

Research Article

Numerical Integration and Synchronization for the 3-Dimensional Metriplectic Volterra System

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The main purpose of this paper is to study the metriplectic system associated to 3-dimensional Volterra model. For this system we investigate the stability problem and numerical integration via Kahan's integrator. Finally, the synchronization problem for two coupled metriplectic Volterra systems is discussed.

1. Introduction

To give a unification of the conservative and dissipative dynamics, Kaufman [1] has introduced the concept of *metriplectic system*.

Let (x_1, x_2, \dots, x_n) be a local coordinate system on \mathbf{R}^n . We consider

$$\dot{x}(t) = P(x(t)) \cdot \nabla H(x(t)) \quad (1.1)$$

be a Hamilton-Poisson system on \mathbf{R}^n with Hamiltonian $H \in C^\infty(\mathbf{R}^n)$, where $x(t) = (x_1(t), \dots, x_n(t))^T$ and $\nabla H(x) = (\partial H/\partial x_1, \dots, \partial H/\partial x_n)^T$.

We add to the Hamilton-Poisson system (1.1) a dissipation term of the form $G(x(t)) \cdot \nabla \tilde{C}(x(t))$, where $G(x)$ is a symmetric matrix which satisfies certain compatibility conditions, and $\tilde{C}(x(t)) = a \cdot C(x(t))$, where $a \in \mathbf{R}$ and $C \in C^\infty(\mathbf{R}^n)$ are a Casimir function (i.e., $P(x) \cdot \nabla C(x) = 0$). One obtains a family of metriplectic systems of the form

$$\dot{x}(t) = P(x(t)) \cdot \nabla H(x(t)) + G(x(t)) \cdot \nabla \tilde{C}(x(t)). \quad (1.2)$$

This family of metriplectic systems have the same Hamiltonian H and the same Casimir function C . For each $a \in \mathbf{R}$, the metriplectic systems (1.2) can be viewed as a perturbation of Hamilton-Poisson system (1.1).

The differential systems of the form (1.2) and their applications have been studied in connection with several dynamical systems derived from mathematical physics; see for instance, [1–4].

Another way for giving rise to a dynamical system of the form of (1.2) is based on the definition of a metriplectic structure on \mathbf{R}^n . These systems can be expressed in terms of Leibniz bracket, see [5–8].

The paper is structured as follows. In Section 2, the metriplectic system associated to 3-dimensional Volterra model (2.8) is constructed. For this dynamical system, the stability of equilibrium states is investigated. In Section 3, we discuss the numerical integration for the system (2.8).

Synchronization problem for dynamical systems has received a great deal of interest due to their application in different fields of science; see [9–12]. For this reason, Section 4 is dedicated to synchronization problem for two coupled metriplectic Volterra systems of the form of (2.8).

2. The Metriplectic System Associated to 3-Dimensional Volterra Model

Let (\mathbf{R}^n, P, H) be a Hamilton-Poisson system given by (1.1). For this system we determine the symmetric matrix $G = (G^{ij})$, where

$$G^{ii}(x) = - \sum_{k=1, k \neq i}^n \left(\frac{\partial H}{\partial x_k} \right)^2, \quad G^{ij}(x) = \frac{\partial H}{\partial x_i} \cdot \frac{\partial H}{\partial x_j}, \quad \text{for } i \neq j. \quad (2.1)$$

If $C \in C^\infty(\mathbf{R}^n)$ is a Casimir function for the configuration (\mathbf{R}^n, P, H) , then we take $\tilde{C} = a \cdot C$, where $a \in \mathbf{R}$ is a parameter.

For P, H, \tilde{C} , and the matrix G determined by relations (2.1), we write the differential system (1.2) in the following tensorial form:

$$\dot{x}_i = P^{ij} \frac{\partial H}{\partial x_j} + G^{ij} \frac{\partial \tilde{C}}{\partial x_j}, \quad i, j = \overline{1, n}. \quad (2.2)$$

System (2.2) is a metriplectic system in \mathbf{R}^n (see [2, 7]), that is the following conditions are satisfied:

$$(i) \quad G(x) \cdot \nabla H(x) = 0; \quad (ii) \quad \left(\nabla \tilde{C}(x) \right)^T \cdot G(x) \cdot \nabla \tilde{C}(x) \leq 0. \quad (2.3)$$

System (2.2) is called the *metriplectic system associated to Hamilton-Poisson system* (1.1) and is denoted by $(\mathbf{R}^n, P, H, G, \tilde{C})$.

