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Research Article

On Dummy Variables of Structure-Preserving Transformations

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A method is given for obtaining equivalence subgroups of a family of differential equations from the equivalence group of simpler equations of a similar form, but in which the arbitrary functions specifying the family element depend on fewer variables. Examples of applications to classical equations are presented, some of which show how the method can actually be used for a much easier determination of the equivalence group itself.

1. Introduction

Denote collectively by $\mathcal{A}$ the set of all arbitrary functions specifying the family element in a collection $\mathcal{F}$ of differential equations of the form

$$\Delta(x, y_{(n)}; \mathcal{A}) = 0, \tag{1.1}$$

where $x = (x^1, \ldots, x^p)$ is the set of independent variables and $y_{(n)}$ denotes y and all its derivatives up to the order n. In the most general case, the function $\mathcal{A}$ may depend on x,y, and the derivatives of y up to a certain order not exceeding n, but quite often $\mathcal{A}$ is simply a function of x, or a constant.

Let G be the Lie pseudo-group of point transformations of the form

$$x = \varphi(z, w), \qquad y = \psi(z, w), \tag{1.2}$$

where $z = (z^1, \ldots, z^p)$ is the new set of independent variables, while $w = w(z)$ is the new dependent variable. The group G is infinite because, as explained in a paper by Tresse

[1, Page. 11], its elements depend in general on arbitrary functions and not on arbitrary constants, and a Lie pseudo-group is to be understood here in the sense of [2, 3], that is, as the infinite-dimensional counterpart of a local Lie group of transformations. We say that G is the group of equivalence transformations of (1.1) if it is the largest Lie pseudo-group of point transformations that map (1.1) into an equation of the same form, that is, if in terms of the same function Δ appearing in (1.1), the transformed equation has an expression of the form

$$\Delta\left(z, w_{(n)}; \mathcal{B}\right) = 0, \tag{1.3a}$$

where

$$\mathcal{B} = T(\mathcal{A}), \tag{1.3b}$$

for a certain function T, is the new set of arbitrary functions specifying the family element in the transformed equation. Equivalently, G will map elements of $\mathcal{F}$ into $\mathcal{F}$, and, when this holds, (1.2) is called an equivalence transformation of the original equation (1.1). By a result of Lie [4], (1.3b) defines another group of transformations G_c induced by G, and acting on the arbitrary functions $\mathcal{A}$ of the original equation. Invariants of the group G_c are termed invariants of the differential equation (1.1). These invariants are functions which depend on the arbitrary functions $\mathcal{A}$ of the original equation, and which have exactly the same expression when they are also formed for the transformed equation, and they play a crucial role in the classification of the family of equation [5, 6].

Early developments in applications of Lie groups for finding equivalence transformations of a given differential equation (DE) started in the work of Lie [7] and were later pursued by Tresse [1] and Ovsiannikov [8]. More recent developments based on Cartan equivalence methods originated in the works of Olver and collaborators [2, 9], and this has led to new methods for computing the differential invariants and Maurer-Cartan structure equations of a Lie pseudo-group.

In this paper, we show that for a given differential equation in which the arbitrary functions defining the family element are functions of independent variables alone, the equivalence group is a subgroup of a differential equation of a similar form, but in which the arbitrary functions also depend on the dependent variable and its derivatives up to a given order. An extension of the theorem to equations with arbitrary functions depending on a subset of the set consisting of all independent variables and the dependent variable and its derivatives up to a given order is provided. Examples of applications to a number of classical equations show how to obtain equivalence subgroups of a more complex family of equations from those of simpler ones.

2. Dummy Variables of the Equivalence Group

If we consider, for instance, the general linear differential equation

$$y^{(n)} + a^1(x) y^{(n-1)} + a^2(x) y^{(n-2)} + \cdots + a^n(x) y = 0, \tag{2.1}$$

the arbitrary functions $\mathcal{A}$ in (1.1) are the coefficients $a^i(x)$, for $i = 1, \ldots, n$. Thus in this case it appears that $\mathcal{A} = \mathcal{A}(x)$ is a function of the independent variable alone. This is quite often the case with various other linear or nonlinear equations. However, more generally, $\mathcal{A}$ arises as a function of the independent variables and the dependent variable y and its derivatives up to a certain order.

