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Research Article

Common Lyapunov Function Based on Kullback–Leibler Divergence for a Switched Nonlinear System

Omar M. Abou Al-Ola,[1] Ken'ichi Fujimoto,[2] and Tetsuya Yoshinaga[2]

[1] *Graduate School of Health Sciences, The University of Tokushima, 3-18-15 Kuramoto, Tokushima 770-8509, Japan*
[2] *Institute of Health Biosciences, The University of Tokushima, 3-18-15 Kuramoto, Tokushima 770-8509, Japan*

Correspondence should be addressed to Ken'ichi Fujimoto, fujimoto@medsci.tokushima-u.ac.jp

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Many problems with control theory have led to investigations into switched systems. One of the most urgent problems related to the analysis of the dynamics of switched systems is the stability problem. The stability of a switched system can be ensured by a common Lyapunov function for all switching modes under an arbitrary switching law. Finding a common Lyapunov function is still an interesting and challenging problem. The purpose of the present paper is to prove the stability of equilibrium in a certain class of nonlinear switched systems by introducing a common Lyapunov function; the Lyapunov function is based on generalized Kullback–Leibler divergence or Csiszár's *I*-divergence between the state and equilibrium. The switched system is useful for finding positive solutions to linear algebraic equations, which minimize the *I*-divergence measure under arbitrary switching. One application of the stability of a given switched system is in developing a new approach to reconstructing tomographic images, but nonetheless, the presented results can be used in numerous other areas.

1. Introduction

A switched system is a dynamical system that consists of a finite number of subsystems and a logical rule that orchestrates switching between these subsystems. Mathematically, these subsystems are usually described by a collection of indexed differential or difference equations; one convenient way to classify switched systems is based on the dynamics of their subsystems, for example, continuous-time or discrete-time and linear or nonlinear. Switching events in switched systems can be classified into state-dependent versus time-dependent and autonomous versus controlled. For the moment, we will concern ourselves with controlled time-dependent switching. Given a family $f_p, p \in P$ of functions from $\mathbb{R}^n$ to $\mathbb{R}^n$, where P is some index set. This gives rise to a family of systems:

$$\frac{dx}{dt} = f_p(x), \quad p \in P, \tag{1.1}$$

evolving on $\mathbb{R}^n$. The functions f_p are assumed to be sufficiently regular. The easiest case to think about is when all these systems are linear and the index set P is finite: $P = \{1, 2, \ldots, m\}$. The switched system with time-dependent switching, generated by the above family, can be described by

$$\frac{dx}{dt} = f_\sigma(x), \tag{1.2}$$

where the switching signal $\sigma : [0, \infty) \to P$ is a piecewise constant function that has a finite number of discontinuities, which we call the switching times, on every bounded time interval and takes a constant value on every interval between two consecutive switching times. The role of σ is to specify, at each time instant t, the index $\sigma(t) \in P$ of the active subsystem, that is, the system from the family in (1.1) that is currently being followed.

Stability analysis is a fundamental problem in the analysis and design of switched systems, and, during the past several years, several methods have been proposed to solve it. It is well known that a necessary condition for asymptotic stability under arbitrary (unconstrained) switching is that all of the individual subsystems are asymptotically stable. However, the above stability condition is not generally sufficient to guarantee asymptotic stability for the switched system under arbitrary switching. Standard Lyapunov theory for the stability of smooth systems, which requires the existence of a Lyapunov function satisfying, with its derivative, some inequalities [1, 2], has a direct extension for the stability of switched systems under arbitrary switching; this extension was obtained by requiring the existence of a common Lyapunov function for all individual subsystems corresponding to the system being considered, that is, the existence of a common Lyapunov function for all subsystems was shown to be a necessary and sufficient condition for a switched system to be asymptotically stable under arbitrary switching law (see [3, 4] and references therein). However, most results have been obtained for cases when switching occurs between linear subsystems, while the common Lyapunov function is set up in the quadratic form [3, 4]. For the nonlinear switched systems, some stability conditions were obtained in [3, 5–7]; however, it should be noted that so far there do not exist common methods of effectively developing the Lyapunov functions for nonlinear systems.

