

## *Research Article*

# **Influence of Initial Stress and Gravity Field on Propagation of Rayleigh and Stoneley Waves in a Thermoelastic Orthotropic Granular Medium**

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The propagation of Rayleigh and Stoneley waves in a thermoelastic orthotropic granular half-space supporting a different layer under the influence of initial stress and gravity field is studied. The frequency equation of Rayleigh waves in the form of twelfth-order determinantal expression and the frequency equation of Stoneley waves in the form of eighth-order determinantal expression are obtained. The standard equation of dispersion is discussed to obtain Rayleigh and Stoneley waves that have complex roots; the real part gives the velocity of Rayleigh or Stoneley waves but the imaginary part gives the attenuation coefficient. Finally, the numerical results have been given and illustrated graphically, and their physical meaning has been explained.

## **1. Introduction**

The propagation of thermoelastic waves in a granular medium under initial stress and gravity field has applications in soil mechanics, earthquake science, geophysics, mining engineering, and so forth. The theoretical outline of the development of the subject from the mid-thirties was given by Paria [1]. The present paper, however, is based on the dynamics of granular media as propounded by Oshima [2, 3]. The medium under consideration is discontinuous such as one composed numerous large or small grains. Unlike a continuous body, each element or grain cannot only translate but also rotate about its centre of gravity. This motion is the characteristics of the medium and has an important effect upon the equation of motion to produce internal friction. It is assumed that the medium contains so many grains that they will never be separated from each other during the deformation, and that the grain

has perfect elasticity. The propagation of Rayleigh waves in granular medium was given by many authors such as Bhattacharyya [4], El-Naggar [5], Ahmed [6, 7], and others. Ahmed [8] discussed Stoneley waves in a nonhomogeneous granular medium under the influence of gravity.

The problem of Stoneley waves plays an important role in the earthquake science, optics, geophysics, and plasma physics. Many authors such as Abd-Alla and Ahmed [9, 10], El-Naggar et al. [11], Das et al. [12], and others investigated the effect of gravity of the propagation of surface waves (Stoneley waves and Rayleigh waves) in an elastic solid medium. Goda [13] illustrated the effect of inhomogeneity and anisotropy on Stoneley waves; Abd-Alla et al. [14] discussed Rayleigh waves in a magnetoelastic half-space of orthotropic material under influence of initial stress and gravity field. Sharma et al. [15] studied the propagation characteristics of Rayleigh waves in transversely isotropic piezothermoelastic half-space with rotation and thermal relaxation. Sharma et al. [16] investigated the problem of propagation characteristics of Rayleigh waves in transversely isotropic piezothermoelastic materials. Sharma and Walia [17] investigated Rayleigh waves in piezothermoelastic materials. Ting [18] illustrated the surface waves in a rotating anisotropic elastic half-space. Abd-Alla and Abo-Dahab [19] investigated Rayleigh waves in magneto-thermo-viscoelastic solid with thermal relaxation times. Recently, Ahmed and Abo-Dahab [20] pointed out Propagation of Love waves in an orthotropic granular layer under initial stress overlying a semi-infinite granular medium. Vinh and Seriani [21] illustrated explicit secular equations of Rayleigh waves in a nonhomogeneous orthotropic elastic medium under the influence of gravity. Vinh and Seriani [22] discussed the explicit secular equations of Stoneley waves in a nonhomogeneous orthotropic elastic medium under the influence of gravity. Vinh [23] discussed the explicit secular equations of Rayleigh waves in elastic media under the influence of gravity and initial stress. Lo [24] investigated the propagation and attenuation of Rayleigh waves in a semi-infinite unsaturated poroelastic medium. El-Naggar [25] investigated the dynamical problem of a generalized thermoelastic granular infinite cylinder under initial stress. Recently, Abd-Alla et al. [26] investigated Rayleigh waves in generalized magneto-thermo-viscoelastic granular medium under the influence of rotation, gravity field.

This paper is devoted to study the effect of gravity field and the initial stress on the propagation of Rayleigh and Stoneley waves in thermoelastic orthotropic granular half-space supporting a different layer under initial stress and gravity field. The frequency equations are obtained: the frequency equation of Rayleigh waves in the form of twelfth-order determinantal expression and the frequency equation of Stoneley waves in the form of eighth-order determinantal expression. The standard equation of dispersion is discussed to obtain the Rayleigh and Stoneley waves that have complex roots; the real part gives the velocity of Rayleigh or Stoneley waves but the imaginary part gives the attenuation coefficient. The results obtained are displayed graphically and their physical meaning has been explained.

## 2. Formulation of the Problem

Let us consider an initially stressed orthotropic granular layer of finite thickness  $H$  overlaying a semi-infinite orthotropic granular medium. The upper surface of the upper layer is assumed to be free and horizontal. We take a set of orthogonal Cartesian axes  $Ox_1x_2x_3$  such that the interface and the free surface of the granular layer resting on the granular half-space of different material are the planes  $x_3 = H$  and  $x_3 = 0$ , respectively, with the origin  $O$  being any point on the interface surface;  $x_3$ -axis is positive in the direction towards the exterior of the

half-space, and  $x_1$ -axis is positive along the direction of Rayleigh waves and Stoneley waves propagation. Let the both media be under initial compression stress  $P$  along  $x_1$ -axis, with the influence of gravity and at initial temperature  $T_0$ . It is assumed that both media exchange heat freely with their surroundings; an initial stress is produced by a slow process of creep, where the shear stresses tend to become small or vanish after a long interval of time.

In view of the two-dimensional nature of the problem, we assume that the state of initial stress is

$$\tau_{11} = \tau_{33} = \tau, \quad \tau_{13} = 0, \quad (2.1)$$

the equilibrium conditions of the initial stress field are given by [12]

$$\frac{\partial \tau}{\partial x_1} = 0, \quad \frac{\partial \tau}{\partial x_3} - \rho g = 0. \quad (2.2)$$

The state of deformation in the granular medium is described by the displacement vector  $\vec{U} = (u_1, 0, u_3)$  of the centre of gravity of a grain and the rotation vector  $\vec{\xi} = (\xi, \eta, \zeta)$  of the grain about its centre of gravity. There exist a stress tensor and a couple stress which are nonsymmetric, that is,

$$\tau_{ij} \neq \tau_{ji}, \quad M_{ij} \neq M_{ji} \quad (i, j = 1, 2, 3). \quad (2.3)$$

The stress tensor  $\tau_{ij}$  can be expressed as the sum of symmetric and antisymmetric tensors

$$\tau_{ij} = \sigma_{ij} + \sigma'_{ij}, \quad (2.4)$$

where

$$\sigma_{ij} = \frac{1}{2}(\tau_{ij} + \tau_{ji}), \quad \sigma'_{ij} = \frac{1}{2}(\tau_{ij} - \tau_{ji}). \quad (2.5)$$

The symmetric tensor  $\sigma_{ij} = \sigma_{ji}$  is related to the symmetric strain tensor

$$e_{ij} = e_{ji} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (2.6)$$

by the Hook's law.

The antisymmetric stress  $\sigma'_{ij}$  is given by

$$\sigma'_{23} = -F \frac{\partial \xi}{\partial t}, \quad \sigma'_{31} = -F \frac{\partial \eta}{\partial t}, \quad \sigma'_{12} = -F \frac{\partial \zeta}{\partial t}, \quad \sigma'_{11} = \sigma'_{22} = \sigma'_{33} = 0, \quad (2.7)$$

the couple stress  $M_{ij}$  is given by

$$\begin{aligned}
 M_{ij} &= Mv_{ij}, \\
 v_{11} &= \frac{\partial \xi}{\partial x_1}, & v_{22} &= 0, & v_{33} &= \frac{\partial \zeta}{\partial x_3}, & v_{23} &= 0, \\
 v_{31} &= \frac{\partial \xi}{\partial x_3}, & v_{12} &= \frac{\partial}{\partial x_1}(\eta + \omega_2), & v_{32} &= \frac{\partial}{\partial x_3}(\eta + \omega_2), \\
 v_{13} &= \frac{\partial \zeta}{\partial x_1}, & v_{21} &= 0,
 \end{aligned} \tag{2.8}$$

where,  $\omega_2 = \partial u_1 / \partial x_3 - \partial u_3 / \partial x_1$ .