Theorem 2.1. *Denote by (A) (1.1) with $\mathcal{A} = \mathcal{A}(x)$ and by (B) the same equation, but in which $\mathcal{A}$ depends on x, y, and the derivatives of y up to a certain order s, that is, in which $\mathcal{A} = \mathcal{A}(x, y_{(s)})$. Similarly, denote by G^A and G^B the equivalence groups for (A) and (B), respectively. Then G^A is a subgroup of G^B.*

Proof. To fix ideas, suppose that the equation has m arbitrary functions $A_1, \ldots, A_m$ specifying the family element in $\mathcal{F}$ (and collectively denoted by $\mathcal{A}$) and that in (B) we have $A_j = A_j(x, y_{(s)})$ for all $j = 1, \ldots, m$. On the other hand, for each multi-index $J = (j_1, \ldots, j_k)$, where $1 \leq j_r \leq p$ for $r = 1, \ldots, k$, we use the notation

$$D_J y \equiv \frac{\partial^{\,k} y(x)}{\partial x^{j_1}, \partial x^{j_2}, \ldots, \partial x^{j_k}}. \tag{2.2}$$

Suppose now that (1.2) is an equivalence transformation of (A). To perform the transformation of (B) by (1.2), we may first ignore the arguments of the function $\mathcal{A}$, in which sense these arguments can be considered as dummy. Then, for every multi-index J with $0 \leq \#J = k \leq n$, each partial derivative $D_J y$ present in the equation is replaced by a function $T_J(z, w_{(k)})$. When all such replacements are made and all remaining occurrences of independent variables are expressed in terms of z and w by another application of (1.2), the resulting equation is the transformed equation (3.12), in which $\mathcal{B}$ collectively denotes the new functions $B_1, \ldots, B_m$. According to (3.15), each transformed function B_j has an expression of the form

$$B_j = F_j(z, A_1, \ldots, A_m), \quad (j = 1, \ldots, m),\ r_j \leq n, \tag{2.3}$$

in terms of the original functions $A_1, \ldots, A_m$, and for a certain function F_j, where F_j does not depend explicitly on the dependent variable w and its derivatives, on the assumption that (1.2) is an equivalence transformation of (A). To complete the transformation of (B), we may now apply again (1.2), to transform the arguments of the functions A_j, and this will transform $A_j(x, y_{(s)})$ into a composition of A_j and a function of z and $w_{(s)}$, and the latter transformation will thus have no effect on the form of the transformed equation. Consequently, the transformed version of (B) has the same form as (B), and this completes the proof of the theorem. □

The proof of Theorem 2.1 suggests the following immediate extension to the case where the function $\mathcal{A}$ in (A) may depend only on a subset of the set of all variables and the derivatives of y up to a given order.

Theorem 2.2. *Denote by (A) (1.1) with $\mathcal{A} = \mathcal{A}(\mathrm{X})$, where X is a subset of the set consisting of all independent variables, of the dependent variable y and its derivatives up to a certain order r. Denote also by (B) the same equation, but in which $\mathcal{A} = \mathcal{A}(x, y_{(s)})$, with $r \leq s$. Similarly, denote by G^A and G^B the equivalence groups for (A) and (B), respectively. Then G^A is a subgroup of G^B.*

Proof. Using the same notation as in the proof of Theorem 2.1, and reasoning in a similar way, if we apply G^A to transform (B) while first ignoring the arbitrary functions' arguments, the form of the equation is preserved, and the new arbitrary functions B_j now take the form

$$B_j = F_j\left(z, w_{(r)}, A_1, \ldots, A_m\right), \quad (j = 1, \ldots, m). \tag{2.4}$$

We can then transform the arguments $(x, y_{(s)})$ in each of the functions A_j appearing in (2.4), to obtain expressions of the form $A_j = A_j(z, w_{(s)})$ in (2.4), and the latter transformation will not affect the form of the equation, as already seen. □

It should however be noted that, as stated, Theorems 2.1 and 2.2 holds only under the assumption that, in (B), the function $\mathscr{A}$ depends explicitly on $x = (x^1, \ldots, x^p)$, y, and all derivatives of y up to the stated order s. This is first because when a derivative of y of a given order s is transformed under a given group, the resulting expression depends in general on all derivatives of the new dependent variable w up to the order s. On the other hand, the transformed function B_j will in general depend on the full set of the new independent variables $z = (z^1, \ldots, z^p)$ and not only on a subset of this set. Thus, if in (B) the function $\mathscr{A}$ does not explicitly depend on all independent variables or on all derivatives of y up to the stated order s, this restriction should be imposed on G^A, and this will yield in general a smaller equivalence subgroup for (B).