Let us construct a metriplectic system of the form of (2.2), starting a Hamilton-Poisson realization of the 3-dimensional Volterra model.

The phase space of the 3-dimensional Volterra model consists of variables x_i , $1 \leq i \leq 3$; see [13, 14]. This is described by the equations

$$\dot{x}_1 = x_1x_2, \quad \dot{x}_2 = -x_1x_2 + x_2x_3, \quad \dot{x}_3 = -x_2x_3. \quad (2.4)$$

It is well known that system (2.4) has a Hamilton-Poisson realization (\mathbf{R}^3, P_V, H_V) with the Casimir $C_V \in C^\infty(\mathbf{R}^3, \mathbf{R})$ (see [14]), where

$$P_V = \begin{pmatrix} 0 & x_1x_2 & 0 \\ -x_1x_2 & 0 & x_2x_3 \\ 0 & -x_2x_3 & 0 \end{pmatrix}, \quad (2.5)$$

$$H_V(x_1, x_2, x_3) = x_1 + x_2 + x_3, \quad C_V(x_1, x_2, x_3) = x_1x_3. \quad (2.6)$$

We apply now relations (2.1) to the function $H = H_V \in C^\infty(\mathbf{R}^3, \mathbf{R})$ given by (2.6). Then the symmetric matrix $G := G_V = (G_V^{ij})$ is

$$G_V = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}. \quad (2.7)$$

We take $H = H_V$ and $C = C_V$ given by (2.6), the skew-symmetric matrix $P = P_V$ given by (2.5) and the symmetric matrix G_V given by (2.7). For the function $\tilde{C}_V = aC_V$ with $a \in \mathbf{R}$, system (2.2) becomes

$$\begin{aligned} \dot{x}_1 &= x_1x_2 + a(x_1 - 2x_3), \\ \dot{x}_2 &= -x_1x_2 + x_2x_3 + a(x_1 + x_3), \\ \dot{x}_3 &= -x_2x_3 + a(-2x_1 + x_3). \end{aligned} \quad (2.8)$$

Proposition 2.1. *The dynamical system $(\mathbf{R}^3, P_V, H_V, G_V, \tilde{C}_V)$ given by (2.8) is a metriplectic system on \mathbf{R}^3 .*

Proof. We have $\partial C_V / \partial x_1 = x_3$, and $\partial C_V / \partial x_2 = 0$, $\partial C_V / \partial x_3 = x_1$. Then

$$P_V \cdot \nabla C_V = \begin{pmatrix} 0 & x_1x_2 & 0 \\ -x_1x_2 & 0 & x_2x_3 \\ 0 & -x_2x_3 & 0 \end{pmatrix} \begin{pmatrix} x_3 \\ 0 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad (2.9)$$

that is, C_V is a Casimir of Hamilton-Poisson system (\mathbf{R}^3, P_V, H_V) .

We prove that conditions (i) and (ii) from (2.3) are verified.

We have $\tilde{C}_V(x) = ax_1x_3$, $\partial H_V/\partial x_1 = \partial H_V/\partial x_2 = \partial H_V/\partial x_3 = 1$ and $\partial \tilde{C}_V/\partial x_1 = ax_3$, $\partial \tilde{C}_V/\partial x_2 = 0$, $\partial \tilde{C}_V/\partial x_3 = ax_1$. Then

$$G_V \cdot \nabla H_V = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\left(\nabla \tilde{C}(x)\right)^T \cdot G(x) \cdot \nabla \tilde{C}(x) = (ax_3, 0, ax_1) \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} ax_3 \\ 0 \\ ax_1 \end{pmatrix} = -2a^2(x_1^2 - x_1x_3 + x_3^2) \leq 0. \quad (2.10)$$

Hence (2.8) is a metriplectic system. \square

System (2.8) is called the *3-dimensional metriplectic Volterra system*. For $a = 0$, it is reduced to Volterra model (2.4).