Most switched systems in practice, however, do not possess a common Lyapunov function, yet they may still be asymptotically stable under some properly chosen switching law. The multiple Lyapunov function technique is a powerful and effective tool for finding such a switching law or identifying a class of useful switching laws [3, 8–10]. The key point in these conditions of multiple Lyapunov function methods is the nonincreasing requirement on any Lyapunov function over the “switched on” time sequence of the corresponding subsystem; the Lyapunov functions in this case are called Lyapunov-like functions. However, this is usually difficult to check in its full generality. Thus, connecting adjacent Lyapunov functions at switching points is a commonly accepted strategy in applying the multiple Lyapunov function methods. To relax this nonincreasing requirement, the concept of generalized Lyapunov-like functions has been addressed [11], where a necessary and sufficient condition for the stability of switched systems in terms of generalized Lyapunov-like functions has been established; this condition tells us how much the corresponding general Lyapunov-like function is allowed to grow on the "switched on" time sequence without violating stability. We also do not need to worry about when and how each subsystem is activated for the first time with this condition. Given the importance of dissipativity concepts for smooth systems where the storage functions induced by dissipativity usually provide natural candidates for Lyapunov functions, a framework of dissipativity theory for switched systems using multiple storage functions and multiple supply rates was set up [12]. Indeed, each subsystem in a switched system is associated with a storage function to describe the energy stored in the subsystem, and it is associated with a supply rate that represents the energy coming from outside the subsystem when the subsystem is active; the exchange of energy between the active subsystem and an inactive subsystem is characterized by cross-supply rates. The stability of the switched system was assured by dissipativity when all supply rates can be made negative, as long as the total exchanged energy between the active subsystem and any inactive subsystems is finite in some sense. Moreover, unlike multiple Lyapunov functions, where a nonincreasing condition on a "switched on" time sequence is a basic assumption even though the Lyapunov function is allowed to increase when the corresponding subsystem is inactive, storage functions are allowed to increase not only on time intervals when the corresponding subsystems are inactive but also on the "switched on" time sequence.

Tomography deals with the problem of determining the shape and dimensional information of an object from a set of projections and is concerned with the reconstruction of cross-sectional images, which permits the interiors of objects to be visualized. Computed tomography (CT) [13–16] is a method of medical imaging employing tomography created by computer processing; it is well known for delivering high-quality images from inside the human body and thus supports physicians in diagnosis. Problems with images reconstructed by a projection operator and from a projection data set generally become ill-posed [17]. Many different reconstruction algorithms are used in medical practice to solve the inverse problem with image reconstruction.

We deal with analyzing the stability of a switched nonlinear system of the form in (1.2) in this study, where all individual subsystems, of the form in (1.1), will be stable, that is, a continuous system with switching signals under arbitrary switching. A common Lyapunov function, based on the generalized Kullback–Leibler divergence [18] or Csiszár's I-divergence [19, 20] measure between the state and the equilibrium, is set up. That Kullback–Leibler divergence can play the role of a common Lyapunov function for a class of switched nonlinear systems is demonstrated for the first time. A particular application of the stability of this switched system, which is to develop a novel approach to reconstructing tomographic images based on the idea of continuous dynamical methods, and an illustrative example are presented.