The six equations of motion are [9, 11, 12]

$$\begin{aligned}
 \frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{31}}{\partial x_3} - \frac{P}{2} \frac{\partial \omega_2}{\partial x_3} - \rho g \frac{\partial u_3}{\partial x_1} &= \rho \frac{\partial^2 u_1}{\partial t^2}, \\
 \frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \tau_{32}}{\partial x_3} &= 0, \\
 \frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{33}}{\partial x_3} - \frac{P}{2} \frac{\partial \omega_2}{\partial x_1} + \rho g \frac{\partial u_1}{\partial x_1} &= \rho \frac{\partial^2 u_3}{\partial t^2}, \\
 \tau_{23} - \tau_{32} + \frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{31}}{\partial x_3} &= 0, \\
 \tau_{31} - \tau_{13} + \frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{32}}{\partial x_3} &= 0, \\
 \tau_{12} - \tau_{21} + \frac{\partial M_{13}}{\partial x_1} + \frac{\partial M_{33}}{\partial x_3} &= 0.
 \end{aligned} \tag{2.9}$$

The components of stress for orthotropic body, under the effect of an initial compression stress  $P$ , are given by [11]

$$\begin{aligned}
 \tau_{11} &= (c_{11} + P) \frac{\partial u_1}{\partial x_1} + (c_{13} + P) \frac{\partial u_3}{\partial x_3} - v_1 T, & \tau_{33} &= c_{13} \frac{\partial u_1}{\partial x_1} + c_{33} \frac{\partial u_3}{\partial x_3} - v_3 T, \\
 \tau_{31} &= c_{55} \left( \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) - F \frac{\partial \eta}{\partial t}, & \tau_{13} &= c_{55} \left( \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) + F \frac{\partial \eta}{\partial t},
 \end{aligned} \tag{2.10}$$

where

$$v_1 = (c_{11} + c_{12})\alpha_1 + c_{13}\alpha_2, \quad v_3 = 2c_{13}\alpha_1 + c_{33}\alpha_2. \tag{2.11}$$

Substituting (2.9) into (2.10), we obtain

$$\begin{aligned}
& (c_{11} + P) \frac{\partial^2 u_1}{\partial x_1^2} + \left( c_{55} + \frac{P}{2} \right) \frac{\partial^2 u_1}{\partial x_3^2} + \left( c_{13} + c_{55} + \frac{P}{2} \right) \frac{\partial^2 u_3}{\partial x_1 \partial x_3} - \frac{\partial}{\partial x_1} (v_1 T) - \rho g \frac{\partial u_3}{\partial x_1} \\
& - F \frac{\partial}{\partial t} \left( \frac{\partial \eta}{\partial x_3} \right) = \rho \frac{\partial^2 u_1}{\partial t^2}, \\
& F \frac{\partial}{\partial t} \left( \frac{\partial \xi}{\partial x_3} - \frac{\partial \zeta}{\partial x_1} \right) = 0, \\
& \left( c_{55} + c_{13} + \frac{P}{2} \right) \frac{\partial^2 u_1}{\partial x_1 \partial x_3} + \left( c_{55} - \frac{P}{2} \right) \frac{\partial^2 u_3}{\partial x_1^2} + c_{33} \frac{\partial^2 u_3}{\partial x_3^2} - \frac{\partial}{\partial x_3} (v_3 T) + \rho g \frac{\partial u_1}{\partial x_1} \\
& + F \frac{\partial}{\partial t} \left( \frac{\partial \eta}{\partial x_1} \right) = \rho \frac{\partial^2 u_3}{\partial t^2}, \\
& -2F \frac{\partial \xi}{\partial t} + \nabla^2 (M \xi) = 0, \quad -2F \frac{\partial \eta}{\partial t} + \nabla^2 \left[ M \left( \eta + \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \right) \right] = 0, \\
& -2F \frac{\partial \zeta}{\partial t} + \nabla^2 (M \zeta) = 0
\end{aligned} \tag{2.12}$$

The heat conduction equation is given by [11]

$$\nabla^2 T - \frac{1}{\chi} \frac{\partial T}{\partial t} - \varepsilon \nabla \cdot \left( \frac{\partial \vec{U}}{\partial t} \right) = 0, \tag{2.13}$$

where  $\chi = (\delta_1 + \delta_2)/2\rho s$ ,  $\varepsilon = T_0(v_1 + v_3)/(\delta_1 + \delta_2)$ .

### 3. Solution of the Problem

By Helmholtz's theorem [27], the displacement vector  $\vec{u}$  can be written in the form of the potentials  $\phi(x_1, x_3, t)$  and  $\psi(x_1, x_3, t)$  which are related to the displacement components  $u_1$  and  $u_3$  by the relations

$$u_1 = \frac{\partial \phi}{\partial x_1} - \frac{\partial \psi}{\partial x_3}, \quad u_3 = \frac{\partial \phi}{\partial x_3} + \frac{\partial \psi}{\partial x_1}. \tag{3.1}$$

Substituting (3.1) into (2.12) and (2.13), we get the following wave equations satisfied by  $\phi$ ,  $\psi$ ,  $\xi$ ,  $\eta$ , and  $\zeta$ :

$$(c_{11} + P) \frac{\partial^2 \phi}{\partial x_1^2} + (c_{13} + 2c_{55} + P) \frac{\partial^2 \phi}{\partial x_3^2} - \rho g \frac{\partial \psi}{\partial x_1} - v_1 T = \rho \frac{\partial^2 \phi}{\partial t^2}, \tag{3.2}$$

$$\frac{\partial}{\partial t} \left( \frac{\partial \zeta}{\partial x_1} - \frac{\partial \xi}{\partial x_3} \right) = 0, \quad (3.3)$$

$$\left( c_{55} - \frac{P}{2} \right) \frac{\partial^2 \psi}{\partial x_1^2} + \left( c_{33} - c_{31} - c_{55} - \frac{P}{2} \right) \frac{\partial^2 \psi}{\partial x_3^2} + \rho g \frac{\partial \phi}{\partial x_1} + F \frac{\partial \eta}{\partial t} = \rho \frac{\partial^2 \psi}{\partial t^2}, \quad (3.4)$$

$$\nabla^2 \xi - S \frac{\partial \xi}{\partial t} = 0, \quad (3.5)$$

$$\nabla^2 \eta - S \frac{\partial \eta}{\partial t} - \nabla^4 \psi = 0, \quad (3.6)$$

$$\nabla^2 \zeta - S \frac{\partial \zeta}{\partial t} = 0, \quad (3.7)$$

$$\nabla^2 T - \frac{1}{\chi} \frac{\partial T}{\partial t} - \varepsilon \nabla^2 \left( \frac{\partial \phi}{\partial t} \right) = 0, \quad (3.8)$$

$$S = \frac{2F}{M}. \quad (3.9)$$

Eliminating  $\eta$  from (3.4) and (3.6), we get

$$\left( \nabla^2 - S \frac{\partial}{\partial t} \right) \left[ \left( c_{55} - \frac{P}{2} \right) \frac{\partial^2 \psi}{\partial x_1^2} + \left( c_{33} - c_{31} - c_{55} - \frac{P}{2} \right) \frac{\partial^2 \psi}{\partial x_3^2} - \rho \frac{\partial^2 \psi}{\partial t^2} + \rho g \frac{\partial \phi}{\partial x_1} \right] + F \nabla^4 \left( \frac{\partial \psi}{\partial t} \right) = 0. \quad (3.10)$$

Also,  $T$  can be eliminated by using (3.8) and (3.9) as follows:

$$\left( \nabla^2 - \frac{1}{\chi} \frac{\partial}{\partial t} \right) \left[ (c_{11} + P) \frac{\partial^2 \varphi}{\partial x_1^2} + (c_{13} + 2c_{55} + P) \frac{\partial^2 \varphi}{\partial x_3^2} - \rho g \frac{\partial \psi}{\partial x_1} - \rho \frac{\partial^2 \varphi}{\partial t^2} \right] - \nu_1 \varepsilon \nabla^2 \left( \frac{\partial \phi}{\partial t} \right) = 0. \quad (3.11)$$

Assuming that

$$(\varphi, \psi) = \{\varphi_1(x_3), \psi_1(x_3)\} \exp \{i(Lx_1 - bt)\}, \quad (3.12)$$

$$(\xi, \eta, \zeta) = \{\xi_1(x_3), \eta_1(x_3), \zeta(x_3)\} \exp \{i(Lx_1 - bt)\}, \quad (3.13)$$

where  $b$  is real positive and  $L$  is in general complex.

Substituting (3.13) into (3.3), (3.5), and (3.7), we get

$$D\xi_1 - iL\xi_1 = 0, \quad (3.14)$$

$$D^2\xi_1 + h^2\xi_1 = 0, \quad (3.15)$$

$$D^2\zeta_1 + h^2\zeta_1 = 0, \quad (3.16)$$

where  $h^2 = ibS - L^2, D \equiv d/dx$ .

