Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2011, Article ID 615612, 12 pages
doi:10.1155/2011/615612

Research Article

Group-Invariant Solutions for Two-Dimensional Free, Wall, and Liquid Jets Having Finite Fluid Velocity at Orifice

R. Naz

Centre for Mathematics and Statistical Sciences, Lahore School of Economics, Lahore 53200, Pakistan

Correspondence should be addressed to R. Naz, rehananaz_qau@yahoo.com

Received 28 January 2011; Accepted 22 June 2011

Academic Editor: Furong Gao

The group-invariant solutions for nonlinear third-order partial differential equation (PDE) governing flow in two-dimensional jets (free, wall, and liquid) having finite fluid velocity at orifice are constructed. The symmetry associated with the conserved vector that was used to derive the conserved quantity for the jets (free, wall, and liquid) generated the group invariant solution for the nonlinear third-order PDE for the stream function. The comparison between results for two-dimensional jet flows having finite and infinite fluid velocity at orifice is presented. The general form of the group invariant solution for two-dimensional jets is given explicitly.

1. Introduction

The governing equations for two-dimensional jet flows are expressed either as the system of two PDEs for the velocity components or by a single nonlinear third-order PDE for the stream function. In [1, 2] the similarity solution and in [3] the group-invariant solution were constructed for the nonlinear third-order PDE for the stream function for two-dimensional free jet with infinite fluid velocity at the orifice. The group-invariant solution for system of equations for the velocity components for the same problem was constructed by Naz et al. [4]. Glauert [5] derived the similarity solution for radial and two-dimensional wall jets having infinite fluid velocity at the orifice.

The general form of similarity solution for the flows having finite velocity at the orifice was suggested by Watson [6], and the similarity solutions for system of equations for velocity components for the radial and two-dimensional liquid jets were derived. The similarity solution for radial and two-dimensional wall jets having finite velocity at orifice was studied by Riley [7], and so our solution has some significance even near axis. The subject of this paper is to find the group-invariant solution for the nonlinear third-order PDE for stream function governing flow in two-dimensional (free, wall, liquid) jets having finite velocity at the orifice which is not considered yet.

The detailed outline of this paper is as follows: In Section 2 the group-invariant solution for two-dimensional free jet is derived. The symmetry associated with the conserved vector which is used to establish the conserved quantity for each jet generates the group-invariant solution for the nonlinear third-order PDE for the stream function. The group-invariant solution for two-dimensional wall and liquid jets is studied in Sections 3 and 4. In Section 5 the comparison between the results for two-dimensional jets, having finite and infinite fluid velocity at orifice, is constructed. The general form of group-invariant solution for two-dimensional free, wall, and liquid jets is given explicitly in Section 6. Finally the conclusions are summarized in Section 7.

2. Group-invariant Solution for Two-Dimensional Free Jet

The flow in two-dimensional free jet is governed by nonlinear third-order PDE for stream function

$$\psi_y\psi_{xy} - \psi_x\psi_{yy} - \nu\psi_{yyy} = 0, \tag{2.1}$$

for an incompressible fluid. The relation between stream function and velocity components is

$$u = \psi_y, \qquad v = -\psi_x. \tag{2.2}$$

The Lie point symmetry generator of (2.1) derived by Mason [3] is

$$X = [(c_1 + c_3)x + c_2]\frac{\partial}{\partial x} + [c_1 y + k(x)]\frac{\partial}{\partial y} + [c_3\psi + c_4]\frac{\partial}{\partial \psi}. \tag{2.3}$$

The boundary conditions and the conserved quantity for two-dimensional free jet in terms of stream function are (see [1–3])

$$y = 0: \quad \psi_x = 0, \quad \psi_{yy} = 0, \tag{2.4}$$

$$y = \pm\infty: \quad \psi_y = 0, \quad \psi_{yy} = 0, \tag{2.5}$$

$$J = 2\rho\int_0^\infty \psi_y^2 dy. \tag{2.6}$$

The conserved vector

$$T^1 = \psi_y^2, \qquad T^2 = -\psi_x\psi_y - \nu\psi_{yy} \tag{2.7}$$

gave the conserved quantity (2.6) for two-dimensional free jet (see [8]). The symmetry associated with the conserved vector which is used to establish the conserved quantity for each jet generates the group-invariant solution for nonlinear third-order PDE [4, 9].

