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Research Article

Using Lie Symmetry Analysis to Solve a Problem That Models Mass Transfer from a Horizontal Flat Plate

W. Sinkala and M. Chaisi

Department of Applied Mathematics, Walter Sisulu University, Private Bag X1, Mthatha 5117, South Africa

Correspondence should be addressed to W. Sinkala, wsinkala@wsu.ac.za

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We use Lie symmetry analysis to solve a boundary value problem that arises in chemical engineering, namely, mass transfer during the contact of a solid slab with an overhead flowing fluid. This problem was earlier tackled using Adomian decomposition method (Fatoorehchi and Abolghasemi 2011), leading to the Adomian series form of solution. It turns out that the application of Lie group analysis yields an elegant form of the solution. After introducing the governing mathematical model and some preliminaries of Lie symmetry analysis, we compute the Lie point symmetries admitted by the governing equation and use these to construct the desired solution as an invariant solution.

1. Introduction

A simplified model for mass transfer phenomenon from a horizontal flat plate fixed along a laminar fluid flow is considered [1]. Imagine a solid slab placed horizontally on the x-axis as shown in Figure 1. The free stream velocity is denoted by u_∞ (m/s), C_{A0} is the initial concentration of molecules of species A (mol/m^3), C_{Ai} is the concentration of molecules of species A at the plate interface (mol/m^3), δ is the momentum (or hydrodynamic) boundary layer thickness (m), and δ_c is the concentration boundary layer thickness (m). Molecules of species A from the solid slab diffuse along the y-axis only to be swept downstream by the fluid flow in the hydrodynamic boundary layer. Subject to a number of assumptions (given in [1]), mass balance for species A over an infinitesimal element based on the Cartesian coordinates leads to the PDE

$$\sqrt{x}C_{yy} = \alpha y C_x, \tag{1.1}$$

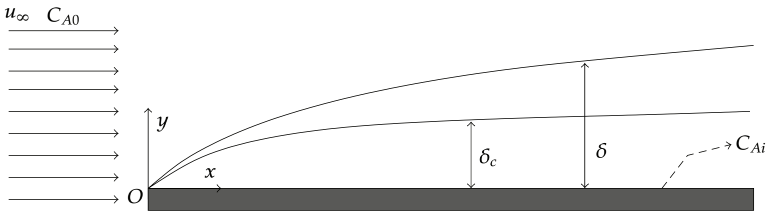

Figure 1: Molecules of species A from the solid slab diffuse along the y-axis only to be swept downstream by fluid flow in the hydrodynamic boundary layer.

for the concentration C of molecules of species A, with the associated boundary conditions

$$\begin{aligned} &\text{BC1}: C(x,y) = C_{A0} \quad \text{when } y = \delta(x) \\ &\text{BC2}: C(x,0) = C_{Ai}, \end{aligned} \tag{1.2}$$

where $\delta(x) = \sqrt{x}/\beta$, and α and β are constants. Precisely

$$\alpha = \frac{3u_\infty^{1.5}}{9.28Dv^{0.5}}, \qquad \beta = \frac{2D}{3u_\infty}\alpha, \tag{1.3}$$

where D is the diffusion coefficient (m^2/s) and v is the kinematic viscosity (m^2/s). The rest of the paper is organised as follows. In Section 2 we present preliminaries of Lie symmetry analysis. Section 3 is the thrust of the paper. In this section we determine Lie point symmetries admitted by the governing PDE and subsequently the basis generator of the one-dimensional Lie symmetry algebra admitted by the boundary value problem (BVP). We then construct the solution to the BVP as an invariant solution. We give concluding remarks in Section 4.