2. Statement of the Problem

Systems of linear equations and matrix inversion play an important and motivating role in linear algebra. Such linear equations appear frequently in applied mathematics to model certain phenomena, for example, in computed tomography (CT) [13–16]. Fundamentally, the problem is to obtain unknown variable $x \in \mathbb{R}_+^J$, with $\mathbb{R}_+$ denoting the set of nonnegative real

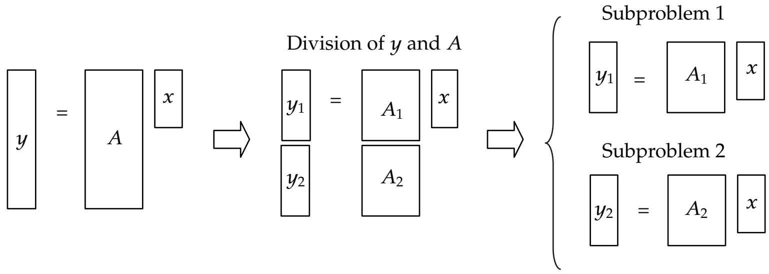

Figure 1: Illustration of the partitioning of y and A for $m = 1, 2$.

numbers, satisfying

$$y = Ax, \tag{2.1}$$

where $y \in \mathbb{R}_+^I \setminus \{0\}$ and $A \in \mathbb{R}_+^{I \times J} \setminus \{0\}$. We say that the system $y = Ax$ is consistent if it has a locally unique solution $e \in \mathbb{R}_+^J$. Equation (2.1) is an ill-posed problem for inconsistent cases, which means that its solution is not unique or does not exist [17]. To find solution x, we formulated a switched nonlinear system consisting of the family of subsystems [21, 22]

$$\frac{dx}{dt} = -\operatorname{diag}(x) A_m^\top (A_m x - y_m) \tag{2.2}$$

or, equivalently,

$$\frac{dx_j}{dt} = -x_j \left(A_m^\top\right)_j (A_m x - y_m), \quad j = 1, 2, \ldots, J, \tag{2.3}$$

where $\operatorname{diag}(x)$ indicates the diagonal matrix of order $J \times J$ in which the diagonal entries starting in the upper left corner are the elements of x, while $A_m \in \mathbb{R}_+^{I_m \times J}$ and $y_m \in \mathbb{R}_+^{I_m}$ are, respectively, a submatrix consisting of I_m partial rows of A and a subvector of y with the same corresponding rows of A_m, for $m = 1, 2, \ldots, M$, with M denoting the total number of divisions. Figures 1 and 2 present this formulation for case $m = 1, 2$.

Proposition 2.1. *If one chooses positive initial value $x^0 \in \mathbb{R}_{++}^J$ in the switched system in* (2.2)*, then the solution $\phi(t, x^0)$ is in $\mathbb{R}_{++}^J$ for all $t \in \mathbb{R}_+$, where $\mathbb{R}_{++}$ represents the set of positive real numbers.*

Proof. We proved in [22] that if we choose positive initial value $x^0 \in \mathbb{R}_{++}^J$ in our continuous-time image reconstruction (CIR) system, then the corresponding solution will be in $\mathbb{R}_{++}^J$ for all $t \in \mathbb{R}_+$. Similarly, if we start from a positive initial value for each individual subsystem in (2.2), the corresponding trajectory will stay in $\mathbb{R}_{++}^J$. Consequently, if we choose a positive initial value for the first subsystem under arbitrary switching in the switched system described by (2.2), an end point of the corresponding trajectory, at a given time of switching, which is the initial point for the second subsystem, will be positive, and so on. □

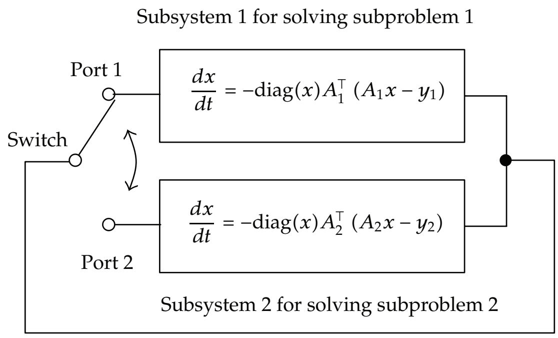

Figure 2: Switched system consisting of two subsystems.