The symmetries associated with a known conserved vector can be determined by [10]

$$X^{[1]}\left(T^1\right) + T^1 D_y\left(\xi^2\right) - T^2 D_y\left(\xi^1\right) = 0, \tag{2.8}$$

$$X^{[1]}\left(T^2\right) + T^2 D_x\left(\xi^1\right) - T^1 D_x\left(\xi^2\right) = 0. \tag{2.9}$$

Equations (2.8) and (2.9) yield

$$T^1\left[c_3 - \frac{1}{2}c_1\right] = 0, \qquad T^2\left[c_3 - \frac{1}{2}c_1\right] = 0, \tag{2.10}$$

and thus for conserved vector (2.7), $c_3 = (1/2)\ c_1$. The Lie point symmetry generator associated with conserved vector (2.7) is

$$X = \left[\frac{3}{2}c_1 x + c_2\right]\frac{\partial}{\partial x} + \left[c_1 y + k(x)\right]\frac{\partial}{\partial y} + \left[\frac{1}{2}c_1\psi + c_4\right]\frac{\partial}{\partial \psi}. \tag{2.11}$$

Now, $\psi = \phi(x, y)$ is a group-invariant solution of (2.1) if

$$X(\psi - \phi(x, y))|_{\psi=\phi} = 0, \tag{2.12}$$

which yields

$$\psi = \left(x + \frac{2c_2}{3c_1}\right)^{1/3} g(\xi) \ - \frac{2c_4}{c_1}, \tag{2.13}$$

$$\xi = \frac{y}{(x + 2c_2/3c_1)^{2/3}} - K(x), \tag{2.14}$$

where

$$K(x) = \frac{2}{3c_1}\int^x \frac{k(x)}{(x + 2c_2/3c_1)^{5/3}}dx. \tag{2.15}$$

The conserved quantity (2.6) is independent of x provided $K(x) = 0$ which yields $k(x) = 0$. Since the stream function is determined up to an arbitrary constant, c_4 can be chosen to be zero. The insertion of (2.13) into (2.1) results in a nonlinear third-order ordinary differential equation (ODE) for $g(\xi)$:

$$3\nu\frac{d^3 g}{d\xi^3} + g\frac{d^2 g}{d\xi^2} + \left(\frac{dg}{d\xi}\right)^2 = 0. \tag{2.16}$$

Equation (2.16) can be transformed to

$$f''' + ff'' + f'^2 = 0, \tag{2.17}$$

with

$$\eta = \frac{A\xi}{3\nu}, \qquad Af = g, \tag{2.18}$$

where A is arbitrary constant, and prime denotes differentiation with respect to η. The boundary conditions and conserved quantity (2.4)–(2.6), in terms of $f(\eta)$, take the following form:

$$f(0) = 0, \qquad f''(0) = 0, \qquad f'(\pm\infty) = 0, \qquad f''(\pm\infty) = 0, \tag{2.19}$$

$$J = \frac{2A^3\rho}{3\nu}\int_0^\infty f'^2 d\eta. \tag{2.20}$$

The solution of (2.17) subject to (2.19) and condition $f(\infty) = 1$ is (see [2, 11, 12])