2. Lie Symmetry Analysis of Differential Equations

There are many good introductions to Lie symmetry analysis of differential equations [2–6]. We will however indulge in a little introduction of the subject for completeness and to whet the appetite of the readers who could be new to the subject. Consider a scalar PDE of order k,

$$F\left(x, u, u_{(1)}, u_{(2)}, \dots, u_{(k)}\right) = 0, \tag{2.1}$$

with n independent variables, $x = (x_1, x_2, \dots, x_n)$, and one dependent variable u, where $u_{(j)}$ represents all the jth order partial derivatives of u with respect to x. An element of the set represented by $u_{(j)}$ is denoted by $u_{i_1 i_2,\dots,i_j} = \partial u/\partial x_{i_1}\partial x_{i_2}, \dots, \partial x_{i_j}$, $i_j = 1, 2, \dots, n$ for $j = 1, 2, \dots, k$. We want to define invariance of (2.1) under a one-parameter Lie group of transformations in the parameter ε

$$\begin{aligned} \tilde{x}_i &= X_i(x, u; \varepsilon) = \varepsilon \xi_i(x, u) + O\left(\varepsilon^2\right), \\ \tilde{u} &= U(x, u; \varepsilon) = \varepsilon \eta(x, u) + O\left(\varepsilon^2\right), \end{aligned} \tag{2.2}$$

$i = 1, 2, \ldots, n$, acting on the (x, u)-space. The group (2.2) has as its infinitesimal generator

$$X = \xi_i(x, u)\partial_{x_i} + \eta(x, u)\partial_u, \tag{2.3}$$

where

$$\begin{aligned} \xi_i(x, u) &= \frac{\partial X_i}{\partial \varepsilon}(x, u; \varepsilon)\bigg|_{\varepsilon=0}, \qquad i = 1, 2, \ldots, n, \\ \eta(x, u) &= \frac{\partial U_i}{\partial \varepsilon}(x, u; \varepsilon)\bigg|_{\varepsilon=0}, \end{aligned} \tag{2.4}$$

and the kth extension given by

$$\begin{aligned} \widetilde{x}_i &= X_i(x, u; \varepsilon) = \varepsilon\xi_i(x, u) + O\left(\varepsilon^2\right), \\ \widetilde{u} &= U(x, u; \varepsilon) = \varepsilon\eta(x, u) + O\left(\varepsilon^2\right) \\ &\vdots \\ \widetilde{u}_{i_1 i_2 \cdots i_k} &= U_{i_1 i_2 \cdots i_k}\left(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(k)}; \varepsilon\right) \\ &= u_{i_1 i_2 \cdots i_k} + \varepsilon\eta^{(k)}_{i_1 i_2 \cdots i_k}\left(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(k)}\right) + O\left(\varepsilon^2\right), \end{aligned} \tag{2.5}$$

where $i = 1, 2, \ldots, n$ and $i_j = 1, 2, \ldots, n$ for $j = 1, 2, \ldots, k$. The kth extension of the group (2.2) is generated by the following (kth extended) infinitesimal generator:

$$X^{(k)} = \xi_i(x, u)\partial_{x_i} + \eta(x, u)\partial_u + \eta^{(1)}_i\left(x, u, u_{(1)}\right)\partial_{u_i} + \cdots + \eta^{(k)}_{i_1 i_2 \cdots i_k}\partial_{i_1 i_2 \cdots i_k}, \tag{2.6}$$

with the explicit formulas for the extended infinitesimals given recursively by

$$\begin{aligned} \eta^{(1)}_i &= D_i\eta - \left(D_i\xi_j\right)u_j, \quad i = 1, 2, \ldots, n, \\ \eta^{(k)}_{i_1 i_2 \cdots i_{k-1}} &= D_{i_k}\eta^{(k-1)}_{i_1 i_2 \cdots i_k} - \left(D_{i_k}\xi_j\right)u_{i_1 i_2 \ldots i_{k-1} j}, \end{aligned} \tag{2.7}$$

$i_j = 1, 2, \ldots, n$ for $j = 1, 2, \ldots, k$ with $k \geq 2$, where D_i is the total derivative operator defined by

$$D_i = \frac{D}{Dx_i} = \frac{\partial}{\partial x_i} + u_i\frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_j} + \cdots + u_{i i_1 i_2 \cdots i_n}\frac{\partial}{\partial u_{i_1 i_2 \cdots i_n}} + \cdots, \tag{2.8}$$

with summation over a repeated index.