Solutions of (3.15) and (3.16) are

$$\xi_1 = A_1 e^{ihx_3} + A_2 e^{-ihx_3}, \quad \zeta_1 = B_1 e^{ihx_3} + B_2 e^{-ihx_3}, \quad (3.17)$$

respectively.

From (3.14) and (3.17), we obtain

$$h(A_1 e^{ihx_3} - A_2 e^{-ihx_3}) - L(B_1 e^{ihx_3} - B_2 e^{-ihx_3}) = 0. \quad (3.18)$$

Equating the coefficients of  $e^{ihx_3}$  and  $e^{-ihx_3}$  to zero in (3.18), we get

$$A_1 = \frac{L}{h} B_1, \quad A_2 = -\frac{L}{h} B_2. \quad (3.19)$$

Substituting (3.12) and (3.13) into (3.10) and (3.11), we get

$$\begin{aligned} & \left[ \left( c_{33} - c_{13} - c_{55} - \frac{P}{2} - ibF \right) D^4 \right. \\ & + \left\{ (ibS - L^2) \left( c_{33} - c_{13} - c_{55} - \frac{P}{2} \right) - L^2 \left( c_{55} - \frac{P}{2} \right) + \rho b^2 + 2ibFL^2 \right\} D^2 \\ & + (ibS - L^2) \left( \rho b^2 - L^2 \left( c_{55} - \frac{P}{2} \right) - ibFL^4 \right) \Big] \psi_1 \\ & + i\rho gL [D^2 - L^2 + ibS] \phi_1 = 0, \end{aligned} \quad (3.20)$$

$$\begin{aligned}
& \left\{ (c_{13} + 2c_{55} + P)D^4 \right. \\
& + \left[ \rho b^2 - L^2(c_{11} + c_{13} + 2c_{55} + 2P) + ibv_1\varepsilon + \frac{ib}{\chi}(c_{13} + 2c_{55} + P) \right] D^2 \\
& + \left[ \left( \frac{ib}{\chi} - L^2 \right) (\rho b^2 - L^2(c_{11} + P)) - ibv_1\varepsilon L^2 \right] \left. \right\} \phi_1 \\
& - i\rho gL \left[ D^2 - L^2 + \frac{ib}{\chi} \right] \psi_1 = 0.
\end{aligned} \tag{3.21}$$

The solution of (3.20) and (3.21) has the form

$$\begin{aligned}
\phi_1 &= A_j e^{-i\lambda_j x_3} + B_j e^{i\lambda_j x_3}, \\
\psi_1 &= E_j e^{-i\lambda_j x_3} + F_j e^{i\lambda_j x_3} \quad (j = 3, 4, 5, 6),
\end{aligned} \tag{3.22}$$

where the constants  $A_j, B_j$  are related with the constants  $E_j, F_j$ , respectively, by means of (3.20) or (3.21), and  $\lambda_j$  ( $j = 3, 4, 5, 6$ ) are taken to be imaginary.

Equating the coefficients of  $e^{-i\lambda_j x_3}$ ,  $e^{i\lambda_j x_3}$  ( $j = 3, 4, 5, 6$ ) to zero, we have using (3.21)

$$E_j = m_j A_j, \quad F_j = m_j B_j \quad (j = 3, 4, 5, 6), \tag{3.23}$$

where

$$\begin{aligned}
m_j &= \frac{1}{i\rho gL \left[ -\lambda_j^2 - L^2 + ib/\chi \right]} \\
& \times \left\{ (c_{13} + 2c_{55} + P)\lambda_j^4 \right. \\
& - \left[ \rho b^2 - L^2(c_{11} + c_{13} + 2c_{55} + 2P) + ibv_1\varepsilon + \frac{ib}{\chi}(c_{13} + 2c_{55} + P) \right] \lambda_j^2 \\
& \left. + \left( \frac{ib}{\chi} - L^2 \right) (\rho b^2 - L^2(c_{11} + P)) - ibv_1\varepsilon L^2 \right\},
\end{aligned} \tag{3.24}$$

where  $\lambda_3, \lambda_4, \lambda_5, \lambda_6$  are the imaginary roots of the equation

$$k_8 \lambda^8 + k_6 \lambda^6 + k_4 \lambda^4 + k_2 \lambda^2 + k_0 = 0, \tag{3.25}$$



$$\begin{aligned}
k_8 &= (c_{13} + 2c_{55} + P) \left( c_{33} - c_{13} - c_{55} - \frac{P}{2} - ibF \right), \\
k_6 &= - \left( \rho b^2 - L^2(c_{11} + c_{13} + 2c_{55} + 2P) + ibv_1\varepsilon + \frac{ib}{\chi}(c_{13} + 2c_{55} + P) \right) \\
&\quad \times \left( c_{33} - c_{13} - c_{55} - \frac{P}{2} - ibF \right) - (c_{13} + 2c_{55} + P) \\
&\quad \times \left[ (ibS - L^2) \left( c_{33} - c_{13} - c_{55} - \frac{P}{2} \right) - L^2 \left( c_{55} - \frac{P}{2} \right) + \rho b^2 + 2ibFL^2 \right], \\
k_4 &= \left( \frac{ib}{\chi} - L^2 \right) \left( \rho b^2 - L^2(c_{11} + P) - ibv_1\varepsilon L^2 \right) \left( c_{33} - c_{13} - c_{55} - \frac{P}{2} - ibF \right) \\
&\quad + (c_{13} + 2c_{55} + P) (ibS - L^2) \left[ \rho b^2 - L^2 \left( c_{55} - \frac{P}{2} \right) - ibFL^4 \right] \\
&\quad + \left[ (ibS - L^2) \left( c_{33} - c_{13} - c_{55} - \frac{P}{2} \right) + \rho b^2 - L^2 \left( c_{55} - \frac{P}{2} \right) + 2ibFL^2 \right] \\
&\quad \times \left[ \rho b^2 - L^2(c_{11} + c_{13} + 2c_{55} + 2P) + ibv_1\varepsilon + \frac{ib}{\chi}(c_{13} + 2c_{55} + P) \right] - \rho^2 g^2 L^2, \\
k_2 &= \left[ (ibS - L^2) \left( c_{33} - c_{13} - c_{55} - \frac{P}{2} \right) - L^2 \left( c_{55} - \frac{P}{2} \right) + \rho b^2 + 2ibFL^2 \right] \\
&\quad \times \left[ \left( \frac{ib}{\chi} - L^2 \right) \left( \rho b^2 - L^2(c_{11} + P) \right) - ibv_1\varepsilon L^2 \right] \\
&\quad + \left[ \rho b^2 - L^2(c_{11} + c_{13} + 2c_{55} + 2P) + ibv_1\varepsilon + \frac{ib}{\chi}(c_{13} + 2c_{55} + P) \right] \\
&\quad \times (ibS - L^2) \left[ \rho b^2 - L^2 \left( c_{55} - \frac{P}{2} \right) - ibFL^4 \right] - \rho^2 g^2 L^2 \left( 2L^2 - ibs - \frac{ib}{\chi} \right). \\
k_0 &= (ibS - L^2) \left[ \rho b^2 - L^2 \left( c_{55} - \frac{P}{2} \right) - ibFL^4 \right] \\
&\quad \times \left[ \left( \frac{ib}{\chi} - L^2 \right) \left( \rho b^2 - L^2(c_{11} + P) \right) - ibv_1\varepsilon L^2 \right] - \rho^2 g^2 L^2 \left( \frac{ib}{\chi} - L^2 \right) (ibS - L^2).
\end{aligned} \tag{3.26}$$

Using (3.2), (3.3), (3.12), (3.13), (3.22), and (3.23), we get

$$\begin{aligned}
\phi &= \left( A_j e^{-i\lambda_j x_3} + B_j e^{i\lambda_j x_3} \right) \exp(i(Lx_1 - bt)), \\
\psi &= m_j \left( A_j e^{-i\lambda_j x_3} + B_j e^{i\lambda_j x_3} \right) \exp(i(Lx_1 - bt)) \quad (j = 3, 4, 5, 6)
\end{aligned} \tag{3.27}$$