System (2.8) can be written in the form $\dot{x}_i = f_i(x)$, $i = \overline{1,3}$, where

$$f_1(x) = x_1x_2 + a(x_1 - 2x_3), \quad f_2(x) = -x_1x_2 + x_2x_3 + a(x_1 + x_3), \quad (2.11)$$

$$f_3(x) = -x_2x_3 + a(-2x_1 + x_3).$$

Proposition 2.2. (i) *The function H_V given by (2.6) is a constant of the motion for the metriplectic Volterra system, that is, it is conserved along the solutions of the dynamics (2.8).*

(ii) *The function \tilde{C}_V decreases along the solutions of system (2.8).*

Proof. (i) We have $dH_V/dt = \dot{x}_1 + \dot{x}_2 + \dot{x}_3 = f_1(x) + f_2(x) + f_3(x) = 0$.

(ii) The derivative of \tilde{C}_V along the solutions of system (2.8) verifies the condition $d\tilde{C}_V/dt \leq 0$. Indeed, $d\tilde{C}_V/dt = a\dot{x}_1x_3 + ax_1\dot{x}_3 = a(x_3f_1(x) + x_1f_3(x)) = -2a^2(x_1^2 - x_1x_3 + x_3^2) \leq 0$. \square

Remark 2.3. If $a \neq 0$, then $\tilde{C}_V = aC_V$ is not a constant of motion for the metriplectic system (2.8).

Proposition 2.4. (i) *If $a \in \mathbf{R}^*$, then the equilibrium states of the dynamics of (2.8) are $e_2^M = (0, M, 0)$ for all $M \in \mathbf{R}$.*

(ii) *For $a = 0$, the equilibrium states of the dynamics of (2.8) are $e_1^{MN} = (M, 0, N)$, and $e_2^M = (0, M, 0)$, $e_3^M = (0, 0, M)$ for all $M, N \in \mathbf{R}$.*

Proof. The equilibria are the solutions of system $f_i(x) = 0$, $i = \overline{1,3}$. \square

Proposition 2.5. *The equilibrium states e_2^M , $M \in \mathbf{R}$ are unstable.*

Proof. Let A be the matrix of the linear part of the system (2.8), that is,

$$A = \begin{pmatrix} x_2 + a & x_1 & -2a \\ -x_2 + a & -x_1 + x_3 & x_2 + a \\ -2a & -x_3 & -x_2 + a \end{pmatrix}. \quad (2.12)$$

The characteristic polynomial of the matrix $A(e_2^M)$ is $p(\lambda) = \lambda(\lambda^2 - 2a\lambda - 3a^2 - M^2)$ with the roots $\lambda_1 = 0$, $\lambda_{2,3} = a \pm \sqrt{4a^2 + M^2}$. Then the assertion follows via Lyapunov's theorem [15]. \square

Remark 2.6. (i) The dynamics of (2.4) and (2.8) have not the same equilibria.

(ii) For $a = 0$, e_1^{MN} , e_2^M , e_3^M have the following behaviors (see [14]): e_1^{MN} is unstable if $M - N < 0$ and spectrally stable if $M - N > 0$; e_2^M is unstable, and e_3^M is unstable if $M \geq 0$ and spectrally stable if $M < 0$.

3. Numerical Integration of the Metriplectic Volterra System (2.8)

For (2.8), Kahan's integrator (see for details [16]) can be written in the following form:

$$\begin{aligned} x_1^{n+1} - x_1^n &= \frac{h}{2} (x_1^{n+1} x_2^n + x_2^{n+1} x_1^n) + ah (x_1^n + x_1^{n+1} - 2x_3^n - 2x_3^{n+1}), \\ x_2^{n+1} - x_2^n &= \frac{h}{2} (-x_1^{n+1} x_2^n - x_2^{n+1} x_1^n + x_3^{n+1} x_2^n + x_2^{n+1} x_3^n) + ah (x_1^n + x_1^{n+1} + x_3^n + x_3^{n+1}), \\ x_3^{n+1} - x_3^n &= -\frac{h}{2} (x_2^{n+1} x_3^n + x_3^{n+1} x_2^n) + ah (x_3^n + x_3^{n+1} - 2x_1^n - 2x_1^{n+1}). \end{aligned} \quad (3.1)$$

Remark 3.1. Taking $a = 0$ in relations (3.1) we obtain the numerical integration for Volterra model (2.4) via Kahan's integrator.