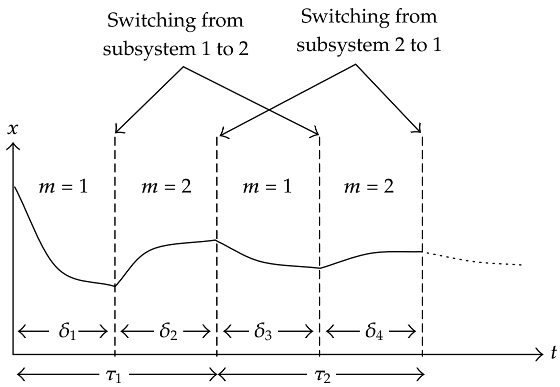

Figure 3: Trajectory of a switched system for $m = 1, 2$.

This behavior is illustrated in Figure 3, where we are switching between two subsystems on the line.

Proposition 2.2. *Point e satisfying* (2.1) *is a common equilibrium for all subsystems in* (2.2).

Proof. Point e satisfies (2.1) if and only if

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_I \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_I \end{pmatrix} e, \tag{2.4}$$

where $A_i \in \mathbb{R}_+^{1\times J}$ and $y_i \in \mathbb{R}_+$. Since the scalar multiplication defined for matrices having scalar elements also applies to partitioned matrices, then $y = Ae$ if and only if $y_1 = A_1 e$, $y_2 = A_2 e, \ldots, y_I = A_I e$, which means that e is a common equilibrium for all subsystems in (2.2) that satisfies $y_m = A_m e$ for all m. This similarly occurs for any submatrices A_m consisting of I_m partial rows of A and the corresponding subvectors y_m of y. □

According to Proposition 2.2, the common equilibria for all subsystems in the switched nonlinear system in (2.2) that satisfy $y_m = A_m x$ for all m are the solutions to the equation $y = Ax$; these equilibria are stable with respect to each subsystem by virtue of existing the Lyapunov functions:

$$V_m(x) = \frac{1}{2}\left\| A_m x - y_m \right\|_2^2. \tag{2.5}$$

Proposition 2.3. *Consider the mth subsystem in the switched system in* (2.2) *with initial value $x_m^0 \in \mathbb{R}_{++}^J$. If there exists locally unique equilibrium $e_m \notin \{0\}$, then $V_m(\phi_m(t, x_m^0))$ described by* (2.5) *decreases in $t \in \mathbb{R}_+$.*

Proof. We have $V_m(x) \geq 0$ with equality if $x = e_m \notin \{0\}$. Its derivative along corresponding solution $\phi_m(t, x_m^0)$ is given by

$$\frac{dV_m}{dt}\left(\phi_m\left(t, x_m^0\right)\right) = -\Lambda_m\left(t, x_m^0\right)^\top \Phi_m\left(t, x_m^0\right)\Lambda_m\left(t, x_m^0\right), \tag{2.6}$$

where $\Lambda_m := A_m^\top (A_m \phi_m - y_m)$ and $\Phi_m(t, x_m^0) := \operatorname{diag}(\phi_m(t, x_m^0))$. According to Proposition 2.1, the derivative of $V_m(\phi_m(t, x_m^0))$ is negative semidefinite for $t \in \mathbb{R}_+$ and any $x_m^0 \in \mathbb{R}_{++}^J$. The $V_m(x)$ can be a Lyapunov function on $\mathbb{R}_{++}^J$. Thus, $x_m^0 \in \mathbb{R}_{++}^J$ exists such that $\phi_m(t, x_m^0) \to e_m$ as $t \to \infty$. □

Proposition 2.4. *The zero equilibrium of the switched system in* (2.2) *is locally unstable.*

Proof. We rewrite the switched system in (2.2) as

$$\frac{dx}{dt} = f_m(x). \tag{2.7}$$

The derivative of f_m with respect to x is

$$\frac{\partial f_m}{\partial x}(x) = -\operatorname{diag}(x) A_m^\top A_m - \operatorname{diag}\left(A_m^\top (A_m x - y_m)\right). \tag{2.8}$$

We can see that all eigenvalues of the derivative at the zero equilibrium

$$\frac{\partial f_m}{\partial x}(0) = \operatorname{diag}\left(A_m^\top y_m\right) \tag{2.9}$$

are nonnegative for all $m = 1, 2, \ldots, M$. □

3. Stability via Common Lyapunov Function

This section addresses the main results where we construct a common Lyapunov function for all subsystems of the switched nonlinear system in (2.2), based on the generalized Kullback–Leibler divergence between the state and the equilibrium. This function will be positive definite and its derivatives, with respect to each subsystem, negative definite.