$$f(\eta) = \tanh\left(\frac{\eta}{2}\right), \tag{2.21}$$

and value of A in terms of J is

$$A = \left(\frac{9\nu J}{2\rho}\right)^{1/3}. \tag{2.22}$$

The final form of group-invariant solution is

$$\begin{aligned}
\psi &= \left[\frac{9\nu J}{2\rho}\left(x + \frac{2c_2}{3c_1}\right)\right]^{1/3} f(\eta), \\
u(x,y) &= \left[\frac{3J^2}{4\rho^2\nu(x + 2c_2/3c_1)}\right]^{1/3} f'(\eta), \\
\eta &= \left[\frac{J}{6\rho\nu^2(x + 2c_2/3c_1)^2}\right]^{1/3} y, \\
X &= \left(\frac{3}{2}x + \frac{c_2}{c_1}\right)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{2}\psi\frac{\partial}{\partial \psi},
\end{aligned} \tag{2.23}$$

is the symmetry that generated the group-invariant solution. Now

$$u(x,0) = \left[\frac{3J^2}{32\rho^2\nu(x + 2c_2/3c_1)}\right]^{1/3} \tag{2.24}$$

is finite at $x = 0$ and so our solution may have some significance even near the axis. By taking $c_2 = 0$, the results [1–4] for infinite velocity at orifice can be rediscovered.

3. Group-invariant Solution for Two-Dimensional Wall Jet

The flow in two-dimensional wall jet is also governed by (2.1). The boundary conditions for the two-dimensional wall jet are [5]

$$y = 0: \quad \psi_x = 0, \quad \psi_y = 0, \tag{3.1}$$

$$y = \infty: \quad \psi_y = 0, \tag{3.2}$$

and the conserved quantity is

$$F = \int_0^\infty \psi_y \left(\int_y^\infty \psi_{y^*}^2 dy^* \right) dy. \tag{3.3}$$

The conserved vector

$$T^1 = \psi\psi_y^2, \qquad T^2 = -\psi\psi_x\psi_y + \frac{\nu}{2}\psi_y^2 - \nu\psi\psi_{yy} \tag{3.4}$$

gave the conserved quantity for two-dimensional wall jet [8], and the symmetry associated with this conserved vector is

$$X = \left[\frac{4}{3}c_1 x + c_2\right]\frac{\partial}{\partial x} + \left[c_1 y + k(x)\right]\frac{\partial}{\partial y} + \frac{1}{3}c_1\psi\frac{\partial}{\partial \psi}. \tag{3.5}$$

The group-invariant solution of (2.1) for two-dimensional wall jet case is

$$\psi = \left(x + \frac{3c_2}{4c_1}\right)^{1/4} g(\xi), \tag{3.6}$$

$$\xi = \frac{y}{(x + 3c_2/4c_1)^{3/4}} - K(x), \tag{3.7}$$

where

$$K(x) = \frac{3}{4c_1}\int^x \frac{k(x)}{(x + 3c_2/4c_1)^{7/4}} dx. \tag{3.8}$$

The conserved quantity (3.3) is independent of x provided $K(x) = 0$ which yields $k(x) = 0$. The substitution of (3.6) into (2.1) gives rise to a nonlinear third-order ODE for $g(\xi)$:

$$\nu\frac{d^3 g}{d\xi^3} + \frac{1}{4}g\frac{d^2 g}{d\xi^2} + \frac{1}{2}\left(\frac{dg}{d\xi}\right)^2 = 0. \tag{3.9}$$

Define $\eta = (A/4\nu)\xi$ and $g = Af$ Equation (3.9) transforms to

$$f''' + ff'' + 2f'^2 = 0. \tag{3.10}$$

Boundary conditions (3.1) and (3.2) and conserved quantity (3.3) take the following form:

$$\begin{gathered} f(0) = 0, \qquad f'(0) = 0, \qquad f'(\infty) = 0, \\ F = \frac{A^4}{4\nu}\int_0^\infty f'\left(\int_\eta^\infty f'^2 d\eta^*\right)d\eta \end{gathered} \tag{3.11}$$

Glauert [5] selected a solution of (3.10) with $f(\infty) = 1$, and after integrating (3.10) twice, the following equation was obtained:

$$\frac{dh}{d\eta} = \frac{1}{3}\left(1 - h^3\right), \quad \text{where } h^2 = f,\ 0 \le h \le 1. \tag{3.12}$$