From (3.2) and (3.27), the temperature  $T$  has the form

$$T = n_j \left( A_j e^{-i\lambda_j x_3} + B_j e^{i\lambda_j x_3} \right) \exp(i(Lx_1 - bt)) \quad (j = 3, 4, 5, 6), \quad (3.28)$$

where

$$n_j = -\frac{1}{v_1} \left[ (c_{13} + 2c_{55} + P)\lambda_j^2 + L^2(c_{11} + P) + iL\rho g m_j - \rho b^2 \right] \quad (3.29)$$

also, from (3.6) and (3.27),  $\eta$  has the form

$$\eta = \Omega_j m_j \left( A_j e^{-i\lambda_j x_3} + B_j e^{i\lambda_j x_3} \right) \exp(i(Lx_1 - bt)) \quad (j = 3, 4, 5, 6), \quad (3.30)$$

where

$$\Omega_j = -\frac{(\lambda_j^2 + L^2)^2}{\lambda_j^2 + L^2 - ibS} \quad (j = 3, 4, 5, 6). \quad (3.31)$$

We use the symbols with a bar for the quantities in the lower medium (except  $x_3, L, b, P, g$ ), and by assuming that the solution of (3.20) and (3.21) satisfies the condition that the corresponding stresses vanish as  $x_3 \rightarrow -\infty$ , we obtain

$$\begin{aligned} \bar{\xi}_1 &= -\frac{L}{h} \bar{B}_2 e^{-i\bar{\lambda} x_3}, & \bar{\zeta}_1 &= \bar{B}_2 e^{-i\bar{\lambda} x_3}, & \bar{\eta}_1 &= \bar{\Omega} \bar{m}_j \bar{A}_j e^{-i\bar{\lambda}_j x_3}, \\ \bar{\phi}_1 &= \bar{A}_j e^{-i\bar{\lambda}_j x_3}, & \bar{\psi}_1 &= \bar{m}_j \bar{A}_j e^{-i\bar{\lambda}_j x_3}, \\ \bar{T} &= \bar{n}_j \bar{A}_j \exp\left(i(Lx_1 - bt - \bar{\lambda}_j x_3)\right) \quad (j = 3, 4, 5, 6). \end{aligned} \quad (3.32)$$

#### 4. Boundary Conditions and Frequency Equation

The boundary conditions on the interface  $x_3 = 0$  are

$$\begin{aligned} \text{(i)} \quad u_1 &= \bar{u}_1, & \text{(ii)} \quad u_3 &= \bar{u}_3, & \text{(iii)} \quad \xi &= \bar{\xi}, \\ \text{(iv)} \quad \eta &= \bar{\eta}, & \text{(v)} \quad \zeta &= \bar{\zeta}, & \text{(vi)} \quad M_{33} &= \bar{M}_{33}, \\ \text{(vii)} \quad M_{31} &= \bar{M}_{31}, & \text{(viii)} \quad M_{32} &= \bar{M}_{32}, & \text{(ix)} \quad \tau_{33} &= \bar{\tau}_{33}, \\ \text{(x)} \quad \tau_{31} &= \bar{\tau}_{31}, & \text{(xi)} \quad \tau_{32} &= \bar{\tau}_{32}, & \text{(xii)} \quad T &= \bar{T}, \\ \text{(xiii)} \quad \frac{\partial T}{\partial x_3} + \theta T &= \frac{\partial \bar{T}}{\partial x_3} + \bar{\theta} \bar{T}. \end{aligned} \quad (4.1)$$

The boundary conditions on the free surface  $x_3 = H$  are

$$\begin{aligned}
 & \text{(xiv)} \quad M_{33} = 0, & \text{(xv)} \quad M_{31} = 0, & \text{(xvi)} \quad M_{32} = 0, \\
 & \text{(xvii)} \quad \tau_{33} = 0, & \text{(xviii)} \quad \tau_{31} = 0, & \text{(xix)} \quad \tau_{32} = 0, \\
 & & \text{(xx)} \quad \frac{\partial T}{\partial x_3} + \theta T = 0,
 \end{aligned} \tag{4.2}$$

where

$$\begin{aligned}
 M_{33} &= M \frac{\partial \zeta}{\partial x_3}, & M_{32} &= M \frac{\partial}{\partial x_3} (\eta - \nabla^2 \psi), & M_{31} &= M \frac{\partial \xi}{\partial x_3}, \\
 \tau_{33} &= c_{13} \frac{\partial^2 \varphi}{\partial x_1^2} + c_{33} \frac{\partial^2 \varphi}{\partial x_3^2} + (c_{33} - c_{13}) \frac{\partial^2 \varphi}{\partial x_1 \partial x_3} - v_3 T, & \tau_{32} &= -F \frac{\partial \zeta}{\partial t}, \\
 \tau_{31} &= c_{55} \left( \frac{\partial^2 \varphi}{\partial x_1^2} - \frac{\partial^2 \varphi}{\partial x_3^2} + 2 \frac{\partial^2 \varphi}{\partial x_1 \partial x_3} \right) - F \frac{\partial \eta}{\partial t},
 \end{aligned} \tag{4.3}$$

$\theta$  is the ratio of the coefficients of heat transfer to the thermal conductivity.

From the boundary conditions (iii), (v), (vi), and (vii), we get

$$\begin{aligned}
 B_1 e^{ihH} - B_2 e^{-ihH} &= -\overline{B_2} e^{-i\bar{h}H}, \\
 B_1 e^{ihH} + B_2 e^{-ihH} &= -\overline{B_2} e^{i\bar{h}H}, \\
 M(B_1 e^{ihH} - B_2 e^{-ihH}) &= -\overline{MB_2} e^{-i\bar{h}H}, \\
 M(B_1 e^{ihH} + B_2 e^{-ihH}) &= -\overline{MB_2} e^{i\bar{h}H},
 \end{aligned} \tag{4.4}$$

hence

$$B_1 = B_2 = \overline{B_2} = 0, \quad \xi = \zeta = \bar{\xi} = \bar{\zeta} = 0. \tag{4.5}$$

The other significant boundary conditions are responsible for the following relations:

$$\begin{aligned}
 \text{(xvi)} \quad & (\lambda_j^2 + \Omega_j + L^2) (e^{-i\lambda_j H} A_j - e^{i\lambda_j H} B_j) = 0, \\
 \text{(xvii)} \quad & (c_{13} L(L + m_j \lambda_j) + c_{33} \lambda_j (\lambda_j - L m_j) + v_3 n_j) e^{-i\lambda_j H} A_j \\
 & + (c_{13} L(L - m_j \lambda_j) + c_{33} \lambda_j (\lambda_j + L m_j) + v_3 n_j) e^{i\lambda_j H} B_j = 0,
 \end{aligned}$$

$$\begin{aligned}
\text{(xviii)} \quad & \left[ c_{55}(\lambda_j^2 - L^2)m_j + 2L\lambda_j + ibF\Omega_j m_j \right] e^{-i\lambda_j H} A_j \\
& + \left[ c_{55}(\lambda_j^2 - L^2)m_j - 2L\lambda_j + ibF\Omega_j m_j \right] e^{i\lambda_j H} B_j = 0, \\
\text{(i)} \quad & (L + m_j \lambda_j) A_j + (L - m_j \lambda_j) B_j = (L + \overline{m_j \lambda_j}) \overline{A_j}, \\
\text{(ii)} \quad & (L m_j - \lambda_j) A_j + (L m_j + \lambda_j) B_j = (L \overline{m_j} - \overline{\lambda_j}) \overline{A_j}, \\
\text{(iv)} \quad & \Omega_j m_j (A_j + B_j) = \overline{\Omega_j m_j} \overline{A_j}, \\
\text{(viii)} \quad & M m_j \lambda_j (\lambda_j^2 + \Omega_j + L^2) (A_j - B_j) = \overline{M m_j \lambda_j} (\overline{\lambda_j^2 + \Omega_j + L^2}), \\
\text{(ix)} \quad & [c_{13}L(L + m_j \lambda_j) + c_{33}\lambda_j(\lambda_j - L m_j) + v_3 n_j] A_j \\
& + [c_{13}L(L - m_j \lambda_j) + c_{33}\lambda_j(\lambda_j + L m_j) + v_3 n_j] B_j \\
& = [c_{13}L(L + \overline{m_j \lambda_j}) + c_{33}\overline{\lambda_j}(\overline{\lambda_j} - L \overline{m_j}) + \overline{v_3 n_j}] \overline{A_j}, \\
\text{(x)} \quad & \left\{ c_{55} \left[ m_j (L^2 + \lambda_j^2) - 2L\lambda_j \right] - ibF m_j \Omega_j \right\} A_j \\
& + \left\{ c_{55} \left[ m_j (L^2 + \lambda_j^2) + 2L\lambda_j \right] - ibF m_j \Omega_j \right\} B_j \\
& = \left\{ \overline{c_{55}} \left[ \overline{m_j} (L^2 + \overline{\lambda_j^2}) - 2L\overline{\lambda_j} \right] - ib\overline{F} \overline{m_j} \overline{\Omega_j} \right\} \overline{A_j}, \\
\text{(xii)} \quad & n_j A_j + n_j B_j = \overline{n_j} \overline{A_j}, \\
\text{(xiii)} \quad & n_j (\theta - i\lambda_j) A_j + n_j (\theta + i\lambda_j) B_j = \overline{n_j} (\overline{\theta} - i\overline{\lambda_j}) \overline{A_j}, \\
\text{(xx)} \quad & n_j (\theta - i\lambda_j) e^{-i\lambda_j H} A_j + n_j (\theta + i\lambda_j) e^{i\lambda_j H} B_j = 0, \quad (j = 3, 4, 5, 6).
\end{aligned} \tag{4.6}$$