On the other hand, the converse is not true in Theorem 2.1, and this is simply because if (A) is transformed by an equivalence transformation of (B), there is no guarantee that in the transformed equation the functions B_j will not depend on w or its derivatives. Similarly, the converse to Theorem 2.2 is not true.

3. Application to Classical Equations

3.1. The General Linear Homogeneous ODE

In the sequel, we will use the notation $f_a = \partial f/\partial a$, for every function f with argument a, hence subscripts in such functions will denote differentiation. We shall also use the notation $\partial_a = \partial/\partial a$ for every variable a, and a transformed equation will be assumed to be expanded as a polynomial in the dependent variable and its derivatives.

If we consider, for instance, (2.1), its equivalence transformations are given by

$$x = S(z), \qquad y = L(z)w, \tag{3.1}$$

where S and L are arbitrary functions. It follows from Theorem 2.1 that, if we assume in (2.1) that the coefficients a^i depend also on y and its derivatives up to a certain order s, then (3.1) remains an equivalence transformation of the resulting equation. Equations of the latter form frequently appear with $s = 0$, so that $a^i = a^i(x, y)$, which gives rise to a nonlinear equation of the form

$$y^{(n)} + a^1(x, y)y^{(n-1)} + a^2(x, y)y^{(n-2)} + \cdots + a^n(x, y)y = 0. \tag{3.2}$$

After some calculations, the equivalence group of (3.2) is found for $n = 3$ to be of the form

$$x = S(z), \qquad y = L(z)w + J(z), \tag{3.3}$$

where S, L, and J are arbitrary functions. It can be shown by induction that (3.3) also holds for all $n \geq 3$ in (3.2). Theorem 2.2 states that for any other variant of (3.2) in which the coefficients a^i are of the form $a^i = a^i(X)$, where $X \subset \{x, y\}$, the equivalence group must be of the predefined form (3.3), where the functions S, L, and J are to be specified. Finding the equivalence group of an equation in a more specific form naturally greatly simplifies calculations. If, for instance, we suppose that $a^i(X) = a^i(y)$, for all i, so that (3.2) reduces to

$$y^{(n)} + a^1(y)y^{(n-1)} + a^2(y)y^{(n-2)} + \cdots + a^n(y)y = 0, \tag{3.4}$$

then since the a^i depend on y alone, L and J must be constant functions, so that the transformation of y reduces to $y = k_3 w + k_4$, for some constants k_3 and k_4. For $n = 3$, under the corresponding transformations of x and y, the term not involving the dependent variable w and its derivatives as a factor in the transformation of (3.4) is $(k_4 a^3 S_z^3)/k_3$. Since this term may however depend on w but not on z, we infer that S_z must be a constant function, so that the transformation of the independent variable x must also be a linear function of the form $x = k_1 z + k_2$. It is readily seen that the set of transformations

$$x = k_1 z + k_2, \qquad y = k_3 w + k_4 \tag{3.5}$$

preserves the form of (3.4) for $n = 3$, and therefore defines its equivalence group. It is also readily found by induction on n that this remains true for all $n \geq 3$. Using (3.3) as the predefined form of the equivalence group leads in a similar manner to a much easier determination of the equivalence group for other possible values of X. In addition, X needs not be the same for all of the coefficients a^i, and thus to apply Theorem 2.2 here we only need to have $a^i = a^i(X^i)$, with $X^i \subset \{x, y\}$. In particular, some of the a^i might be constants.

