($N = L = 50$)	0.122549	0.418613	0.510310	0.067142	0.341086
IEMM ($N = L = 100$)	0.122564	0.418769	0.510325	0.067291	0.341125
IEMM ($N = L = 200$)	0.122569	0.418827	0.510331	0.067347	0.341139
IEMM ($N = L = 300$)	0.122570	0.418843	0.510332	0.067362	0.341143
TM [12] ($L = 0,\ N = 10$)	0.120324	0.418204	0.510261	0.062841	0.340930
TM [12] ($L = 0,\ N = 20$)	0.120437	0.419265	0.510367	0.063865	0.341176
TM [12] ($L = 0,\ N = 50$)	0.120469	0.419594	0.510400	0.064160	0.341247
TM [12] ($L = 0,\ N = 100$)	0.120474	0.419642	0.510405	0.064202	0.341257
TM [12] ($L = 0,\ N = 200$)	0.120475	0.419654	0.510406	0.064213	0.341259
TM [12] ($L = 0,\ N = 300$)	0.120475	0.419657	0.510406	0.064215	0.341260
Plane-wave ($N = L = 0$)	0.085914	0.345402	0.503897	0.013111	0.303414
DEMM ($N = 100$)	0.122564	0.418769	0.510325	0.067291	0.341125
DEMM ($N = 200$)	0.122569	0.418827	0.510331	0.067347	0.341139
DEMM ($N = 300$)	0.122570	0.418843	0.510332	0.067362	0.341143
Evans and Linton [19]	0.1223	0.4192	0.5103	—	—

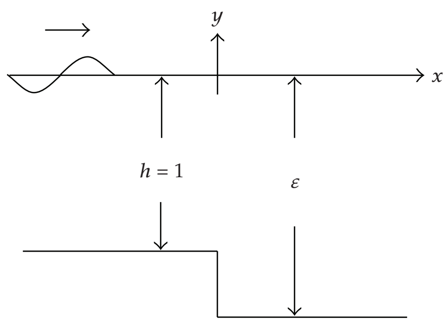


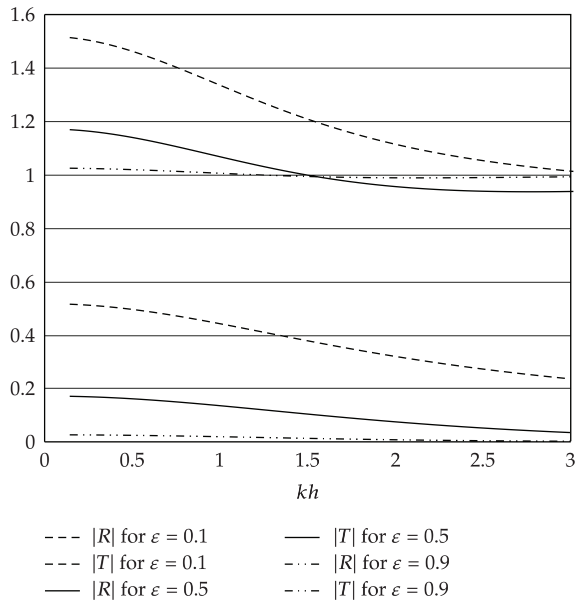
Figure 2: Schematic sketch of water wave scattering over a step.

the IEMM is introduced in Section 3. On the other hand, the TM method and plane-wave approximation are described in Section 4. For completeness, the DEMM of Takano [9] is briefly reviewed in the Appendix for solving water wave scattering over a series of steps since all previous studies [9–11] do not consider this general configuration. In all of these cases, the reflection coefficients are tabulated and compared with the results in the literatures.