Elimination of  $A_j, B_j, \overline{A_j}$  ( $j = 3, 4, 5, 6$ ) gives the wave velocity equation in the form

$$\det \cdot (d_{rc}) = 0 \quad (r, c = 1, 2, \dots, 12), \tag{4.7}$$

where the nonvanishing entries of the twelfth-order determinant of  $d_{rc}$  are given by

$$\begin{aligned}
d_{11} &= (\lambda_3^2 + \Omega_3 + L^2) e^{-i\lambda_3 H}, & d_{12} &= (\lambda_4^2 + \Omega_4 + L^2) e^{-i\lambda_4 H}, \\
d_{13} &= (\lambda_5^2 + \Omega_5 + L^2) e^{-i\lambda_5 H}, & d_{14} &= (\lambda_6^2 + \Omega_6 + L^2) e^{-i\lambda_6 H},
\end{aligned}$$

$$\begin{aligned}
d_{15} &= -(\lambda_3^2 + \Omega_3 + L^2)e^{i\lambda_3 H}, & d_{16} &= -(\lambda_4^2 + \Omega_4 + L^2)e^{i\lambda_4 H}, \\
d_{17} &= -(\lambda_5^2 + \Omega_5 + L^2)e^{i\lambda_5 H}, & d_{18} &= -(\lambda_6^2 + \Omega_6 + L^2)e^{i\lambda_6 H}, \\
d_{21} &= (c_{13}L(L + m_3\lambda_3) + c_{33}\lambda_3(\lambda_3 - Lm_3) + v_3n_3)e^{-i\lambda_3 H}, \\
d_{22} &= (c_{13}L(L + m_4\lambda_4) + c_{33}\lambda_4(\lambda_4 - Lm_4) + v_3n_4)e^{-i\lambda_4 H}, \\
d_{23} &= (c_{13}L(L + m_5\lambda_5) + c_{33}\lambda_5(\lambda_5 - Lm_5) + v_3n_5)e^{-i\lambda_5 H}, \\
d_{24} &= (c_{13}L(L + m_6\lambda_6) + c_{33}\lambda_6(\lambda_6 - Lm_6) + v_3n_6)e^{-i\lambda_6 H}, \\
d_{25} &= (c_{13}L(L - m_3\lambda_3) + c_{33}\lambda_3(\lambda_3 + Lm_3) + v_3n_3)e^{i\lambda_3 H}, \\
d_{26} &= (c_{13}L(L - m_4\lambda_4) + c_{33}\lambda_4(\lambda_4 + Lm_4) + v_3n_4)e^{i\lambda_4 H}, \\
d_{27} &= (c_{13}L(L - m_5\lambda_5) + c_{33}\lambda_5(\lambda_5 + Lm_5) + v_3n_5)e^{i\lambda_5 H}, \\
d_{28} &= (c_{13}L(L - m_6\lambda_6) + c_{33}\lambda_6(\lambda_6 + Lm_6) + v_3n_6)e^{i\lambda_6 H}, \\
d_{31} &= [c_{55}(\lambda_3^2 - L^2)m_3 + 2L\lambda_3 + ibF\Omega_3m_3]e^{-i\lambda_3 H}, \\
d_{32} &= [c_{55}(\lambda_4^2 - L^2)m_4 + 2L\lambda_4 + ibF\Omega_4m_4]e^{-i\lambda_4 H}, \\
d_{33} &= [c_{55}(\lambda_5^2 - L^2)m_5 + 2L\lambda_5 + ibF\Omega_5m_5]e^{-i\lambda_5 H}, \\
d_{34} &= [c_{55}(\lambda_6^2 - L^2)m_6 + 2L\lambda_6 + ibF\Omega_6m_6]e^{-i\lambda_6 H}, \\
d_{35} &= [c_{55}(\lambda_3^2 - L^2)m_3 - 2L\lambda_3 + ibF\Omega_3m_3]e^{i\lambda_3 H}, \\
d_{36} &= [c_{55}(\lambda_4^2 - L^2)m_4 - 2L\lambda_4 + ibF\Omega_4m_4]e^{i\lambda_4 H}, \\
d_{37} &= [c_{55}(\lambda_5^2 - L^2)m_5 - 2L\lambda_5 + ibF\Omega_5m_5]e^{i\lambda_5 H}, \\
d_{38} &= [c_{55}(\lambda_6^2 - L^2)m_6 - 2L\lambda_6 + ibF\Omega_6m_6]e^{i\lambda_6 H}, \\
d_{41} &= L + m_3\lambda_3, & d_{42} &= L + m_4\lambda_4, \\
d_{43} &= L + m_5\lambda_5, & d_{44} &= L + m_6\lambda_6, \\
d_{45} &= L - m_3\lambda_3, & d_{46} &= L - m_4\lambda_4, \\
d_{47} &= L - m_5\lambda_5, & d_{48} &= L - m_6\lambda_6, \\
d_{49} &= -(L + \overline{m_3}\overline{\lambda_3}), & d_{410} &= -(L + \overline{m_4}\overline{\lambda_4}), \\
d_{411} &= -(L + \overline{m_5}\overline{\lambda_5}), & d_{412} &= -(L + \overline{m_6}\overline{\lambda_6}), \\
d_{51} &= Lm_3 - \lambda_3, & d_{52} &= Lm_4 - \lambda_4, \\
d_{53} &= Lm_5 - \lambda_5, & d_{54} &= Lm_6 - \lambda_6,
\end{aligned}$$