We say that (2.1) admits (or is invariant under) the Lie group of point transformations (2.2) if (2.1) has the same form in the new variables $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n)$ and $\widetilde{u}$, that is,

$$F\left(\widetilde{x}, \widetilde{u}, \widetilde{u}_{(1)}, \widetilde{u}_{(2)}, \ldots, \widetilde{u}_{(k)}\right) = 0. \tag{2.9}$$

When this is the case we sometimes (loosely) simply say that X (the infinitesimal generator defined in (2.3)) is a symmetry of (2.1). Invariance of a differential equation under a given Lie group of transformations is neatly characterised by the infinitesimal criterion. Equation (2.1) is invariant under the Lie group of transformations (2.2) if and only if

$$X^{(k)}F\left(x,u,u_{(1)},u_{(2)},\ldots,u_{(k)}\right)=0 \quad \text{when } F\left(x,u,u_{(1)},u_{(2)},\ldots,u_{(k)}\right)=0. \tag{2.10}$$

The infinitesimal criterion (2.10) provides a key to the explicit determination of symmetry groups admitted by differential equations. Using a straightforward algorithm based on (2.10) one obtains infinitesimals of the Lie group of point transformations that leaves a given differential equation invariant. The tedious algebraic calculations involved in this process are today done easily, often automatically, thanks to powerful Computer Algebra Systems (CAS) like Maple and Mathematica, Maxima and Reduce, and the many specific packages for performing symmetry analysis of differential equations [7–13].

An admissible Lie group characterises symmetry properties of a differential equation and is used for, among other things, complete integration (in the case of ODEs) or construction of special exact solutions of the differential equation (see, e.g., [14–17]). For boundary (initial) value problems (BIVPs) there is a principle, the invariance principle, which states that if a BIVP is invariant under a given group, then one should seek the solution to the problem among the functions invariant under the admitted group. To state this principle concretely consider a BIVP

$$F\left(x,u,u_{(1)},u_{(2)},\ldots,u_{(k)}\right)= 0, \tag{2.11}$$

$$u|_s = h(x), \tag{2.12}$$

where s is a given manifold. Suppose that (2.11) admits m one-parameter symmetries

$$X_i=\xi_j(x,u)\partial_{x_j}+\eta(x,u)\partial_u, \quad i=1,\ldots,m. \tag{2.13}$$

We say that (2.11)-(2.12) is invariant under a symmetry

$$X=\sum_{i=1}^{m}\varepsilon_i X_i, \tag{2.14}$$

for some constants ε_i, provided that

(i) the manifold s is invariant under X;

(ii) the boundary (initial) condition $u|_s = h(x)$ is invariant under X restricted to s.

We now state the invariance principle [15, 18]: if the BIVP (2.11)-(2.12) admits one-parameter symmetries X_μ, one should seek the solution of the problem among the functions invariant under X_μ. Successful application of the invariance principle to solve BIVPs has been reported in a number of papers (see, e.g., [17, 19–21]).

3. Solution of the BVP (1.1)-(1.2)

Using LIE [8, 13] we obtain that (1.1) admits an infinite dimensional Lie symmetry algebra spanned by the following operators:

$$X_1 = \left(\frac{1}{\sqrt{x}}\right)\partial_x, \qquad X_2 = y\partial_y + 2x\partial_x,$$
$$X_3 = \partial_C, \qquad X_4 = yx^{3/2}\partial_y + x^{5/2}\partial_x - C\left(\frac{\alpha y^3}{4} + x^{3/2}\right)\partial_C, \tag{3.1}$$

$$X_\phi = \phi(x,y)\partial_C, \tag{3.2}$$

where $\phi(x,y)$ is an arbitrary solution of (1.1).