Equation (3.12) yields

$$\eta = \log\frac{\sqrt{1 + h + h^2}}{1 - h} + \sqrt{3}\tan^{-1}\frac{\sqrt{3}h}{2 + h}. \tag{3.13}$$

The conserved quantity gave the unknown constant A as

$$A = (40\nu F)^{1/4}. \tag{3.14}$$

Thus we finally obtain

$$\begin{aligned} \psi &= \left[40F\nu\left(x + \frac{3c_2}{4c_1}\right)\right]^{1/4} f(\eta), \\ u(x,y) &= \left[\frac{5F}{2\nu(x + 3c_2/4c_1)}\right]^{1/2} f'(\eta), \\ \eta &= \left[\frac{5F}{32\nu^3(x + 3c_2/4c_1)^3}\right]^{1/4} y, \\ \mathrm{X} &= \left(\frac{4}{3}x + \frac{c_2}{c_1}\right)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{3}\psi\frac{\partial}{\partial \psi}. \end{aligned} \tag{3.15}$$

The results obtained here for $3c_2/4c_1 = l$ agree with Riley [7], and l can be determined from [13]. By taking $c_2 = 0$, the results for infinite velocity at orifice obtained by Glauert [5] can be rediscovered.

4. Group-Invariant Solution for Two-Dimensional Liquid Jet

The governing equation for two-dimensional liquid jet in terms of stream function is (2.1). The boundary conditions and conserved quantity for two-dimensional liquid jet are [6]

$$y = 0: \quad \psi_x = 0, \quad \psi_y = 0, \tag{4.1}$$

$$y = \phi(x): \quad \psi_{yy} = 0, \tag{4.2}$$

$$M = \int_0^{\phi(x)} \psi_y dy. \tag{4.3}$$

The conserved vector

$$T^1 = \psi_y, \qquad T^2 = -\psi_x \tag{4.4}$$

gave conserved quantity for two-dimensional liquid jet [8].

Equations (2.8) and (2.9) yield the following Lie point symmetry generator associated with the conserved vector (4.4):

$$X = [c_1 x + c_2]\frac{\partial}{\partial x} + [c_1 y + k(x)]\frac{\partial}{\partial y} + c_4 \frac{\partial}{\partial \psi}. \tag{4.5}$$

The group-invariant solution for two-dimensional liquid jet is

$$\psi = g(\xi) + \ln\left(x + \frac{c_2}{c_1}\right)^{c_4/c_1}, \quad \xi = \frac{y}{x + c_2/c_1} - K(x), \tag{4.6}$$

where

$$K(x) = \frac{1}{c_1}\int^x \frac{k(x)}{(x + c_2/c_1)^2} dx. \tag{4.7}$$

The conserved quantity is independent of x only if $K(x) = 0$ which gives $k(x) = 0$. The stream function contains an additive constant so we may choose $c_4 = 0$ without loss of generality.

The substitution of (4.6), with $k(x) = 0$, $c_4 = 0$, into (2.1) yields a nonlinear third-order ordinary differential equation for $g(\xi)$:

$$\nu \frac{d^3 g}{d\xi^3} + \left(\frac{dg}{d\xi}\right)^2 = 0. \tag{4.8}$$

Equation (4.8) takes the following form:

$$f''' + 3f'^2 = 0, \tag{4.9}$$

where $\eta = A/3\nu\xi$ and $g = Af$. Boundary conditions (4.1) and (4.2) are

$$f(0) = 0, \qquad f'(0) = 0, \qquad f''(1) = 0, \tag{4.10}$$

where the free surface is chosen to be $\eta = 1$. The conserved quantity (4.3) becomes

$$M = \int_0^1 Af' d\eta. \tag{4.11}$$

Equation (4.9) yields (see [11, 12, 14])

$$\frac{dt}{d\eta} = \left[2\left(1 - t^3\right)\right]^{1/2}, \qquad t = f'. \tag{4.12}$$