$$\begin{aligned}
d_{55} &= Lm_3 + \lambda_3, & d_{56} &= Lm_4 + \lambda_4, \\
d_{57} &= Lm_5 + \lambda_5, & d_{58} &= Lm_6 + \lambda_6, \\
d_{59} &= -L\bar{m}_3 + \bar{\lambda}_3, & d_{510} &= -L\bar{m}_4 + \bar{\lambda}_4, \\
d_{511} &= -L\bar{m}_5 + \bar{\lambda}_5, & d_{512} &= -L\bar{m}_6 + \bar{\lambda}_6, \\
d_{61} &= \Omega_3 m_3, & d_{62} &= \Omega_4 m_4, & d_{63} &= \Omega_5 m_5, & d_{64} &= \Omega_6 m_6, \\
d_{65} &= \Omega_3 m_3, & d_{66} &= \Omega_4 m_4, & d_{67} &= \Omega_5 m_5, & d_{68} &= \Omega_6 m_6, \\
d_{69} &= -\bar{\Omega}_3 \bar{m}_3, & d_{610} &= -\bar{\Omega}_4 \bar{m}_4, & d_{611} &= -\bar{\Omega}_5 \bar{m}_5, & d_{612} &= -\bar{\Omega}_6 \bar{m}_6, \\
d_{71} &= Mm_3 \lambda_3 (\lambda_3^2 + \Omega_3 + L^2), & d_{72} &= Mm_4 \lambda_4 (\lambda_4^2 + \Omega_4 + L^2), \\
d_{73} &= Mm_5 \lambda_5 (\lambda_5^2 + \Omega_5 + L^2), & d_{74} &= Mm_6 \lambda_6 (\lambda_6^2 + \Omega_6 + L^2), \\
d_{75} &= -Mm_3 \lambda_3 (\lambda_3^2 + \Omega_3 + L^2), & d_{76} &= -Mm_4 \lambda_4 (\lambda_4^2 + \Omega_4 + L^2), \\
d_{77} &= -Mm_5 \lambda_5 (\lambda_5^2 + \Omega_5 + L^2), & d_{78} &= -Mm_6 \lambda_6 (\lambda_6^2 + \Omega_6 + L^2), \\
d_{79} &= -\bar{M} \bar{m}_3 \bar{\lambda}_3 (\bar{\lambda}_3^2 + \bar{\Omega}_3 + L^2), & d_{710} &= -\bar{M} \bar{m}_4 \bar{\lambda}_4 (\bar{\lambda}_4^2 + \bar{\Omega}_4 + L^2), \\
d_{711} &= -\bar{M} \bar{m}_5 \bar{\lambda}_5 (\bar{\lambda}_5^2 + \bar{\Omega}_5 + L^2), & d_{712} &= -\bar{M} \bar{m}_6 \bar{\lambda}_6 (\bar{\lambda}_6^2 + \bar{\Omega}_6 + L^2), \\
d_{81} &= c_{13} L (L + m_3 \lambda_3) + c_{33} \lambda_3 (\lambda_3 - Lm_3) + v_3 n_3, \\
d_{82} &= c_{13} L (L + m_4 \lambda_4) + c_{33} \lambda_4 (\lambda_4 - Lm_4) + v_3 n_4, \\
d_{83} &= c_{13} L (L + m_5 \lambda_5) + c_{33} \lambda_5 (\lambda_5 - Lm_5) + v_3 n_5, \\
d_{84} &= c_{13} L (L + m_6 \lambda_6) + c_{33} \lambda_6 (\lambda_6 - Lm_6) + v_3 n_6, \\
d_{85} &= c_{13} L (L - m_3 \lambda_3) + c_{33} \lambda_3 (\lambda_3 + Lm_3) + v_3 n_3, \\
d_{86} &= c_{13} L (L - m_4 \lambda_4) + c_{33} \lambda_4 (\lambda_4 + Lm_4) + v_3 n_4, \\
d_{87} &= c_{13} L (L - m_5 \lambda_5) + c_{33} \lambda_5 (\lambda_5 + Lm_5) + v_3 n_5, \\
d_{88} &= c_{13} L (L - m_6 \lambda_6) + c_{33} \lambda_6 (\lambda_6 + Lm_6) + v_3 n_6, \\
d_{89} &= -\left[ \bar{c}_{13} L (L + \bar{m}_3 \bar{\lambda}_3) + \bar{c}_{33} \bar{\lambda}_3 (\bar{\lambda}_3 - L\bar{m}_3) + \bar{v}_3 \bar{n}_3 \right], \\
d_{810} &= -\left[ \bar{c}_{13} L (L + \bar{m}_4 \bar{\lambda}_4) + \bar{c}_{33} \bar{\lambda}_4 (\bar{\lambda}_4 - L\bar{m}_4) + \bar{v}_3 \bar{n}_4 \right], \\
d_{811} &= -\left[ \bar{c}_{13} L (L + \bar{m}_5 \bar{\lambda}_5) + \bar{c}_{33} \bar{\lambda}_5 (\bar{\lambda}_5 - L\bar{m}_5) + \bar{v}_3 \bar{n}_5 \right], \\
d_{812} &= -\left[ \bar{c}_{13} L (L + \bar{m}_6 \bar{\lambda}_6) + \bar{c}_{33} \bar{\lambda}_6 (\bar{\lambda}_6 - L\bar{m}_6) + \bar{v}_3 \bar{n}_6 \right], \\
d_{91} &= c_{55} \left[ m_3 (L^2 + \lambda_3^2) - 2L\lambda_3 \right] - ibFm_3 \Omega_3,
\end{aligned}$$

$$\begin{aligned}
d_{92} &= c_{55} \left[ m_4 (L^2 + \lambda_4^2) - 2L\lambda_4 \right] - ibFm_4\Omega_4, \\
d_{93} &= c_{55} \left[ m_5 (L^2 + \lambda_5^2) - 2L\lambda_5 \right] - ibFm_5\Omega_5, \\
d_{94} &= c_{55} \left[ m_6 (L^2 + \lambda_6^2) - 2L\lambda_6 \right] - ibFm_6\Omega_6, \\
d_{95} &= c_{55} \left[ m_3 (L^2 + \lambda_3^2) + 2L\lambda_3 \right] - ibFm_3\Omega_3, \\
d_{96} &= c_{55} \left[ m_4 (L^2 + \lambda_4^2) + 2L\lambda_4 \right] - ibFm_4\Omega_4, \\
d_{97} &= c_{55} \left[ m_5 (L^2 + \lambda_5^2) + 2L\lambda_5 \right] - ibFm_5\Omega_5, \\
d_{98} &= c_{55} \left[ m_6 (L^2 + \lambda_6^2) + 2L\lambda_6 \right] - ibFm_6\Omega_6, \\
d_{99} &= - \left\{ \overline{c_{55}} \left[ \overline{m_3} (L^2 + \overline{\lambda_3^2}) - 2L\overline{\lambda_3} \right] - ib\overline{F}\overline{m_3}\overline{\Omega_3} \right\}, \\
d_{910} &= - \left\{ \overline{c_{55}} \left[ \overline{m_4} (L^2 + \overline{\lambda_4^2}) - 2L\overline{\lambda_4} \right] - ib\overline{F}\overline{m_4}\overline{\Omega_4} \right\}, \\
d_{911} &= - \left\{ \overline{c_{55}} \left[ \overline{m_5} (L^2 + \overline{\lambda_5^2}) - 2L\overline{\lambda_5} \right] - ib\overline{F}\overline{m_5}\overline{\Omega_5} \right\}, \\
d_{912} &= - \left\{ \overline{c_{55}} \left[ \overline{m_6} (L^2 + \overline{\lambda_6^2}) - 2L\overline{\lambda_6} \right] - ib\overline{F}\overline{m_6}\overline{\Omega_6} \right\}, \\
d_{101} &= n_3, & d_{102} &= n_4, & d_{103} &= n_5, & d_{104} &= n_6, \\
d_{105} &= n_3, & d_{106} &= n_4, & d_{107} &= n_5, & d_{108} &= n_6, \\
d_{109} &= \overline{n_3}, & d_{1010} &= \overline{n_4}, & d_{1011} &= \overline{n_5}, & d_{1012} &= \overline{n_6}, \\
d_{111} &= n_3(\theta - i\lambda_3), & d_{112} &= n_4(\theta - i\lambda_4), & d_{113} &= n_5(\theta - i\lambda_5), \\
d_{114} &= n_6(\theta - i\lambda_6), & d_{115} &= n_3(\theta + i\lambda_3), & d_{116} &= n_4(\theta + i\lambda_4), \\
d_{117} &= n_5(\theta + i\lambda_5), & d_{118} &= n_6(\theta + i\lambda_6), & d_{119} &= -\overline{n_3}(\overline{\theta} - i\overline{\lambda_3}), \\
d_{1110} &= -\overline{n_4}(\overline{\theta} - i\overline{\lambda_4}), & d_{1111} &= -\overline{n_5}(\overline{\theta} - i\overline{\lambda_5}), & d_{1112} &= -\overline{n_6}(\overline{\theta} - i\overline{\lambda_6}), \\
d_{121} &= n_3(\theta - i\lambda_3)e^{-i\lambda_3H}, & d_{122} &= n_4(\theta - i\lambda_4)e^{-i\lambda_4H}, \\
d_{123} &= n_5(\theta - i\lambda_5)e^{-i\lambda_5H}, & d_{124} &= n_6(\theta - i\lambda_6)e^{-i\lambda_6H}, \\
d_{125} &= n_3(\theta + i\lambda_3)e^{i\lambda_3H}, & d_{126} &= n_4(\theta + i\lambda_4)e^{i\lambda_4H}, \\
d_{127} &= n_5(\theta + i\lambda_5)e^{i\lambda_5H}, & d_{128} &= n_6(\theta + i\lambda_6)e^{i\lambda_6H}.
\end{aligned}
\tag{4.8}$$

Equation (4.7) determines the wave velocity equation for the Rayleigh waves in an orthotropic thermoelastic granular medium under the influence of initial stress and gravity field.