The generalized Kullback–Leibler divergence of two nonnegative vectors α and β is defined as

$$\mathrm{KL}(\alpha, \beta) = \sum_j \beta_j \log \frac{\beta_j}{\alpha_j} + \alpha_j - \beta_j. \tag{3.1}$$

The standard Kullback–Leibler divergence only consists of the first term and is only defined for probability distributions, that is, the sum of each vector is 1. The last two terms are necessary so that the generalized divergence has, for arbitrary nonnegative vectors α and β, the property of being nonnegative and zero exactly when α and β are equal. This divergence is called Csiszár's I-divergence measure, which leads to effective methods of selection to solve optimization problems with nonnegativity constraints [19, 20]. The divergence $\mathrm{KL}(\alpha, \beta)$ for the α and β vectors of nonnegative real numbers is nonnegative with $\mathrm{KL}(\alpha, \beta) = 0$ if and only if $\alpha = \beta$. Note that it suffices to prove this when α and β are scalars, because $\mathrm{KL}(\alpha, \beta) = \sum_j \mathrm{KL}(\alpha_j, \beta_j)$. Thus, we let $r = \alpha/\beta$ and

$$\begin{aligned} \mathrm{KL}(\alpha, \beta) &= \beta \log \frac{\beta}{\alpha} + \alpha - \beta \\ &= \beta(-\log r + r - 1) \\ &= \beta \int_1^r \frac{s-1}{s} ds. \end{aligned} \tag{3.2}$$

It is clear that $\int_1^r (s-1)/s\, ds$ is nonnegative and equal to zero if and only if $r = 1$.

Now, all the individual subsystems in (2.2) are asymptotically stable (Proposition 2.3). The existence of a common Lyapunov function for the family of subsystems in (2.2) guarantees stability in the corresponding switched system for arbitrary switching signals; for example, see [3].

Theorem 3.1. *If the system $y = Ax$ is consistent, the switched nonlinear system corresponding to the family of systems in* (2.2) *is uniformly asymptotically stable.*

Proof. Consider the following function, which is based on the generalized Kullback–Leibler divergence between the state x and the equilibrium e:

$$\begin{aligned} V &= \mathrm{KL}(x, e) \\ &= \sum_{j=1}^{J} e_j \log \frac{e_j}{x_j} + x_j - e_j \\ &= \sum_{j=1}^{J} \int_{e_j}^{x_j} \frac{v - e_j}{v} dv, \end{aligned} \tag{3.3}$$

which is positive definite. Using (2.3) to get its derivative, after some manipulations, in the form

$$\begin{aligned}\dot{V}\big|_{(2.2)} &= \sum_{j=1}^{J} \frac{x_j - e_j}{x_j}\dot{x}_j \\ &= -\left\| A_m x - y_m \right\|_2^2 \\ &\leq 0\end{aligned} \tag{3.4}$$

with equality if $x = e$, for $m = 1, 2, \ldots, M$. Thus, all subsystem in the family in (2.2) share a common Lyapunov function given by (3.3), and, therefore, the corresponding switched system is uniformly asymptotically stable; the terminology uniform is employed here to indicate the uniformity with respect to the switching signals. □

4. Applications in Computed Tomography

This section presents a concise review from the field of computed tomography (CT), on the reason for which we have investigated the switched nonlinear system in (2.2) and its stability, as well as an illustrative example of a switched nonlinear system. Needless to say, the generality in (2.1) makes it possible to use the presented results on the switched nonlinear system in (2.2) in numerous other areas.