3.2. The Linear Hyperbolic Equation

Consider the PDE in two independent variables t and x of the form

$$u_{tx} + a^1(t, x)u_t + a^2(t, x)u_x + a^3(t, x)u = 0, \tag{3.6}$$

generally referred to as the linear hyperbolic second-order equation. It is well known (see e.g., [10, 11]) that the group G of equivalence transformations of this equation is given in terms of new independent variables y and z and dependent variables w, by invertible transformations of the form

$$t = R(y), \qquad x = S(z), \qquad u = L(y, z)w, \tag{3.7}$$

where $R,\ S$, and L are arbitrary functions satisfying the nonvanishing Jacobian condition

$$R_y S_z L \neq 0, \tag{3.8}$$

where $R_y = \partial_y R$ and $S_z = \partial_z S$. Consider now a nonlinear extension of (3.6) of the form

$$u_{tx} + a^1(t,x,u)u_t + a^2(t,x,u)u_x + a^3(t,x,u)u = 0, \tag{3.9}$$

in which the dependent variable u also appears as argument in the arbitrary coefficients. We undertake the somewhat lengthy calculation of the equivalence group of (3.9) which is not available in the literature, to illustrate how the knowledge of such a group can be utilized for a much easier determination of the equivalence group of equations of a similar form. The equivalence group H of (3.9) must be sought in the form

$$t = R(y,z,w), \qquad x = S(y,z,w), \qquad u = T(y,z,w), \tag{3.10}$$

for some functions R, S, and T to be specified and which must satisfy the non-vanishing Jacobian condition

$$(-R_z S_y + R_y S_z)T_w + (R_z S_w - R_w S_z)T_y + (-R_y S_w + R_w S_y)T_z \neq 0. \tag{3.11}$$

However, we notice that under (3.10), u_t is transformed into

$$u_t = \frac{-S_z T_y + S_y T_z + (S_w T_z - S_z T_w)w_y + (S_y T_w - S_w T_y)w_z}{R_z S_y - R_y S_z + (R_z S_w - R_w S_z)w_y + (R_w S_y - R_y S_w)w_z}. \tag{3.12}$$

The occurrence of derivatives of w in the denominator of this transformed expression for u_t will give rise in the expression of u_{tx} to undesired terms in w_{zz} and w_{yy} as well as nonlinear terms of the form $w_z^i w_y^j$, where i and j are some natural numbers, and they should therefore disappear. This disappearance of derivatives is translated into the conditions

$$R_z S_w - R_w S_z = 0, \qquad -R_y S_w + R_w S_y = 0, \tag{3.13}$$

under which the Jacobian condition (3.11) is reduced to

$$(-R_z S_y + R_y S_z)T_w \neq 0. \tag{3.14}$$

If we suppose that $R_w = 0$ and $S_w \neq 0$, then it follows from (3.13) that $R_y = R_z = 0$, which contradicts (3.14). Thus we have $R_w = 0$ if and only if $S_w = 0$. On the other hand, if we assume that $R_w \neq 0$, then using (3.13) we may write

$$S_y = R_y\left(\frac{S_w}{R_w}\right), \qquad R_z = S_z\left(\frac{R_w}{S_w}\right), \tag{3.15}$$

which also contradicts (3.14), so that $R_w = S_w = 0$, and the resulting expressions in (3.10) take the form

$$t = R(y, z), \qquad x = S(y, z), \qquad u = T(y, z, w). \tag{3.16}$$

Under the new change of variables, the terms in w_{yy} and w_{zz} in the transformed equation are given by

$$R_z S_z (-R_z S_y + R_y S_z) T_w w_{yy}, \qquad R_y S_y (-R_z S_y + R_y S_z) T_w w_{zz}, \tag{3.17}$$

respectively. On account of (3.14), the vanishing of these terms is reduced to the condition

$$R_z S_z = R_y S_y = 0, \tag{3.18}$$

which on account of the Jacobian condition (3.8) readily yields

$$R_z = S_y = 0. \tag{3.19}$$

Consequently, (3.10) reduces to

$$t = R(y), \qquad x = S(z), \qquad u = T(y, z, w). \tag{3.20}$$

Under this new set of transformations, (3.9) takes the form

$$\begin{aligned} &\frac{w_y w_z T_{w,w}}{T_w} + w_z \left(a^2 R_y + \frac{T_{y,w}}{T_w} \right) + w_y \left(a^1 S_z + \frac{T_{z,w}}{T_w} \right) \\ &\quad + \frac{a^3 T R_y S_z + a^1 S_z T_y + a^2 R_y T_z + T_{y,z}}{T_w} + w_{y,z} = 0, \end{aligned} \tag{3.21}$$

which shows that T must be linear in w. Consequently, the required change of variables must be of the form