The final form of solution of (4.9) in parametric form is (see [14])

$$\eta = \frac{2}{3\sqrt{2}}\left[{}_2F_1\left[\frac{1}{2}, \frac{2}{3}, \frac{3}{2}, 1\right] - \left(1 - t^3\right)^{1/2} \times {}_2F_1\left[\frac{1}{2}, \frac{2}{3}, \frac{3}{2}, 1 - t^3\right]\right], \tag{4.13}$$

where ${}_2F_1$ is the hypergeometric function of first kind. We can tabulate the values of η for given values of parameter $t = f'$ from (4.13), and conserved quantity M yields the constant A.

Thus finally we obtain

$$\begin{aligned}
\psi(x, y) &= \frac{3\sqrt{3}M}{\pi} f(\eta), \\
u(x, y) &= \frac{9M^2}{\nu\pi^2(x + c_2/c_1)} f'(\eta), \\
\eta &= \frac{\sqrt{3}M}{\nu\pi(x + c_2/c_1)} y, \\
X &= \left(x + \frac{c_2}{c_1}\right)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.
\end{aligned} \tag{4.14}$$

Now $u(x, 0)$ is finite at $x = 0$. The results obtained here for $c_2/c_1 = l$ agree with those concluded by Watson [6] by the similarity solution method, and the procedure to obtain l is discussed there.

5. Comparison between Two-Dimensional Jets with Finite Velocity at Orifice and Infinite Velocity at Orifice

The comparison between two-dimensional jets with finite fluid velocity at orifice and infinite velocity at orifice is constructed in Table 1. Table 1 shows that by formally taking $c_2 = 0$, the stream function ψ, variable η, and symmetry that generates group-invariant solution for infinite velocity case can be deduced from those of the finite velocity case.

Table 1: Comparison between two-dimensional jets with finite velocity at orifice and infinite velocity at orifice.

	Finite velocity at orifice	Infinite velocity at orifice
2-D free jet	$\psi = \left[\frac{9\nu J}{2\rho}\left(x + \frac{2c_2}{3c_1}\right)\right]^{1/3} f(\eta)$	$\psi = \left[\frac{9\nu J}{2\rho}x\right]^{1/3} f(\eta)$
	$u(x,y) = \left[\frac{3J^2}{4\rho^2\nu(x+2c_2/3c_1)}\right]^{1/3} f'(\eta)$	$u(x,y) = \left[\frac{3J^2}{4\rho^2\nu(x+2c_2/3c_1)}\right]^{1/3} f'(\eta)$
	$\eta = \left[\frac{J}{6\rho\nu^2(x+2c_2/3c_1)^2}\right]^{1/3} y$	$\eta = \left[\frac{J}{6\rho\nu^2}\right]^{1/3}\frac{y}{x^{2/3}}$
	$f''' + ff'' + f'^2 = 0$	$f''' + ff'' + f'^2 = 0$
	$X = \left(\frac{3}{2}x + \frac{c_2}{c_1}\right)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{2}\psi\frac{\partial}{\partial \psi}$	$X = \frac{3}{2}x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{2}\psi\frac{\partial}{\partial \psi}$
2-D wall jet	$\psi = \left[40F\nu\left(x + \frac{3c_2}{4c_1}\right)\right]^{1/4} f(\eta)$	$\psi = [40F\nu x]^{1/4} f(\eta)$
	$\eta = \left[\frac{5F}{32\nu^3(x+3c_2/4c_1)^3}\right]^{1/4} y$	$\eta = \left[\frac{5F}{32\nu^3}\right]^{1/4}\frac{y}{x^{3/4}}$
	$u(x,y) = \left[\frac{5F}{2\nu(x+3c_2/4c_1)}\right]^{1/2} f'(\eta)$	$u(x,y) = \left[\frac{5F}{2\nu x}\right]^{1/2} f'(\eta)$
	$f''' + ff'' + 2f'^2 = 0$	$f''' + ff'' + 2f'^2 = 0$
	$X = \left(\frac{4}{3}x + \frac{c_2}{c_1}\right)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{3}\psi\frac{\partial}{\partial \psi}$	$X = \frac{4}{3}x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{3}\psi\frac{\partial}{\partial \psi}$
2-D liquid jet	$\psi(x,y) = \frac{3\sqrt{3}M}{\pi} f(\eta)$	$\psi(x,y) = \frac{3\sqrt{3}M}{\pi} f(\eta)$
	$\eta = \frac{\sqrt{3}M}{\nu\pi(x+c_2/c_1)} y$	$\eta = \frac{\sqrt{3}M}{\nu\pi}\frac{y}{x}$
	$u(x,y) = \frac{9M^2}{\nu\pi^2(x+c_2/c_1)} f'(\eta)$	$u(x,y) = \frac{9M^2}{\nu\pi^2 x} f'(\eta)$
	$f''' + 3f'^2 = 0$	$f''' + 3f'^2 = 0$
	$X = \left(x + \frac{c_2}{c_1}\right)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$	$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$