Proposition 3.2. *Kahan's integrator (3.1) preserves the constant of motion H_V of the dynamics of (2.8).*

Proof. Indeed, adding all equations (3.1) we obtain

$$x_1^{n+1} + x_2^{n+1} + x_3^{n+1} = x_1^n + x_2^n + x_3^n. \quad (3.2)$$

Hence $H_V(x_1^{n+1}, x_2^{n+1}, x_3^{n+1}) = H_V(x_1^n, x_2^n, x_3^n)$. \square

For the initial conditions $x_1(0) = 1$, $x_2(0) = 2$, and $x_3(0) = 1$, the solutions of Volterra model (2.4) (using Kahan's integrator (3.1) with $a = 0$), are represented in the system of coordinates $Ox_1x_2x_3$ in Figure 1.

For the same initial conditions, the solutions of the metriplectic Volterra system (2.8) for $a = 1$ (using Kahan's integrator (3.1) with $a = 1$), are represented in the system of coordinates $Ox_1x_2x_3$ in Figure 2.

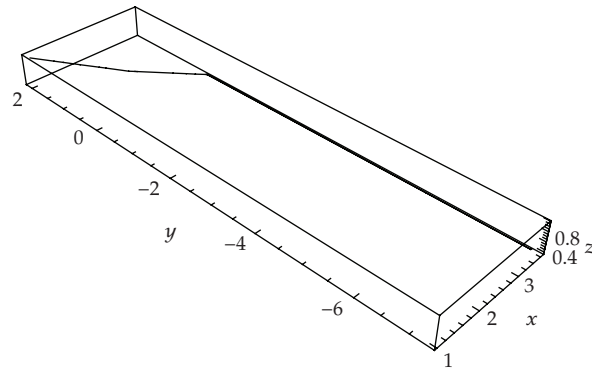


Figure 1: Kahan's integrator for Volterra model (2.4).

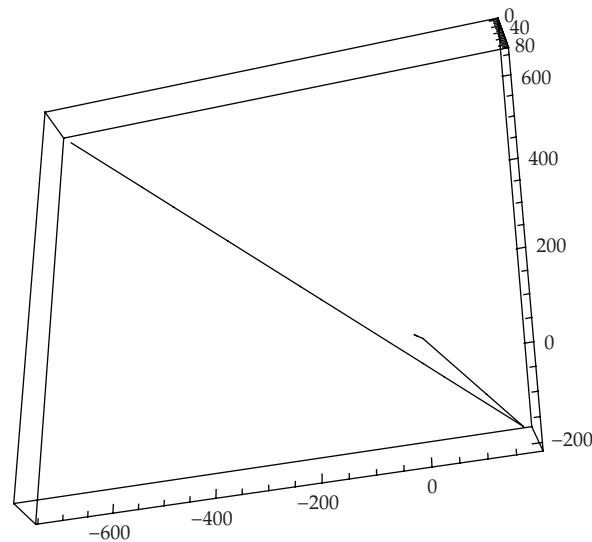


Figure 2: Kahan's integrator for the metriplectic Volterra system (2.8) with $a = 1$.

Remark 3.3. Using Runge-Kutta 4 steps integrator, we obtain almost the same result; see Figure 3.

4. The Synchronization of Two Metriplectic Volterra Systems

In this section we apply Pecora and Carroll method for constructing the drive-response configuration (see [12]).

Let us build the configuration with the drive system given by the metriplectic Volterra system (2.8), and a response system (this is obtained from (2.8) by replacing x_i with y_i and adding u_i for $i = \overline{1,3}$). Suppose that these systems are coupled. More precisely, the second system is driven by the first one, but the behavior of the first system is not affected by the second one.