$$t = R(y), \qquad x = S(z), \qquad u = L(y, z) w + J(y, z), \tag{3.22}$$

and (3.22) transforms (3.9) into

$$\begin{aligned} &\frac{J a^3 R_y S_z}{L} + \left(\frac{L_z}{L} + a^1 S_z \right) w_y + \left(\frac{L_y}{L} + a^2 R_y \right) w_z \\ &\quad + \frac{w (a^2 L_z R_y + a^1 L_y S_z + L a^3 R_y S_z + L_{y,z})}{L} + w_{y,z} = 0, \end{aligned} \tag{3.23}$$

which is of the required form since we may in this case write the constant term $J a^3 R_y S_z / L$ in the transformed equation in the form $w(J a^3 R_y S_z / (L w))$. Therefore, (3.22) defines the equivalence group H of (3.9). It differs from the equivalence group G of (3.6) only by the additional term $J(y,z)$ in the transformation of u, which has the effect of adding a sort of nonhomogeneous term to the transformed equation.

Now that the equivalence group (3.22) of (3.9) is available, we can clearly demonstrate how Theorem 2.2 can be used for a much easier determination of equivalence groups for variants of (3.9), in which the arbitrary coefficients a^i have arguments of the form $X^i \subset \{t,x,u\}$. Indeed, since by Theorem 2.2 any such group is a subgroup of H, its equivalence transformations must therefore be sought in the form (3.22), which greatly simplifies calculations. We show this by considering two examples.

To begin with, consider a variant of (3.9) of the form

$$u_{tx} + a^1(x)u_t + a^2(t)u_x + a^3(t,x)u = 0. \tag{3.24}$$

Since the coefficient $a^3(t,x)$ of u does not involve u itself, we may assume that $J(y,z) = 0$ in (3.22), and we thus look for equivalence transformations of (3.24) in the form

$$t = R(y), \qquad x = S(z), \qquad u = L(y,z)w, \tag{3.25}$$

and under which the coefficients θ^y of w_y and θ^z of w_z take the form

$$\theta^y = a^1 S_z + \frac{L_z}{L}, \qquad \theta^z = a^2 R_y + \frac{L_y}{L}. \tag{3.26}$$

Since we have

$$0 = \frac{\partial\theta^y}{\partial y} = \frac{\partial\theta^z}{\partial z}, \qquad \frac{\partial\theta^y}{\partial y} = \frac{\partial\theta^z}{\partial z} = \frac{LL_{y,z} - L_y L_z}{L^2}, \tag{3.27}$$

L must satisfy the condition $\partial_y(L_z/L) = 0$, and hence $L = e^{f(y)+g(z)}$, for some arbitrary functions f and g. This new expression for L reduces (3.25) to

$$t = R(y), \qquad x = S(z), \qquad u = e^{f(y)+g(z)}w, \tag{3.28}$$

and the latter change of variables transforms (3.24) into

$$\begin{aligned} &w\Big(f_y\big(g_z + a^1 S_z\big) + R_y\big(a^2 g_z + a^3 S_z\big)\Big) \\ &\quad + \big(g_z + a^1 S_z\big)w_y + \big(f_y + a^2 R_y\big)w_z + w_{y,z} = 0, \end{aligned} \tag{3.29}$$

which is of the prescribed form, and this shows that (3.28) represents the equivalence transformations of (3.24).

For the second example of determination with a variant of (3.9), consider the equation

$$u_{tx} + a^1(t)u_t + a^2(t)u_x + a^3(t,x)u = 0, \tag{3.30}$$

in which the coefficient of u_t and u_x depend on t alone. Here again, the equivalence group is to be sought in the form (3.25), and under this change of variables, the coefficients θ^y of w_y and θ^z of w_z take the form

$$\theta^y = a^1 S_z + \frac{L_z}{L}, \qquad \theta^z = a^2 R_y + \frac{L_y}{L}, \tag{3.31}$$

where $a^1 = a^1(R(y))$ and $a^2 = a^2(R(y))$, and the conditions $0 = \partial_z\theta^y$ and $0 = \partial_z\theta^z$ lead to the differential equations

$$\frac{-L_z^2 + LL_{zz}}{L^2} + a^1 S_{zz} = 0, \qquad \frac{-L_y L_z + LL_{y,z}}{L^2} = 0. \tag{3.32}$$