## 5. Stoneley Waves

To investigate the possibility Stoneley waves in thermoelastic granular medium under the influence of initial stress and gravity field, we replace the layer by a half-space ( $H \rightarrow \infty$ ), in the preceding problem, in this case  $e^{i\lambda_j H} \rightarrow 0$ , and the coefficients of  $e^{-i\lambda_j H}$  ( $j = 3, 4, 5, 6$ ) must vanish. Hence, the wave velocity equation (4.7) reduces to

$$\det \cdot (d'_{rc}) = 0 \quad (r, c = 1, 2, \dots, 8), \quad (5.1)$$

where

$$\begin{aligned} d'_{11} &= L + m_3 \lambda_3, & d'_{12} &= L + m_4 \lambda_4, \\ d'_{13} &= L + m_5 \lambda_5, & d'_{14} &= L + m_6 \lambda_6, \\ d'_{15} &= L - m_3 \lambda_3, & d'_{16} &= L - m_4 \lambda_4, \\ d'_{17} &= L - m_5 \lambda_5, & d'_{18} &= L - m_6 \lambda_6, \\ d'_{21} &= L m_3 - \lambda_3, & d'_{22} &= L m_4 - \lambda_4, \\ d'_{23} &= L m_5 - \lambda_5, & d'_{24} &= L m_6 - \lambda_6, \\ d'_{25} &= L m_3 + \lambda_3, & d'_{26} &= L m_4 + \lambda_4, \\ d'_{27} &= L m_5 + \lambda_5, & d'_{28} &= L m_6 + \lambda_6, \\ d'_{31} &= \Omega_3 m_3, & d'_{32} &= \Omega_4 m_4, & d'_{33} &= \Omega_5 m_5, & d'_{34} &= \Omega_6 m_6, \\ d'_{35} &= -\Omega_3 m_3, & d'_{36} &= -\Omega_4 m_4, & d'_{37} &= -\Omega_5 m_5, & d'_{38} &= -\Omega_6 m_6, \\ d'_{41} &= M m_3 \lambda_3 (\lambda_3^2 + \Omega_3 + L^2), & d'_{42} &= M m_4 \lambda_4 (\lambda_4^2 + \Omega_4 + L^2), \\ d'_{43} &= M m_5 \lambda_5 (\lambda_5^2 + \Omega_5 + L^2), & d'_{44} &= M m_6 \lambda_6 (\lambda_6^2 + \Omega_6 + L^2), \\ d'_{45} &= -M m_3 \lambda_3 (\lambda_3^2 + \Omega_3 + L^2), & d'_{46} &= -M m_4 \lambda_4 (\lambda_4^2 + \Omega_4 + L^2), \\ d'_{47} &= -M m_5 \lambda_5 (\lambda_5^2 + \Omega_5 + L^2), & d'_{48} &= -M m_6 \lambda_6 (\lambda_6^2 + \Omega_6 + L^2), \\ d'_{51} &= c_{13} L (L + m_3 \lambda_3) + c_{33} \lambda_3 (\lambda_3 - L m_3) + \nu_3 n_3, \\ d'_{52} &= c_{13} L (L + m_4 \lambda_4) + c_{33} \lambda_4 (\lambda_4 - L m_4) + \nu_3 n_4, \\ d'_{53} &= c_{13} L (L + m_5 \lambda_5) + c_{33} \lambda_5 (\lambda_5 - L m_5) + \nu_3 n_5, \\ d'_{54} &= c_{13} L (L + m_6 \lambda_6) + c_{33} \lambda_6 (\lambda_6 - L m_6) + \nu_3 n_6, \\ d'_{55} &= c_{13} L (L - m_3 \lambda_3) + c_{33} \lambda_3 (\lambda_3 + L m_3) + \nu_3 n_3, \end{aligned}$$

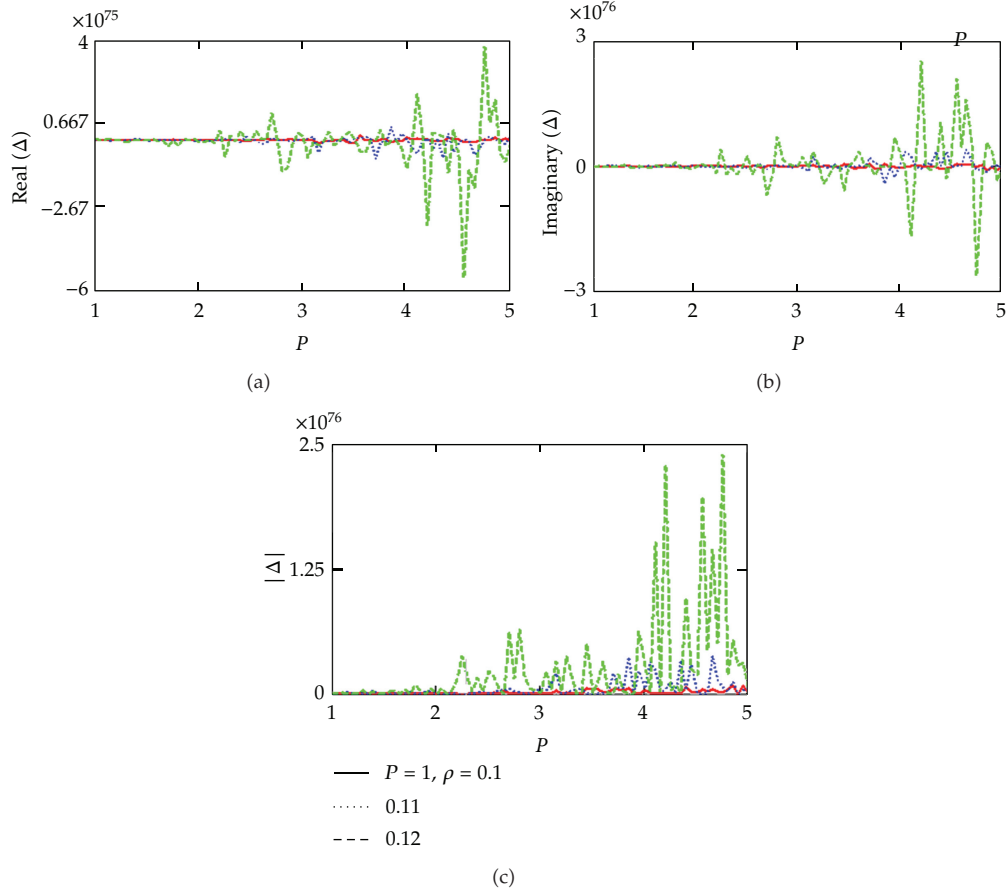


$$\begin{aligned}
d'_{56} &= c_{13}L(L - m_4\lambda_4) + c_{33}\lambda_4(\lambda_4 + Lm_4) + v_3n_4, \\
d'_{57} &= c_{13}L(L - m_5\lambda_5) + c_{33}\lambda_5(\lambda_5 + Lm_5) + v_3n_5, \\
d'_{58} &= c_{13}L(L - m_6\lambda_6) + c_{33}\lambda_6(\lambda_6 + Lm_6) + v_3n_6, \\
d'_{61} &= c_{55} \left[ m_3(L^2 + \lambda_3^2) - 2L\lambda_3 \right] - ibFm_3\Omega_3, \\
d'_{62} &= c_{55} \left[ m_4(L^2 + \lambda_4^2) - 2L\lambda_4 \right] - ibFm_4\Omega_4, \\
d'_{63} &= c_{55} \left[ m_5(L^2 + \lambda_5^2) - 2L\lambda_5 \right] - ibFm_5\Omega_5, \\
d'_{64} &= c_{55} \left[ m_6(L^2 + \lambda_6^2) - 2L\lambda_6 \right] - ibFm_6\Omega_6, \\
d'_{65} &= c_{55} \left[ m_3(L^2 + \lambda_3^2) + 2L\lambda_3 \right] - ibFm_3\Omega_3, \\
d'_{66} &= c_{55} \left[ m_4(L^2 + \lambda_4^2) + 2L\lambda_4 \right] - ibFm_4\Omega_4, \\
d'_{67} &= c_{55} \left[ m_5(L^2 + \lambda_5^2) + 2L\lambda_5 \right] - ibFm_5\Omega_5, \\
d'_{68} &= c_{55} \left[ m_6(L^2 + \lambda_6^2) + 2L\lambda_6 \right] - ibFm_6\Omega_6, \\
d'_{71} &= n_3, & d'_{72} &= n_4, & d'_{73} &= n_5, & d'_{74} &= n_6, \\
d'_{75} &= n_3, & d'_{76} &= n_4, & d'_{77} &= n_5, & d'_{78} &= n_6, \\
d'_{81} &= n_3(\theta - i\lambda_3), & d'_{82} &= n_4(\theta - i\lambda_4), & d'_{83} &= n_5(\theta - i\lambda_5), \\
d'_{84} &= n_6(\theta - i\lambda_6), & d'_{85} &= n_3(\theta + i\lambda_3), & d'_{86} &= n_4(\theta + i\lambda_4), \\
d'_{87} &= n_5(\theta + i\lambda_5), & d'_{88} &= n_6(\theta + i\lambda_6).
\end{aligned} \tag{5.2}$$

Equation (5.1) determines the wave velocity equation for the Stoneley waves in an orthotropic thermoelastic granular medium under the influence of initial stress and gravity field.