In the light of the invariance principle, the starting point in the "search" for the solution of (1.1)-(1.2) is the determination of the subalgebra of the Lie algebra spanned by the symmetries in (3.1) that leaves the boundary conditions (1.2) invariant. To do this we construct a special linear combination,

$$X = \sum_{i=1}^{4} \varepsilon_i X_i, \tag{3.3}$$

of the symmetries in (3.1), with the constants ε_i's suitably chosen so that the boundary conditions (1.2) are invariant under (3.3), that is, we require that

$$X(C - C_{A0}) = 0 \quad \text{when } y = \delta(x),$$
$$X(C - C_{Ai}) = 0 \quad \text{when } y = 0. \tag{3.4}$$

From (3.4) we obtain that

$$\varepsilon_3 - \varepsilon_4\left(1 + \frac{\alpha}{4\beta^3}\right)x^{3/2} = 0, \tag{3.5}$$

from which it follows that $\varepsilon_3 = \varepsilon_4 = 0$. Therefore (3.3) reduces to

$$\Gamma = \left(2x + \frac{\kappa}{\sqrt{x}}\right)\partial_x + y\partial_y, \tag{3.6}$$

the general form of the symmetry admitted by both the PDE (1.1) and the boundary conditions (1.2), where κ is an arbitrary constant. We can now assume the existence of an invariant solution $C = \phi(x,y)$ of the BVP (1.1)-(1.2), the form of which is determined by functions invariant under the group generated by (3.6). The characteristic equations of (3.6),

$$\frac{dx}{2x + \kappa/\sqrt{x}} = \frac{dy}{y} = \frac{dC}{0}, \tag{3.7}$$

yield the integrals

$$J_1 = \frac{y^3}{2x^{3/2} + \kappa}, \\ J_2 = C, \tag{3.8}$$

from the first and second equations, respectively. These integrals are the similarity variables of the BVP and the invariant solution is expressed in terms of them in the form

$$J_2 = \phi(J_1), \tag{3.9}$$

or

$$C(x,y) = \phi\left(\frac{y^3}{2x^{3/2} + \kappa}\right), \tag{3.10}$$

where ϕ is an arbitrary function. Upon substituting (3.10) into (1.1) we obtain the following second-order linear ODE:

$$3z\phi''(z) + (2 + \alpha z)\phi'(z) = 0. \tag{3.11}$$

The solution of (3.11) is

$$\phi(z) = K_1 + K_2\Gamma\left(\frac{1}{3}, \frac{\alpha z}{3}\right), \tag{3.12}$$

where K_1 and K_2 are arbitrary constants and $\Gamma(a,z)$ is the incomplete gamma function defined by

$$\Gamma(a,z) = \int_z^\infty t^{a-1}\mathrm{e}^{-t}\mathrm{d}t. \tag{3.13}$$

We now have that all invariant solutions of (1.1) arising from (3.6) are embedded in the family of solutions (3.10). The values of K_1 and K_2 in (3.12) are determined from the boundary conditions (1.2):

$$C(x,y)|_{y=\delta(x)} = C_{A0} \Longrightarrow K_1 + K_2\Gamma\left(\frac{1}{3}, \varphi(x)\right) = C_{A0}, \\ C(x,0) = C_{Ai} \Longrightarrow K_1 + K_2\Gamma\left(\frac{1}{3}\right) = C_{Ai}, \tag{3.14}$$

where

$$\varphi(x) = \frac{\alpha x^{3/2}}{3\beta^3(2x^{3/2} + \kappa)}, \qquad \Gamma(z) = \int_0^\infty t^{z-1}\mathrm{e}^{-t}\mathrm{d}t. \tag{3.15}$$