4.1. Reason of Proposing Switching

The basic problem in CT is to calculate pixel values $x \in \mathbb{R}_+^J$ satisfying (2.1), where $y \in \mathbb{R}_+^I \setminus \{0\}$ is the projection value, and $A \in \mathbb{R}_+^{I\times J} \setminus \{0\}$ is a normalized projection operator. To find solution x, we proposed a novel approach to reconstructing tomographic images based on the idea of continuous dynamical methods; the method consists of a continuous-time image reconstruction (CIR) system described by the nonlinear continuous system:

$$\frac{dx}{dt} = -\operatorname{diag}(x)A^{\top}(Ax - y), \tag{4.1}$$

where the notations are as above. The main benefit of using the CIR system is that it easily allows theoretical analysis of the convergence and the stability of a solution [22].

We have extended our CIR method by introducing subsets of projections as in block iterative methods [23] to what is called a block CIR system [21, 22, 24]. The switched system with a piecewise smooth vector field, which describes our block CIR system, is given by (2.2). A convenient way of thinking about this formulation is facilitated from Figures 1–3, in the case of $m = 1, 2$. According to the many simulations and numerical discussions we took part in, we noticed that our block CIR system could yield very good-quality image reconstructions; for example, see [21, 22, 24]. The goodness of our results from the block CIR approach tempts us to show that this approach works well theoretically, based on dynamical systems theory. We introduced an illustrative idea to search for a common Lyapunov function for all subsystems in our block CIR system (2.2) as a trial of showing its stability as a switched system; the idea was efficient in solving many practical problems [21]. Because of this, the present paper introduces a common Lyapunov function, given by (3.3) that can be applied to all problems in our block CIR system that is described by (2.2).

4.2. Example

Let us take a switched system having an explicit solution to illustrate the theoretical results. Consider matrix A and vector y as

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{4.2}$$

So, we study the switched nonlinear system in (2.2) with $I = J = M = 2$, which is defined by

$$\frac{dx_1}{dt} = x_1(y_1 - x_1), \tag{4.3}$$

$$\frac{dx_2}{dt} = 0, \tag{4.4}$$

$$\frac{dx_1}{dt} = x_1(y_2 - (x_1 + x_2)), \tag{4.5}$$

$$\frac{dx_2}{dt} = x_2(y_2 - (x_1 + x_2)). \tag{4.6}$$

The solution to the first subsystem, consisting of (4.3) and (4.4), takes the form

$$\begin{aligned} x_1 &= \frac{y_1}{1 + (y_1/x_1^0 - 1)e^{-y_1 t}}, \\ x_2 &= x_2^0, \end{aligned} \tag{4.7}$$

which is positive when we start from positive initial values $x_1(0) = x_1^0 > 0$ and $x_2(0) = x_2^0 > 0$. The solution to the second subsystem, consisting of (4.5) and (4.6), takes the form

$$\begin{aligned} x_1 &= \frac{y_2 x_1^{0*}}{\left(x_1^{0*} + x_2^{0*}\right) + \left(y_2 - \left(x_1^{0*} + x_2^{0*}\right)\right)e^{-y_2 t}}, \\ x_2 &= \frac{y_2 x_2^{0*}}{\left(x_1^{0*} + x_2^{0*}\right) + \left(y_2 - \left(x_1^{0*} + x_2^{0*}\right)\right)e^{-y_2 t}}, \end{aligned} \tag{4.8}$$

which is positive when we start from positive initial values $x_1(0) = x_1^{0*} > 0$ and $x_2(0) = x_2^{0*} > 0$. Thus in either subsystem if we start from a positive initial state, then the corresponding solution including a switching point to the other subsystem, will be positive. Hence, under