It follows from the arbitrariness of the coefficient a^1 in the first equation of (3.32) that $S = k_1 z + k_2$ for some constants k_1 and k_2, while

$$-L_z^2 + LL_{zz} = 0. \tag{3.33}$$

The latter equation has solution $L = g(y)e^{zf(y)}$, where $g \neq 0$ and f are arbitrary functions. A substitution of this expression for L in the second equation of (3.32) leads to $gf_y e^{2zf} = 0$, and hence $f = k_3$ is a constant function, so that the corresponding change of variables (3.25) now takes the form

$$t = R(y), \qquad x = k_1 z + k_2, \qquad u = g(y)e^{k_3 z} w. \tag{3.34}$$

The transformation of (3.30) under (3.34) yields the equation

$$\begin{aligned} &\left(a^1 k_1 + \frac{L_z}{L}\right) w_y + \left(a^2 R_y + \frac{L_y}{L}\right) w_z + w_{y,z} \\ &\quad + \frac{w\left(Ak_1 L_y + k_1 L a^3 R_y + a^2 L_z R_y + L_{y,z}\right)}{L} = 0, \end{aligned} \tag{3.35}$$

which is of the required form, and thus (3.34) represents the group of equivalence transformations of (3.30).

3.3. Some Applications

It is worthwhile making some comments at this point on some applications of equivalence transformations with regards to the integration of differential equations. By essence, equivalence transformations are a key tool by means of which equations can be classified and then represented in each equivalence class by a simpler and more tractable canonical form. If we consider, for instance, the variant (3.24) of the linear hyperbolic equation it follows from the form of the transformed equation (3.29) that if in the corresponding equivalence transformations (3.28) we set

$$f = k_1 - \int a^2(R) R_y dy, \qquad g = k_2 - \int a^1(S) S_z dz, \tag{3.36}$$

for some constants k_1 and k_2 and then set $R_y S_z = 1$, then the transformed equation is reduced to

$$b(y,z)w + w_{y,z} = 0, \quad \text{with } b(y,z) = a^3 - a^1 a^2, \tag{3.37}$$

which gives a much simpler canonical form for all equations of the form (3.24). In particular, if in the latter equation we have

$$a^3 = a^1 a^2, \tag{3.38}$$

then (3.24) is always reducible to the linear wave equation

$$w_{y,z} = 0. \tag{3.39}$$

The complete classification of families of equations can be realized by means of invariant functions of these equations, which as already mentioned are defined by the equivalence transformations. For instance, using the functions

$$H = a_t^1 + a^1 a^2 - a^3, \qquad K = a_x^2 + a^1 a^2 - a^3 \tag{3.40}$$

known as Laplace invariants [10], Ovsiannikov obtained the contact invariants $P = H/K$ and $Q = (\ln |H|)_{t,x}/H$ and used them to achieve the classification of a subfamily of the linear hyperbolic equation (3.24) [8, 12]. Similar classifications of differential equations based on invariant functions have also been carried out by other more recent writers on the topic [5, 6, 13, 14]. Using Cartan's equivalence method in the form developed by Fels and Olver [9], Morozov [13] gave a complete solution to the equivalence problem for a number of families of partial differential equations, under the pseudo-group of contact transformations.

Significant simplifications of the equation, which may include a reduction of its order, are often achieved by the simple vanishing of the invariants of the equation. For instance, the condition $a^3 = a^1 a^2$ obtained in (3.38) by a direct analysis of the equivalence transformations of (3.24) and which have led to the reduction of this equation to the linear wave equation $w_{y,z} = 0$, actually corresponds exactly to the vanishing of the invariant $P = H/K$ of this equation. Similar reductions or derivations of integrability properties of differential equations based on the vanishing of invariant functions were carried out by Laguerre [15], followed by Brioschi [16] for third-and fourth-order ODEs, as well as by Liouville [17] and many others.

4. Concluding Remarks

In this paper, we considered two families (A) and (B) of DEs of a similar form depending on arbitrary functions, where the arguments of each of the arbitrary functions in (A) are a subset of that for the corresponding arbitrary function in (B). We then showed in Theorem 2.2 that the equivalence group G^A of (A) must be a subgroup of the equivalence group G^B of (B). We also showed through some examples of applications how the theorem can be used either for an easier determination of equivalence subgroups or just equivalence transformations of a complex family of DEs or for finding the equivalence group G^A of type (A) DEs when G^B is known.

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