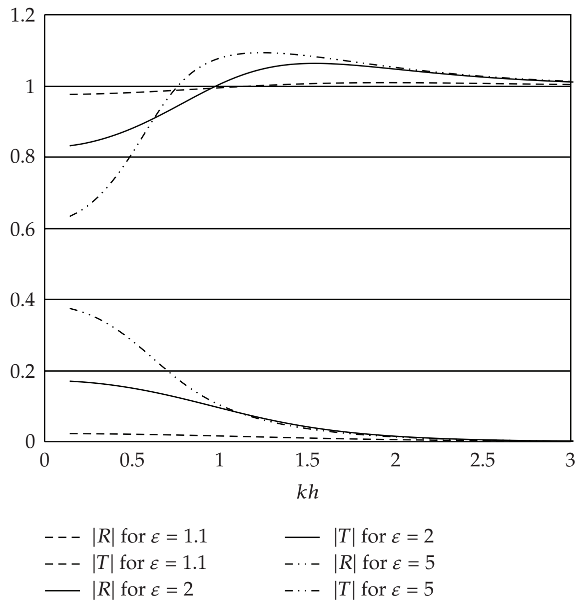
If the derived methods were verified well, they can be applied to the problem of water wave scattering over Roseau's curved profile. This result demonstrates the applicability of our method for solving water wave scattering by an arbitrary bottom topography.

5.1. Water Wave Scattering over a Step

We first consider the problem of water wave scattering over a step as depicted in Figure 2. Miles [12] and Evans and Linton [19] have solved the problem using the variational formulation and the intermediate mapping technique, respectively. Table 1 presents the reflection coefficients obtained by the IEMM, TM method, plane-wave approximation, and



(a)



(b)

Figure 3: Reflection and transmission coefficients of water wave scattering over a step. (a) up steps; (b) down steps.

DEMM and a comparison with the solutions of Evans and Linton [19]. In Table 1, we note that the convergence of the reflection coefficients obtained by the IEMM is significant. Furthermore, it is very interesting to observe that the solutions of the IEMM and DEMM are equal to each other up to the sixth decimal places. However, only $(L+3)(M-1)$ linear equations are needed to be solved in the IEMM and this number is quite smaller than that of

Table 2: Reflection by a rectangle obstacle and comparison with the solution of Mei and Black [15].

$K,\ b/H$	0.17, 2	0.06, 4	0.03, 6
IEMM ($N = L = 10$)	0.318114	0.328488	0.330839
IEMM ($N = L = 50$)	0.318261	0.328517	0.330844
IEMM ($N = L = 100$)	0.318271	0.328519	0.330845
IEMM ($N = L = 200$)	0.318275	0.328520	0.330845
IEMM ($N = L = 300$)	0.318277	0.328520	0.330845
DEMM ($N = L = 300$)	0.318277	0.328520	0.330845
TM ($L = 0,\ N = 50$)	0.318266	0.328301	0.330648
TM ($L = 0,\ N = 100$)	0.318268	0.328302	0.330649
TM ($L = 0,\ N = 200$)	0.318268	0.328302	0.330649
Mei and Black [15] ($L = 5,\ N = 25$)	0.318330	0.328524	0.330841
Plane-wave ($N = L = 0$)	0.286598	0.319753	0.327476

Table 3: kH values for full transmission by a rectangle obstacle.

$b/H = 6$	0.173	0.355	0.556	0.785	1.043	1.32	1.606
$b/H = 4$	0.252	0.532	0.868	1.26			
$b/H = 2$	0.472	1.099					

$(2N + 2)(M - 1)$ in the DEMM. For some specific settings described in Section 4, our codes can reproduce the solutions of the TM method and plane-wave approximation. It is shown that they are in good agreement with the results listed in [19] to the third decimal places. This indicates that the solutions of IEMM or DEMM should converge to the analytical solution if no round-off errors are cumulated.

Figure 3 gives the reflection and transmission coefficients against kh for various depths of the downstream shelf ε. This result provides a very good example for the comparison of other methods.

5.2. Water Wave Scattering over a Rectangle Obstacle

Then, we consider the water wave scattering over a rectangle obstacle as shown in Figure 4. Here, $H/h = 2$ is assumed in this example. This problem has also been solved by Mei and Black [15]. They solved the problem using Miles' variational formulation without considering the wide-spacing approximation. Their formulation is in exact agreement with our formulation of IEMM except the different definition of the coordinate system. In Table 2, the reflection coefficients for various widths of the rectangle obstacle are obtained by the IEMM, TM method, plane-wave approximation, and DEMM. A comparison with the solution of Mei and Black [15] is also presented. A good agreement between the IEMM and DEMM can also be observed. Note that the reflection coefficients obtained by the IEMM converge perfectly to the fifth decimal places.