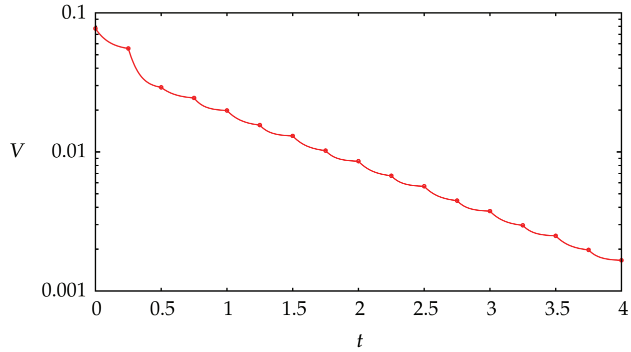

Figure 4: Common Lyapunov function given by (3.3) for Section 4.2 (Example).

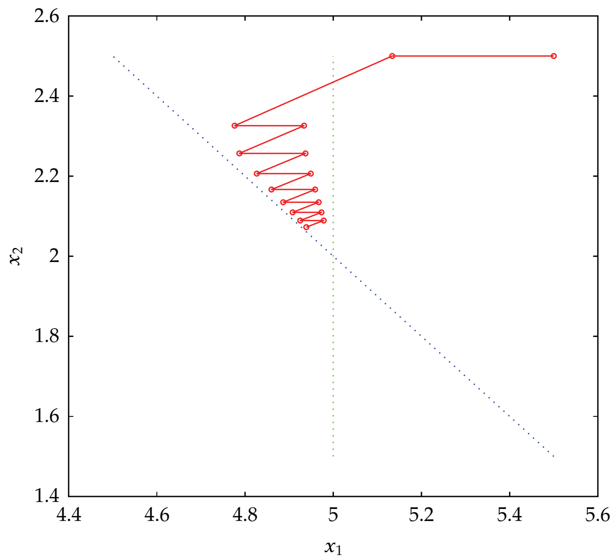

Figure 5: Trajectory in phase plane approaches the common equilibrium point $(5,2)^{\top}$ for Section 4.2 (Example).

arbitrary switching, if we choose a positive initial value for the first subsystem, an end point of the corresponding trajectory, at a given time of switching, which is the initial point for the second subsystem, will be positive and so the trajectory of the second subsystem will stay positive. We will switch to the first subsystem with a positive switching point at a given time. Repeating this procedure will generate a trajectory in $\mathbb{R}^2_{++}$ of the corresponding switched system, which coincides with what we have already proved in the general case in Proposition 2.1.

The two subsystems have the point $(y_1, y_2 - y_1)^{\top}$ as a common equilibrium point (see Proposition 2.2). This point is stable for each of the two subsystems by virtue of existing two Lyapunov functions given by (2.5), with $m = 1, 2$, which coincides with Proposition 2.3. The existence of the common Lyapunov function, for these two subsystems, given by (3.3), guarantees the stability of the corresponding switched system for arbitrary switching signals as mentioned in Theorem 3.1. Now, in case of $y = (5,7)^{\top}$ in (4.2), we introduce two figures, the first one, Figure 4, shows the value of the common Lyapunov function given by (3.3), while the second one, Figure 5, shows a trajectory in the phase plane, with initial state $(5.5, 2.5)^{\top}$, which approaches the common equilibrium point of the two subsystems $(5,2)^{\top}$.

5. Conclusion

In this work, to solve the linear system $y = Ax$, we investigated a certain class of nonlinear switched systems, which we were interested in. Several basic properties of this nonlinear switched system were established. Since the construction of Lyapunov functions has always been a challenging task and an important problem in dynamical systems and control theory, we suggested a common Lyapunov function for the stability of an equilibrium in this switched nonlinear system, based on the generalized Kullback–Leibler divergence between the state and the equilibrium. To ensure practical relevance, the stability of the given switched system was used to develop a new approach to reconstructing tomographic images. However, the presented results can be used in numerous other areas, because of the frequent appearance of linear systems in applied mathematics used in modeling various phenomena.

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