## 6. Special Cases

The transcendental equations (4.7) and (5.1), in the determinant form, have complex roots. The real part gives the velocity of Rayleigh waves and Stoneley waves, respectively, while the imaginary part gives the attenuation due to the granular nature of the medium. It is clear from the frequency equations (4.7) and (5.1) that the phase velocity depends on the initial stress  $P$ , the friction  $F$ , the gravity field, and the coupling factor  $\varepsilon$ .



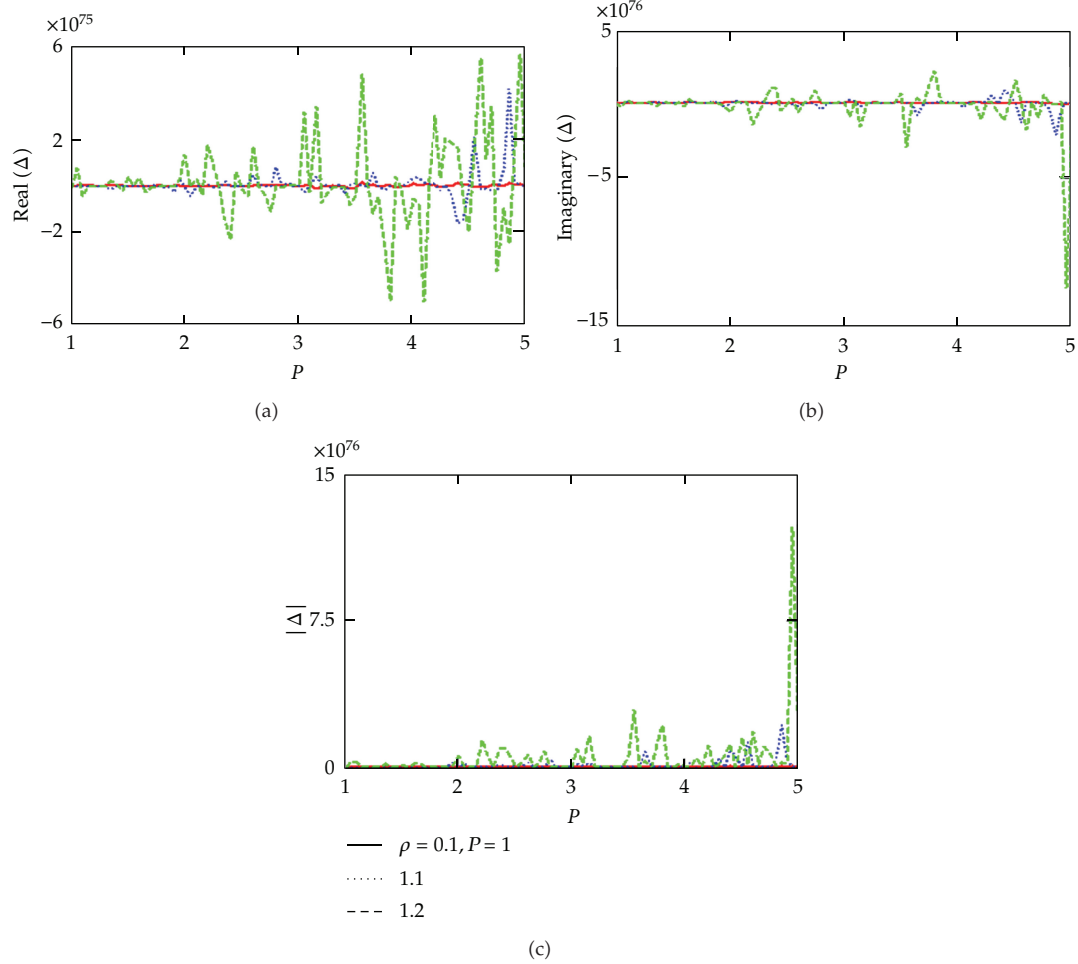
**Figure 1:** Effect of the density  $\rho$  on (real, imaginary, and magnitude) of Stoneley waves determinant with respect to initial stress.

When the gravity field is neglected, there is no coupling between the constants  $A_j, B_j$ , and  $E_j, F_j$  ( $j = 3, 4, 5, 6$ ), the equations (3.10), (3.11) become, respectively,

$$\begin{aligned} \left( \nabla^2 - S \frac{\partial}{\partial t} \right) \left[ \left( c_{55} - \frac{P}{2} \right) \frac{\partial^2 \psi}{\partial x_1^2} + \left( c_{33} - c_{31} - c_{55} - \frac{P}{2} \right) \frac{\partial^2 \psi}{\partial x_3^2} - \rho \frac{\partial^2 \psi}{\partial t^2} \right] + F \nabla^4 \left( \frac{\partial \psi}{\partial t} \right) &= 0, \\ \left( \nabla^2 - \frac{1}{\chi} \frac{\partial}{\partial t} \right) \left[ (c_{11} + P) \frac{\partial^2 \phi}{\partial x_1^2} + (c_{13} + 2c_{55} + P) \frac{\partial^2 \phi}{\partial x_3^2} - \rho \frac{\partial^2 \phi}{\partial t^2} \right] - \nu_1 \varepsilon \nabla^2 \left( \frac{\partial \phi}{\partial t} \right) &= 0. \end{aligned} \quad (6.1)$$

Equation (6.1) are in agreement with the corresponding equations obtained by Ahmed [7].

In the absence of initial stress and when there is no coupling between the temperature and strain fields, (4.7) takes the form as obtained by Oshima [2].



**Figure 2:** Effect of the parameter  $b$  on (real, imaginary, and magnitude) of Stoneley waves determinant with respect to initial stress.

### 7. Numerical Results and Discussions

For a numerical computational work, Potash material is considered as an upper granular media with thickness  $H$  and  $Fe$  as lower granular media [28]

$$\begin{aligned}
 \rho &= 1.098, & M &= 0.4, & F &= 0.5, & Q_1 &= 0.4, & Q'_2 &= 0.6, \\
 \rho' &= 4.499, & M' &= 0.6, & F' &= 0.6, & Q'_1 &= 0.6, & Q'_2 &= 0.8.
 \end{aligned}
 \tag{7.1}$$

A Mathcad program is used to invert the transform in order to obtain the results in the physical domain. From Figure 1, It is obvious that the determinant of Stoneley waves velocity increases and decreases with an increasing of the various values of the initial stress  $P$ , also with an increasing of the density  $\rho$ ; the real and imaginary parts of the determinant of Stoneley waves frequency equation increase. It is seen that magnitude of determinant of the

frequency equation for Stoneley waves takes a large decreasing with an increasing of the initial stress and density.

Figure 2 displays the influence of parameter  $b$  on the frequency equation of Stoneley waves. It is concluded that the determinant of frequency equation of Stoneley waves increases with an increasing of the initial stress  $P$ . Also, it is clear that  $\text{Re}(\Delta)$  increases with an increasing of parameter  $b$  but  $\text{Im}(\Delta)$  decreases.

Finally, it appears that the magnitude of  $\Delta$  increases with the increased values of the parameter  $b$ .

## 8. Conclusion

From the results obtained, we concluded the following.

- (1) The determinant of frequency of Rayleigh and Stoneley waves is affected by the influences of initial stress, density, gravity, and orthotropic of material and very pronounced on the waves propagation phenomena that indicates their utilitarian aspects in diverse fields as Geophysics, Geology, Acoustics, Plasma, and so forth.
- (2) From Figures 1 and 2, it is obvious that the magnitude of Stoneley waves increases clearly with the influences of the density  $\rho$  and parameter  $b$ .
- (3) If the media considered are isotropic, the relevant results obtained deduce to the results obtained by El-Naggar [25].

## Nomenclature

- $c_{ij}$ : Are the elastic constants  
 $e_{ij}$ : Are the components of strain tensor  
 $F$ : Is the coefficient of fraction  
 $g$ : Is the acceleration due to the gravity  
 $M$ : Is the third elastic constant  
 $P$ : Is the initial stress  
 $s$ : Is the specific heat per unit mass  
 $T$ : Is the temperature change about the initial temperature  $T_0$   
 $v$ : Is the phase speed  
 $\alpha_1$ : Is the thermal expansion coefficient in the planes of orthotropic  
 $\alpha_2$ : Is the thermal expansion coefficient along the  $x_3$ -axis  
 $\rho$ : Is the density  
 $\tau$ : Is a function of depth  
 $\tau_{ij}$ : Are the components of stress tensor  
 $\omega$ : Is the frequency.

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