Figure 5 describes the comparison of the reflection coefficients against kH obtained by the TM method, IEMM, and Mei and Black's method [15]. Good agreement between the later two methods can be observed. The reflection and transmission coefficients against kH are given in Figure 6. Full transmission for some specific kH can be observed as addressed in [15]. These specific kH values are tabulated in Table 3.

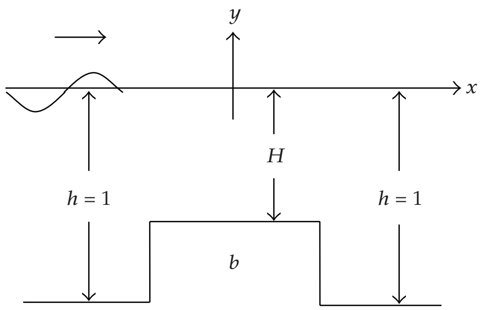


Figure 4: Schematic sketch of water wave scattering over a rectangle obstacle.

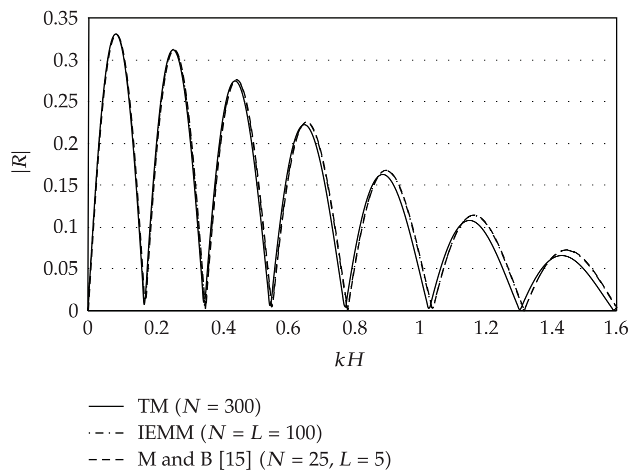


Figure 5: Comparison of reflection coefficients of water wave scattering over a rectangle obstacle.

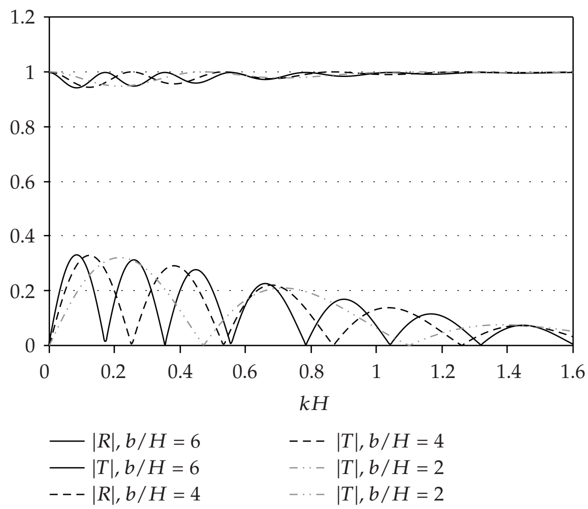


Figure 6: Reflection and transmission coefficients of water wave scattering over a rectangle obstacle.

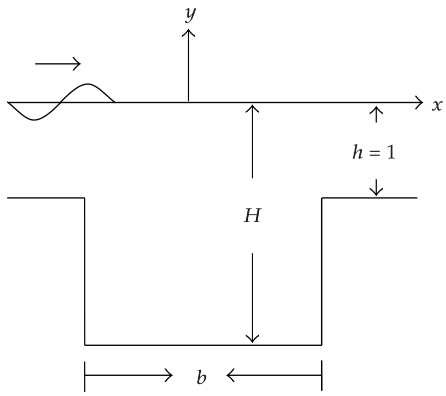


Figure 7: Schematic sketch of water wave scattering over a trench.