We now solve the equations (3.14) simultaneously for K_1 and K_2. We obtain that K_1 and K_2 are constants only if we set

$$\kappa = 0, \tag{3.16}$$

in which case

$$K_1 = \frac{C_{A0}\Gamma(1/3) - C_{Ai}\,\Gamma(1/3, \alpha/6\beta^3)}{\Gamma(1/3) - \Gamma(1/3, \alpha/6\beta^3)},$$
$$K_2 = \frac{C_{A0} - C_{Ai}}{\Gamma(1/3, \alpha/6\beta^3) - \Gamma(1/3)}. \tag{3.17}$$

Finally from (3.10), (3.12), and (3.17) we have that the BVP (1.1)-(1.2) is solved by

$$C(x,y) = \frac{C_{A0}\Gamma(1/3) + (C_{Ai} - C_{A0})\,\Gamma(1/3, \alpha y^3/6x^{3/2}) - C_{Ai}\,\Gamma(1/3, \alpha/6\,\beta^3)}{\Gamma(1/3) - \Gamma(1/3, \alpha/6\,\beta^3)}. \tag{3.18}$$

4. Concluding Remarks

When a PDE is richly endowed with Lie point symmetries (i.e., when the PDE admits the Lie symmetry algebras $sl(2,\mathbb{R})\bigoplus_s A_1$ or $sl(2,\mathbb{R})\bigoplus_s W$, where W is the Heisenberg-Weyl [19]) the symmetry analysis approach usually provides a neat solution (when it exists) to the associated BIVP. Algorithms for obtaining admitted Lie point symmetries and for using the symmetries to construct invariant solutions are well developed and quite routine. It is instructive to compare the solution obtained in this paper, (3.18), with the series-form solution obtained by Fatoorehchi and Abolghasemi [1] via Adomian decomposition:

$$\frac{C(x,y) - C_{Ai}}{C_{A0} - C_{Ai}} = \lambda\beta\Bigg(w_0 y x^{-0.5} - \frac{\alpha}{24} w_1 y^4 x^{-2} + \frac{\alpha^2}{504} w_2 y^7 x^{-3.5} - \frac{\alpha^3}{12960} w_3 y^{10} x^{-5}$$
$$+ \frac{\alpha^4}{404352} w_1 y^{13} x^{-6.5} - \frac{\alpha^5}{14929920} w_5 y^{16} x^{-8} + \cdots \Bigg), \tag{4.1}$$

where w_i's and λ are "suitable" constants. As reported in [1] one can also solve (1.1)-(1.2) by the combination of variables method, the starting point of which is the assumption that

$$C(x,y) = \varphi(y^n x^m) \tag{4.2}$$

for suitable numbers m and n and some function ψ. This approach also leads to the exact solution

$$\frac{C(x,y)-C_{Ai}}{C_{A0}-C_{Ai}} = \frac{\sqrt[3]{\alpha/3}}{\Gamma(4/3)}\left(yx^{-0.5} - \frac{\alpha y^4 x^{-2}}{3\times 4} + \frac{\alpha^2 y^7 x^{-3.5}}{9\times 2!\times 7} - \frac{\alpha^3 y^{10} x^{-5}}{27\times 3!\times 10} + \frac{\alpha^4 y^{13} x^{-6.5}}{81\times 4!\times 13} - \frac{\alpha^5 y^{16} x^{-8}}{243\times 5!\times 16} + \cdots\right). \quad (4.3)$$

We remark here that Fatoorehchi and Abolghasemi [1] only determine the constants w_i's and λ by comparing (4.1) with the exact solution (4.3). These constants are determined to be

$$\lambda = \frac{4.64/\sqrt[3]{9.28}}{\Gamma(4/3)\mathrm{Sc}^{1/3}}, \qquad w_i = 2^i, \quad (4.4)$$

where Sc is the Schmidt number.

Finally, it is noteworthy that in the combination of variables approach to the solution of (1.1)-(1.2) it turns out that $m = -1/2$ and $n = 1$ [1]. This makes (3.10) and (4.2) equivalent in the light of (3.16).