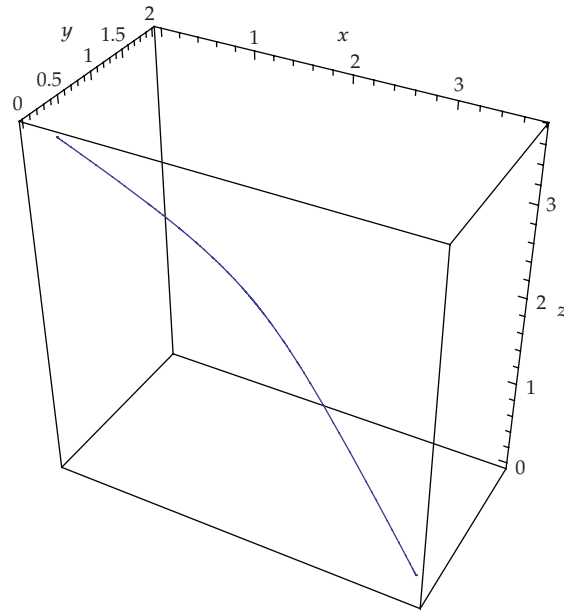


Figure 3: Runge-Kutta's integrator for Volterra model (2.4).

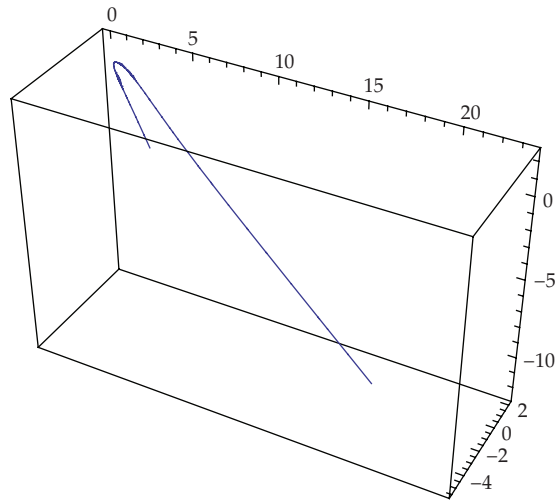


Figure 4: Runge-Kutta's integrator for the metriplectic Volterra system (2.8) with $a = 1$.

Therefore, the drive and response systems are given by

$$\begin{aligned}
 \dot{x}_1 &= x_1x_2 + a(x_1 - 2x_3), \\
 \dot{x}_2 &= -x_1x_2 + x_2x_3 + a(x_1 + x_3), \\
 \dot{x}_3 &= -x_2x_3 + a(-2x_1 + x_3),
 \end{aligned}
 \tag{4.1}$$

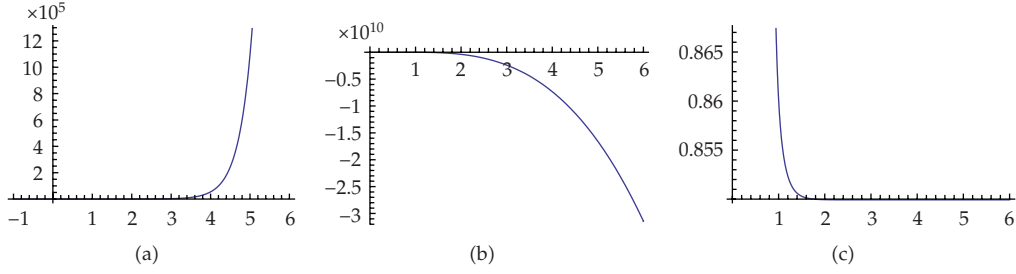


Figure 5: Synchronization of systems (4.1) and (4.2) for $a = 1$. The solutions $x_1(t), y_1(t)$ and the evolution of error $e_1(t)$.