Table 4: Reflection by a trench and comparison with the solution of Kirby and Dalrymple [10].

kh	0.341	0.723	1.296
IEMM ($N = L = 32$)	0.459568	0.295852	0.0300908
IEMM ($N = L = 300$)	0.459672	0.295807	0.0297184
DEMM [10] ($N = L = 2$)	0.455857	0.295939	0.0399653
DEMM [10] ($N = L = 4$)	0.458108	0.295293	0.0327490
DEMM [10] ($N = L = 8$)	0.458944	0.295921	0.0319301
DEMM [10] ($N = L = 16$)	0.459415	0.295879	0.0305612
DEMM [10] ($N = L = 32$)	0.459568	0.295852	0.0300908
DEMM [10] ($N = L = 100$)	0.459655	0.295817	0.0297851
DEMM [10] ($N = L = 200$)	0.459669	0.295810	0.0297332
DEMM [10] ($N = L = 300$)	0.459672	0.295807	0.0297184
TM ($L = 0,\ N = 300$)	0.460213	0.285417	0.0134597
Plane-wave ($N = L = 0$)	0.389926	0.188591	0.0140776

5.3. Water Wave Scattering over a Trench

The last typical example of water wave scattering over a trench is defined in Figure 7. Here, $b = 10$ is assumed since Kirby and Dalrymple [10] have solved this problem by the DEMM. Table 4 gives the reflection coefficients corresponding to different trench depths obtained by all the methods mentioned above. The first three decimals of the reflection coefficients obtained by our implementation are the same as those obtained by Kirby and Dalrymple [10]. Furthermore, the significant convergence of the DEMM can also be observed. In the table, only a few reflection coefficients obtained by the IEMM are addressed since they are the same with the DEMM up to the sixth decimals. Furthermore, the reflection and transmission coefficients against kh are given in Figure 8. In the figure, the full transmission can also be reached and its corresponding kh values are addressed in Table 5.

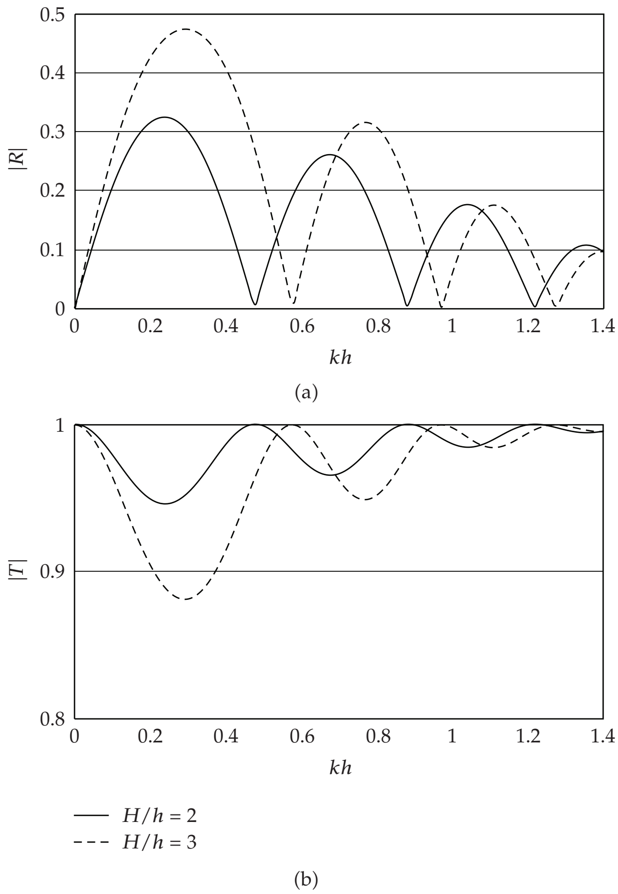


Figure 8: (a) Reflection and (b) transmission coefficients of water wave scattering over a trench.

5.4. Roseau's Explicit Solution

Finally, we apply our method of step approximation to water wave scattering over Roseau's curved profiles as depicted in Figure 9. The bottom profiles are written in the form

$$\beta x = \ln \mu + \frac{\varepsilon - 1}{2} \ln\left(1 + 2\mu \cos \beta + \mu^2\right), \tag{5.1}$$

where

$$\mu = \left(\tan \frac{\beta(y+1)}{1-\varepsilon}\right)\left(\sin \beta - \left(\tan \frac{\beta(y+1)}{1-\varepsilon}\right) \cos \beta\right)^{-1}. \tag{5.2}$$

Equation (5.1) defines a step of depths from 1 to ε with $\beta \in (0, \pi/2)$ being the steepness of the step. In the present investigation, three typical cases of mild ($\beta = 0.1$), middle ($\beta = 1.0$), and steep ($\beta = 1.0$) slopes are all considered. These values are selected according to Evans

Table 5: kh values for full transmission by a trench.