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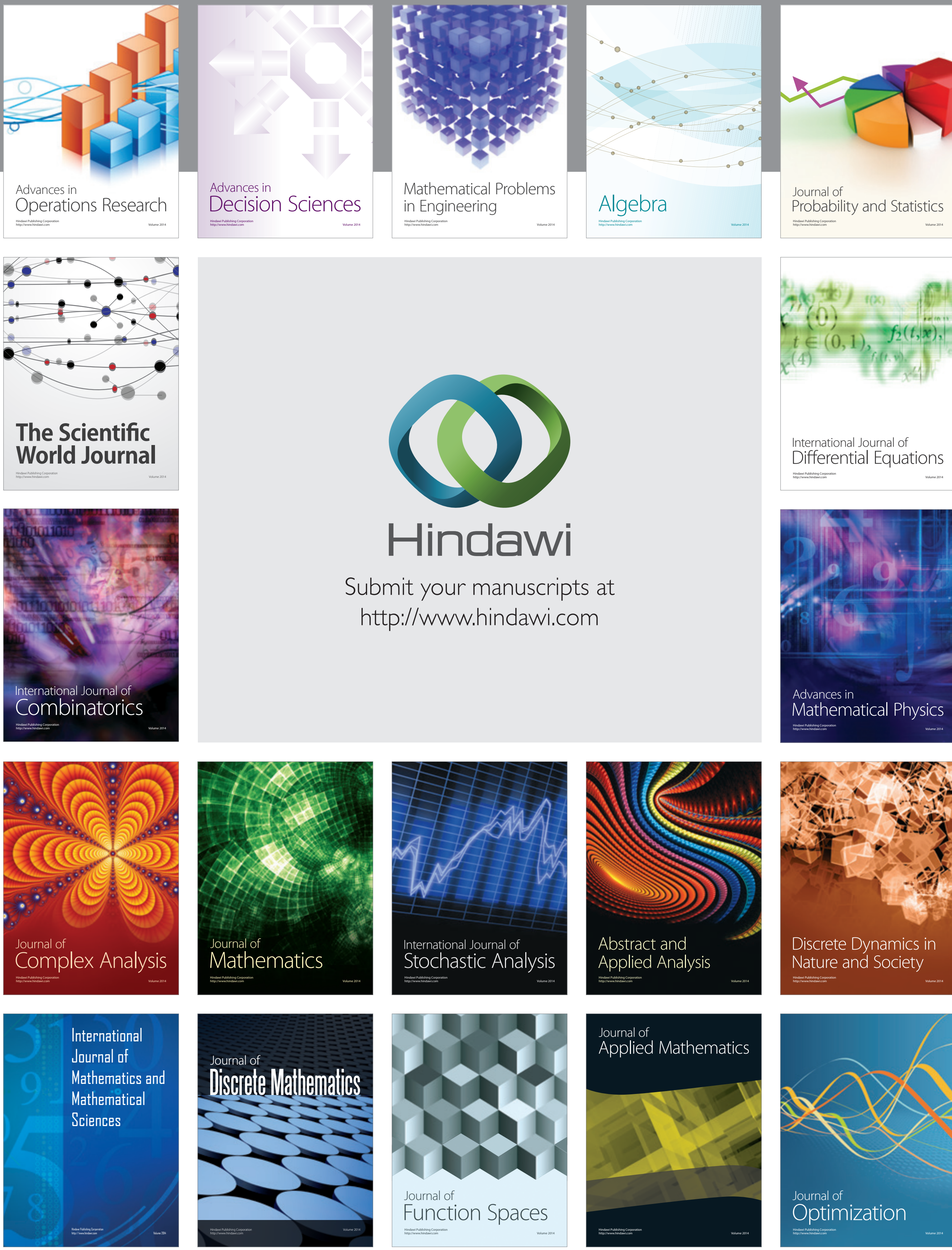
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