6. General Form of Group-invariant Solutions for Two-Dimensional Jets

The flow in two-dimensional free, wall, and liquid jets is governed by nonlinear third-order PDE (2.1). The symmetry generator (2.3) associated with the conserved vector that gives conserved quantity for jet flow gives the following conditions on constant:

$$c_3 = \left(\frac{1}{\alpha} - 1\right)c_1, \tag{6.1}$$

where

$$\alpha = \begin{cases} \dfrac{2}{3} & \text{for two-dimensional free jet,} \\ \dfrac{3}{4} & \text{for two-dimensional wall jet,} \\ 1 & \text{for two-dimensional liquid jet.} \end{cases} \tag{6.2}$$

We choose $c_4 = 0$ because the stream function is determined up to an arbitrary constant. The expression for group-invariant solutions for two-dimensional jet flows is

$$\begin{aligned} \psi &= \left(x + \alpha\frac{c_2}{c_1}\right)^{1-\alpha} g(\xi), \\ \xi &= \frac{y}{(x + \alpha(c_2/c_1))^{\alpha}} - K(x), \end{aligned} \tag{6.3}$$

where

$$K(x) = \frac{\alpha}{c_1}\int^{x} \frac{k(x)}{(x + \alpha(c_2/c_1))^{\alpha+1}} dx. \tag{6.4}$$

The condition where conserved quantity is independent of x yields $k(x) = 0$ in each of free, wall, and liquid jets. Using (6.3)–(6.4), (2.1) yields

$$\nu\frac{d^3 g}{d\xi^3} + (1-\alpha)g\frac{d^2 g}{d\xi^2} + (2\alpha - 1)\left(\frac{dg}{d\xi}\right)^2 = 0. \tag{6.5}$$

Define the transformations

$$\eta = \begin{cases} (1-\alpha)\dfrac{A}{\nu}\xi & \text{for two-dimensional free and wall jets,} \\ \dfrac{A}{3\nu}\xi & \text{for two-dimensional liquid jet,} \end{cases} \qquad g = Af, \tag{6.6}$$

where A is a constant. The final form of a group-invariant solution is

$$\begin{aligned} \psi &= \left(x + \alpha\frac{c_2}{c_1}\right)^{1-\alpha} Af(\eta), \\ \eta &= \begin{cases} (1-\alpha)\dfrac{Ay}{\nu(x + \alpha(c_2/c_1))^{\alpha}} & \text{for two-dimensional free and wall jets,} \\ \dfrac{Ay}{3\nu(x + c_2/c_1)} & \text{for two-dimensional liquid jet.} \end{cases} \end{aligned} \tag{6.7}$$

For two-dimensional free and wall jets, (6.5) yields

$$f''' + ff'' + \frac{2\alpha - 1}{1 - \alpha} f'^2 = 0, \tag{6.8}$$

and for two-dimensional liquid jet, we have

$$f''' + 3f'^2 = 0. \tag{6.9}$$

The symmetry

$$X = \left(x + \alpha \frac{c_2}{c_1} \right) \frac{\partial}{\partial x} + \alpha y \frac{\partial}{\partial y} + (1 - \alpha)\psi \frac{\partial}{\partial \psi} \tag{6.10}$$

yielded the group-invariant solution.