$H/h = 2$	0.477	0.882	1.218
$H/h = 3$	0.576	0.971	1.274

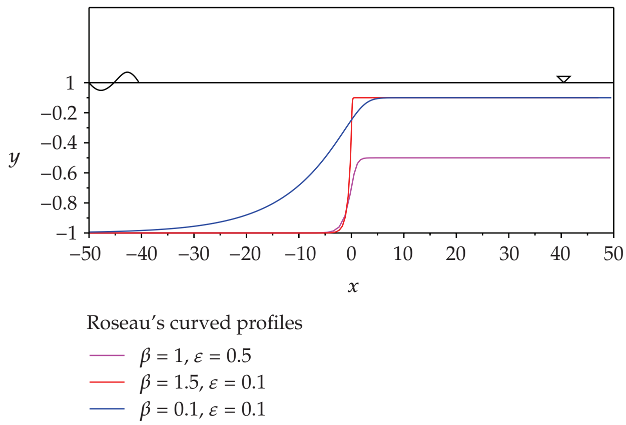


Figure 9: Roseau's curved profiles.

and Linton [19]. The analytical reflection coefficient was derived by Roseau [1] and can be expressed as

$$|R| = \left| \frac{\sinh \pi\beta^{-1}(k_- - \varepsilon k_+)}{\sinh \pi\beta^{-1}(k_- + \varepsilon k_+)} \right|, \tag{5.3}$$

where the plus and minus signs follow from the requirements of progressive and transmission waves. Their wavenumbers are defined by

$$K = k_- \tanh k_- = k_+ \tanh \varepsilon k_+. \tag{5.4}$$

Table 6 gives the reflection and transmission coefficients obtained by the IEMM and TM method and its comparison with the results of Evans and Linton [19] and the analytical solutions obtained by (5.3). In the numerical calculations, we assume $N = 30$ since it is sufficient for most of our studies. In the table, it can be seen that more steps (bigger M) generally produce better results. The best accuracy of all the three cases are basically up to four decimals. For mild-slope case, the results obtained by the IEMM and TM method are basically the same because the wide-spacing assumption is reasonable. Furthermore, the increases of L have little help to the numerical results. On the other hand, for middle-slope case, the results of TM method basically converge to another value due to the fact that the wide-spacing approximation is not valid in this slope. Finally, for the steep-slope case the solutions corresponding to $L = 0$ are not accurate enough. This implies that the value of L should be increased to improve the result.

Table 6: Comparison with the explicit solution of Roseau and the method of Evans and Linton [19] (most accurate solutions are bold).