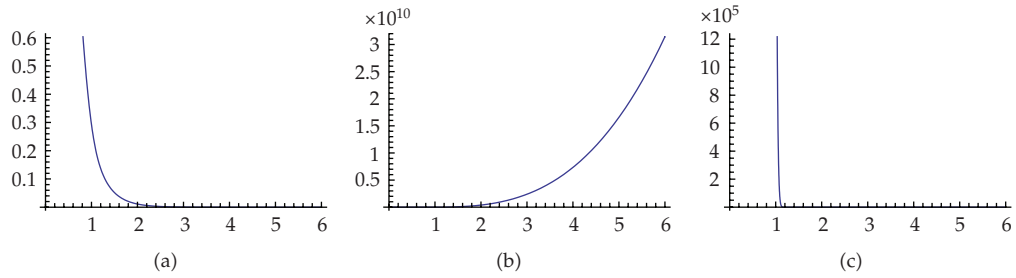


Figure 6: Synchronization of systems (4.1) and (4.2) for $a = 1$. The solutions $x_2(t), y_2(t)$ and the evolution of error $e_2(t)$.

respectively

$$\begin{aligned}
 \dot{y}_1 &= y_1 y_2 + a(y_1 - 2y_3) + u_1, \\
 \dot{y}_2 &= -y_1 y_2 + y_2 y_3 + a(y_1 + y_3) + u_2, \\
 \dot{y}_3 &= -y_2 y_3 + a(-2y_1 + y_3) + u_3,
 \end{aligned} \tag{4.2}$$

where $u_1(t), u_2(t)$, and $u_3(t)$ are three control functions.

We define the synchronization error system as the subtraction of the metriplectic Volterra model (4.1) from the controlled metriplectic Volterra model (4.2):

$$\begin{aligned}
 e_1(t) &= y_1(t) - x_1(t), \\
 e_2(t) &= y_2(t) - x_2(t), \\
 e_3(t) &= y_3(t) - x_3(t).
 \end{aligned} \tag{4.3}$$

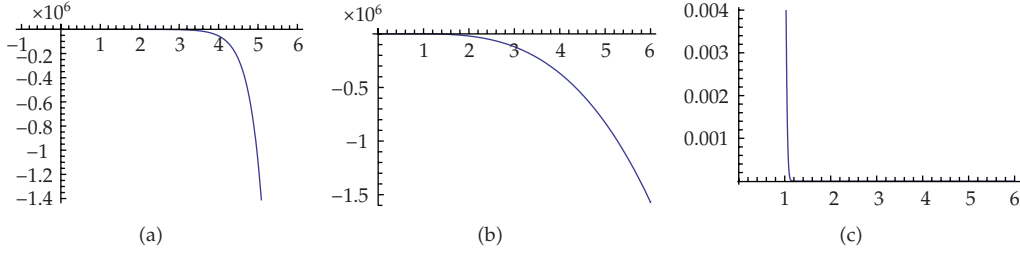


Figure 7: Synchronization of systems (4.1) and (4.2) for $a = 1$. The solutions $x_3(t), y_3(t)$ and the evolution of error $e_3(t)$.

By subtracting (4.2) from (4.1) and using notation (4.3) we can get

$$\begin{aligned}\dot{e}_1 &= e_1 e_2 + (x_2 + a)e_1 + x_1 e_2 - 2ae_3 + u_1, \\ \dot{e}_2 &= -e_1 e_2 + e_2 e_3 + (a - x_2)e_1 + (x_3 - x_1)e_2 + (a + x_2)e_3 + u_2, \\ \dot{e}_3 &= -e_2 e_3 - 2ae_1 - x_3 e_2 + (a - x_2)e_3 + u_3.\end{aligned}\quad (4.4)$$

We define the active control inputs $u_1(t), u_2(t)$, and $u_3(t)$ as follows:

$$\begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix} = K \cdot \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix} + \begin{pmatrix} -e_1(t)e_2(t) \\ e_1(t)e_2(t) - e_2(t)e_3(t) \\ e_2(t)e_3(t) \end{pmatrix}, \quad (4.5)$$

where

$$K = \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix} \quad (4.6)$$

and $k_{ij}, i, j = \overline{1,3}$ are real functions which depend on $x_1(t), x_2(t)$, and $x_3(t)$. Then the differential system of errors (4.7) is given by

$$\begin{aligned}\dot{e}_1 &= e_1 e_2 + (x_2 + a + k_{11})e_1 + (x_1 + k_{12})e_2 + (k_{13} - 2a)e_3, \\ \dot{e}_2 &= -e_1 e_2 + e_2 e_3 + (a - x_2 + k_{21})e_1 + (x_3 - x_1 + k_{22})e_2 + (a + x_2 + k_{23})e_3, \\ \dot{e}_3 &= -e_2 e_3 + (k_{31} - 2a)e_1 + (k_{23} - x_3)e_2 + (a - x_2 + k_{33})e_3.\end{aligned}\quad (4.7)$$