7. Conclusions

The group-invariant solutions for two-dimensional free, wall, and liquid jets were derived for finite velocity at orifice. For two-dimensional free jet, a Lie point symmetry was associated with the conserved vector that generated the conserved quantity for two-dimensional free jet. This symmetry generated the group-invariant solution for the nonlinear third-order PDE for stream function subject to certain boundary conditions. The nonlinear third-order PDE was transformed to the nonlinear third-order ODE. Using certain transformations we deduced the same nonlinear third-order ODE as was obtained for two-dimensional free jet having infinite fluid velocity at orifice. The analogue analysis was done for the two-dimensional wall and liquid jets. A detailed comparison of results for finite and infinite velocity at orifice was constructed. The general form of group-invariant solution for the two-dimensional jets was derived.

Acknowledgment

R. Naz is thankful to The Lahore School of Economics for providing funding to complete this research work.

References

[1] H. Schlichting, "Laminare strahlausbreitung," *Zeitschrift für Angewandte Mathematik und Mechanik*, vol. 13, pp. 260–263, 1933.
[2] W. G. Bickley, "The plane jet," *Philosophical Magazine*, vol. 23, pp. 727–731, 1937.
[3] D. P. Mason, "Group invariant solution and conservation law for a free laminar two-dimensional jet," *Journal of Nonlinear Mathematical Physics*, vol. 9, supplement 2, pp. 92–101, 2002.
[4] R. Naz, D. P. Mason, and F. M. Mahomed, "Conservation laws via the partial Lagrangian and group invariant solutions for radial and two-dimensional free jets," *Nonlinear Analysis*, vol. 10, no. 6, pp. 3457–3465, 2009.
[5] M. B. Glauert, "The wall jet," *Journal of Fluid Mechanics*, vol. 1, pp. 625–643, 1956.
[6] E. J. Watson, "The radial spread of a liquid jet over a horizontal plane," *Journal of Fluid Mechanics*, vol. 20, pp. 481–499, 1964.

[7] N. Riley, “An applicaton of Mangler’s transformation,” *The Quarterly Journal of Mechanics and Applied Mathematics*, vol. 14, pp. 197–202, 1961.

[8] R. Naz, D. P. Mason, and F. M. Mahomed, “Conservation laws and conserved quantities for laminar two-dimensional and radial jets,” *Nonlinear Analysis*, vol. 10, no. 5, pp. 2641–2651, 2009.

[9] I. Naeem and R. Naz, “Wall jet on a hemi-spherical shell: conserved quantities and group invariant solution,” *International Journal of Nonlinear Science*, vol. 7, no. 2, pp. 149–158, 2009.

[10] A. H. Kara and F. M. Mahomed, “Relationship between symmetries and conservation laws,” *International Journal of Theoretical Physics*, vol. 39, no. 1, pp. 23–40, 2000.

[11] N. Riley, “Asymptotic expansions in radial jets,” *Journal of Mathematical Physics*, vol. 41, pp. 132–146, 1962.

[12] N. Riley, “Radial jets with swirl, part I. Incompressible flow,” *The Quarterly Journal of Mechanics and Applied Mathematics*, vol. 15, pp. 435–458, 1962.

[13] M. B. Glauert, *Symposium uber Grenzschichtforschung*, Springer, 1957, edited by H. Gortler.

[14] R. Naz, F. M. Mahomed, and D. P. Mason, “Symmetry solutions of a third-order ordinary differential equation which arises from Prandtl boundary layer equations,” *Journal of Nonlinear Mathematical Physics*, vol. 15, supplement 1, pp. 179–191, 2008.