K, ε, β	0.1, 0.1, 0.1		1, 0.5, 1		0.1, 0.1, 1.5	
	$\lvert R\rvert$	$\lvert T\rvert$	$\lvert R\rvert$	$\lvert T\rvert$	$\lvert R\rvert$	$\lvert T\rvert$
IEMM ($M = 11,\ L = 0$)	0.1144	1.7271	0.0748	1.0435	0.4747	1.5301
IEMM ($M = 21,\ L = 0$)	0.2857	1.6660	0.0056	1.0464	0.4723	1.5323
IEMM ($M = 51,\ L = 0$)	0.0046	1.7385	0.0066	1.0464	0.4791	1.5260
IEMM ($M = 101,\ L = 0$)	0.0015	1.7385	0.0071	1.0464	0.4802	1.5250
IEMM ($M = 201,\ L = 0$)	0.0017	1.7385	0.0073	1.0464	0.4807	1.5245
IEMM ($M = 301,\ L = 0$)	**0.0017**	**1.7385**	**0.0073**	**1.0464**	0.4809	1.5243
IEMM ($M = 11,\ L = 3$)	0.1128	1.7274	0.0757	1.0434	0.4751	1.5298
IEMM ($M = 21,\ L = 3$)	0.2853	1.6662	0.0058	1.0464	0.4722	1.5324
IEMM ($M = 51,\ L = 3$)	0.0046	1.7385	0.0067	1.0464	0.4782	1.5268
IEMM ($M = 101,\ L = 3$)	0.0015	1.7385	0.0071	1.0464	0.4788	1.5262
IEMM ($M = 201,\ L = 3$)	0.0017	1.7385	0.0073	1.0464	0.4791	1.5260
IEMM ($M = 301,\ L = 3$)	**0.0017**	**1.7385**	**0.0073**	**1.0464**	**0.4791**	**1.5259**
TM ($M = 51,\ N = 30,\ L = 0$)	0.0046	1.7385	0.0068	1.0464	0.4804	1.5248
TM ($M = 101,\ N = 30,\ L = 0$)	0.0015	1.7385	0.0074	1.0464	0.4820	1.5233
TM ($M = 201,\ N = 30,\ L = 0$)	0.0017	1.7385	0.0078	1.0464	0.4828	1.5224
TM ($M = 301,\ N = 50,\ L = 0$)	**0.0017**	**1.7385**	0.0080	1.0464	0.4831	1.5221
Evans and Linton [19]	0.0019	1.7385	0.0073	1.0464	0.4791	1.5259
Exact [1]	0.0018	1.7385	0.0073	1.0464	0.4792	1.5258

This study demonstrates the applicability of the IEMM for solving problems of water wave scattering over smoothly varying bottom profiles for both mild and steep slopes. At the same time, it is clear that the DEMM is not suitable to this example since its system matrix is too large. In addition, the IEMM could avoid the pseudoconvergence of the TM method.

6. Discussions and Conclusion

An indirect eigenfunction marching method (IEMM) is derived to solve problems of water wave scattering over a series of steps. In the solution procedure, the solution is represented by the horizontal velocity above the steps and a system of linear equations is resulted by using the Galerkin method. Furthermore, it is demonstrated that the IEMM can be exactly reduced to the transfer-matrix (TM) method when the wide-spacing approximation is applied.

Numerical methods were carried out to validate the applicability for the proposed IEMM and TM method and comparisons were made with the direct eigenfunction marching method (DEMM) and other available results in the literatures. The numerical results demonstrated that the IEMM has the same accuracy when compared with the DEMM for problems of water wave scattering over a step, a rectangle obstacle, and a trench. However, the IEMM has a smaller dimension of matrix to save computer time. For water wave scattering caused by Roseau's curved profile, the IEMM also provides accurate reflection and transmission coefficients up to four decimals whilst the DEMM fails to work due to a very large system matrix. Furthermore, the IEMM can avoid the pseudoconvergence of the TM method for problems of steep-slope bottom profiles.

Appendix

The direct eigenfunction marching method of Takano [9] is governed by the following four equations:

$$\begin{aligned}
&\mathbf{i}\kappa_m\left(A_m e^{\mathbf{i}\kappa_m x_m} - B_m e^{-\mathbf{i}\kappa_m x_m}\right)\langle \chi_m \mid \overline{\chi}_m\rangle + \sum_{n=1}^{N} k_{m,n}\left(C_{m,n} - D_{m,n}e^{-k_{m,n}(x_m - x_{m-1})}\right)\langle \psi_{m,n} \mid \overline{\chi}_m\rangle \\
&\quad = \mathbf{i}\kappa_{m+1}\left(A_{m+1}e^{\mathbf{i}\kappa_{m+1}x_m} - B_m e^{-\mathbf{i}\kappa_{m+1}x_m}\right)\langle \chi_{m+1} \mid \overline{\chi}_m\rangle \\
&\qquad + \sum_{n=1}^{N} k_{m+1,n}\left(C_{m+1,n}e^{-k_{m+1,n}(x_{m+1}-x_m)} - D_{m+1,n}\right)\langle \psi_{m+1,n} \mid \overline{\chi}_m\rangle,
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
&\mathbf{i}\kappa_m\left(A_m e^{\mathbf{i}\kappa_m x\,m} - B_m e^{-\mathbf{i}\kappa_m x\,m}\right)\left\langle \chi_m \mid \overline{\varphi}_{m,l}\right\rangle \\
&\qquad + \sum_{n=1}^{N} k_{m,n}\left(C_{m,n} - D_{m,n}e^{-k_{m,n}(x_m - x_{m-1})}\right)\left\langle \psi_{m,n} \mid \overline{\varphi}_{m,l}\right\rangle \\
&\quad = \mathbf{i}\kappa_{m+1}\left(A_{m+1}e^{\mathbf{i}\kappa_{m+1}x\,m} - B_m e^{-\mathbf{i}\kappa_{m+1}x\,m}\right)\left\langle \chi_{m+1} \mid \overline{\varphi}_{m,l}\right\rangle \\
&\qquad + \sum_{n=1}^{N} k_{m+1,n}\left(C_{m+1,n}e^{-k_{m+1,n}(x_{m+1}-x_m)} - D_{m+1,n}\right)\left\langle \psi_{m+1,n} \mid \overline{\varphi}_{m,l}\right\rangle \quad \text{for } l = 1,2,3,\ldots,N,
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
&\left(A_m e^{\mathbf{i}\kappa_m x_m} + B_m e^{-\mathbf{i}\kappa_m x_m}\right)\langle \widetilde{\chi}_m \mid \chi_m\rangle + \sum_{n=1}^{N}\left(C_{m,n} + D_{m,n}e^{-k_{m,n}(x_m - x_{m-1})}\right)\langle \widetilde{\chi}_m \mid \psi_{m,n}\rangle \\
&\quad = \left(A_{m+1}e^{\mathbf{i}\kappa_{m+1}x_m} + B_{m+1}e^{-\mathbf{i}\kappa_{m+1}x_m}\right)\langle \widetilde{\chi}_m \mid \chi_{m+1}\rangle \\
&\qquad + \sum_{n=1}^{N}\left(C_{m+1,n}e^{-k_{m+1,n}(x_{m+1}-x_m)} + D_{m+1,n}\right)\langle \widetilde{\chi}_m \mid \psi_{m+1,n}\rangle,
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
&\left(A_m e^{\mathbf{i}\kappa_m x_m} + B_m e^{-\mathbf{i}\kappa_m x_m}\right)\langle \widetilde{\varphi}_{m,l} \mid \chi_m\rangle + \sum_{n=1}^{N}\left(C_{m,n} + D_{m,n}e^{-k_{m,n}(x_m - x_{m-1})}\right)\langle \widetilde{\varphi}_{m,l} \mid \psi_{m,n}\rangle \\
&\quad = \left(A_{m+1}e^{\mathbf{i}\kappa_{m+1}x_m} + B_{m+1}e^{-\mathbf{i}\kappa_{m+1}x_m}\right)\langle \widetilde{\varphi}_{m,l} \mid \chi_{m+1}\rangle \\
&\qquad + \sum_{n=1}^{N}\left(C_{m+1,n}e^{-k_{m+1,n}(x_{m+1}-x_m)} + D_{m+1,n}\right)\langle \widetilde{\varphi}_{m,l} \mid \psi_{m+1,n}\rangle \quad \text{for } l = 1,2,3,\ldots,N,
\end{aligned} \tag{A.4}$$

where

$$\overline{\chi}_m(y) = \begin{cases} \chi_{m+1}(y) & \text{if } h_m < h_{m+1}, \\ \chi_m(y) & \text{if } h_{m+1} < h_m, \end{cases}$$
$$\overline{\varphi}_{m,l}(y) = \begin{cases} \varphi_{m+1,l}(y) & \text{if } h_m < h_{m+1}, \\ \varphi_{m,l}(y) & \text{if } h_{m+1} < h_m. \end{cases} \tag{A.5}$$

In (A.1)~(A.4), there are $(M-1)(2N+2)$ unknowns, $A_2, \ldots, A_M$, $B_1, \ldots, B_{M-1}$, $D_2, \ldots, D_M$, $C_1, \ldots, C_{M-1}$, and can be solved by the above $(M-1)(2N+2)$ equations.

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