If we choose

$$K = \begin{pmatrix} -a - x_2 - x_2^2 & 0 & 0 \\ x_2 - a & x_1 - x_3 - x_1^2 & 0 \\ 2a & x_3 & x_2 - a - x_3^2 \end{pmatrix}, \quad (4.8)$$

then the active controls defined by (4.5) become

$$\begin{aligned} u_1 &= -e_1 e_2 - (a + x_2 + x_2^2) e_1, \\ u_2 &= e_1 e_2 - e_2 e_3 + (x_2 - a) e_1 + (x_1 - x_3 - x_1^2) e_2, \\ u_3 &= e_2 e_3 + 2a e_1 + x_3 e_2 + (-a + x_2 - x_3^2) e_3. \end{aligned} \quad (4.9)$$

Using (4.8), the system of errors (4.7) becomes

$$\begin{aligned} \dot{e}_1 &= -x_2^2 e_1 + x_1 e_2 - 2a e_3, \\ \dot{e}_2 &= -x_1^2 e_2 + (a + x_2) e_3, \\ \dot{e}_3 &= -x_3^2 e_3. \end{aligned} \quad (4.10)$$

Proposition 4.1. *The equilibrium state $(0, 0, 0)$ of the differential system (4.10) is asymptotically stable.*

Proof. An easy computation shows that the all conditions of Lyapunov-Malkin theorem [17] are satisfied, and so we have that the equilibrium state $(0, 0, 0)$ is asymptotically stable. \square

Numerical simulations are carried out using the software MATHEMA-TICA 6. We consider the case $a = 1$. The fourth-order Runge-Kutta integrator is used to solve systems (4.1), (4.2), and (4.10) with the control functions $u_1(t), u_2(t), u_3(t)$ given by (4.9).

The initial values of the drive system (4.1) and response system (4.2) are $x_1(0) = 1, x_2(0) = 2, x_3(0) = 1$ and $y_1(0) = 1, y_2(0) = 2, y_3(0) = 3$. These choices result in initial errors of $e_1(0) = 0.001, e_2(0) = 0.01, e_3(0) = 0.002$.

The dynamics of the metriplectic Volterra system (4.1) to be synchronized with the dynamic of (4.2) accompanied with the control functions given by (4.9) and the dynamics of synchronization errors given by (4.10) are shown in Figures 5, 6, and 7.

According to numerical simulations, by a good choice of parameters the synchronization error states $e_1(t), e_2(t), e_3(t)$ converge to zero, and hence the synchronization between two coupled metriplectic Volterra systems is achieved.

Remark 4.2. Taking $a = 0$ in (4.1), (4.2), (4.8), and (4.9), we obtain the synchronization between two coupled Volterra models of the form of (2.4).

5. Conclusion

It is well known that many nonlinear differential systems like the Euler equations of fluid dynamics, the soliton equations can be written in the Hamiltonian form. An interesting example of nonlinear lattice equations is Volterra lattice (see [18]) which is a model for vibrations of the particles on lattices. Also the behavior of viscoelastic materials is an example where the dynamics is governed by Volterra equations. The metriplectic systems will be successfully used in mathematical physics, fluid mechanics, and information security; see for instance [4, 5, 10, 12].

In this paper we have build a metriplectic system on \mathbf{R}^3 associated to Volterra model. For the metriplectic Volterra system (2.8), we have presented some relevant geometrical and dynamics properties and the numerical integration. Finally, using the Pecora and Carroll method, the synchronization problem for two coupled metriplectic Volterra systems of the form of (2.8) is discussed. This technique is realized since a suitable control has been chosen to achieve synchronization.

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