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Research Article

New Meir-Keeler Type Tripled Fixed-Point Theorems on Ordered Partial Metric Spaces

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In this paper, we prove some new Meir-Keeler type tripled fixed-point theorems on a partially ordered complete partial metric space. Also, as application, some results of integral type are given.

1. Introduction and Preliminaries

In the last century, the theory of fixed points has appeared as a crucial technique in the study of nonlinear phenomena. Particularly, the tools in fixed-point theory have an application in such diverse fields as biology, chemistry, physics, economics, computer sciences, and engineering.

Recently, fixed-point theorems are considered on partial metric spaces on which self-distance of some points may not be zero. This phenomenon was discovered by Matthews [1] when he considered the tools of metric spaces in the field of semantics and domain theory in computer science (see, e.g., [2, 3]). After the initial results of Mathews, other papers have been released on partial metric spaces (see e.g., [4–20]).

Another important development is reported in fixed-point theory via ordered metric spaces. Fixed-point theorems in ordered sets were discussed by Ran and Reurings [21]. Subsequently, many results in this direction were given (see, e.g., [22–31]).

In this paper, we combine two recent trends, partial metric spaces and ordered sets, and discuss the existence and uniqueness of some new Meir-Keeler type tripled fixed-point theorems in the context of partially ordered partial metric spaces.

Let X be a nonempty set. A partial metric is a function $p : X \times X \to [0, \infty)$ satisfying the following conditions:

($P1$) if $p(x,x) = p(x,y) = p(y,y)$, then $x = y$,

($P2$) $p(x,y) = p(y,x)$,

($P3$) $p(x,x) \leq p(x,y)$,

($P4$) $p(x,z) + p(y,y) \leq p(x,y) + p(y,z)$,

for all $x, y, z \in X$. Then, (X, p) is called a partial metric space.

If p is a partial metric p on X, then the function $d_p : X \times X \to [0, \infty)$ given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y) \tag{1.1}$$

is a metric on X. Each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. Similarly, closed p-ball is defined as $B_p[x,\varepsilon] = \{y \in X : p(x,y) \leq p(x,x) + \varepsilon\}$. For more details, see [1, 5].

Definition 1.1 (see, e.g., [1, 5, 15]). Let (X, p) be a partial metric space.

(i) A sequence $\{x_n\}$ in X converges to $x \in X$ whenever $\lim_{n\to\infty} p(x, x_n) = p(x,x)$.

(ii) A sequence $\{x_n\}$ in X is called Cauchy whenever $\lim_{n,m\to\infty} p(x_n, x_m)$ exists (and finite).

(iii) (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$, that is, $\lim_{n,m\to\infty} p(x_n, x_m) = p(x,x)$.

Lemma 1.2 (see, e.g., [1, 5, 15]). *Let (X, p) be a partial metric space.*

(a) *A sequence $\{x_n\}$ is Cauchy if and only if $\{x_n\}$ is a Cauchy sequence in the metric space (X, d_p).*

(b) *(X, p) is complete if and only if the metric space (X, d_p) is complete. Moreover,*

$$\lim_{n\to\infty} d_p(x, x_n) = 0 \Longleftrightarrow \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m) = p(x,x). \tag{1.2}$$

Lemma 1.3 (see, e.g., [4, 15, 16]). *Let (X, p) be a partial metric space. Then,*

(A) *if $p(x,y) = 0$, then $x = y$,*

(B) *if $x \neq y$, then $p(x,y) > 0$.*

Remark 1.4. If $x = y$, $p(x,y)$ may not be 0.

Lemma 1.5 (see, e.g., [4, 15, 16]). *Let $x_n \to z$ as $n \to \infty$ in a partial metric space (X, p) where $p(z,z) = 0$. Then, $\lim_{n\to\infty} p(x_n, y) = p(z,y)$ for every $y \in X$.*

$(X, p, \leq)$ is called a partially ordered partial metric space if $(X, \leq)$ is a partially ordered set and (X, p) is a partial metric space. Further, if (X, p) is a complete partial metric space, then $(X, p, \leq)$ is called a partially ordered complete partial metric space. Hereafter, we assume that $X \neq \emptyset$ and we use the notation

$$X^k = \underbrace{X \times X \times \cdots \times X}_{k\text{-many}}. \tag{1.3}$$

Also, take the mapping $P : X^3 \times X^3 \to [0, +\infty)$ such that

$$P(\mathbf{x}, \mathbf{y}) := \max\{p(x_1, y_1), p(x_2, y_2), p(x_3, y_3)\}, \tag{1.4}$$

where $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3) \in X^3$.

Let $(X, \leq)$ be a partially ordered set. We consider the following partial order (also denoted by $\leq$) on the product space X^3:

$$(u, v, w) \leq (x, y, z) \quad \text{iff } x \geq u,\ y \leq v,\ z \geq w, \tag{1.5}$$

where $(u, v, w), (x, y, z) \in X^3$. Moreover, we say that (x, y, z) is equal to (u, v, r) if and only if $x = u, = v$, and $z = r$. In the sequel, we need the following definitions.

Definition 1.6 (see [32]). Let $(X, \leq)$ be a partially ordered set and $F : X^3 \to X$ a given mapping. We say that F has the mixed monotone property if $F(x, y, z)$ is monotone nondecreasing in x and z, and it is monotone nonincreasing in y, that is, for any $x, y, z \in X$,

$$\begin{gathered} x_1, x_2 \in X,\ x_1 \leq x_2 \Longrightarrow F(x_1, y, z) \leq F(x_2, y, z), \\ y_1, y_2 \in X,\ y_1 \leq y_2 \Longrightarrow F(x, y_1, z) \geq F(x, y_2, z), \\ z_1, z_2 \in X,\ z_1 \leq z_2 \Longrightarrow F(x, y, z_1) \leq F(x, y, z_2). \end{gathered} \tag{1.6}$$

Definition 1.7 (see [32]). An element $(x, y, z) \in X^3$ is called a tripled fixed point of $F : X^3 \to X$ if

$$F(x, y, z) = x, \qquad F(y, x, y) = y, \qquad F(z, y, x) = z. \tag{1.7}$$

Berinde and Borcut [32] proved the following theorem.

Theorem 1.8. *Let $(X, \leq)$ be a partially ordered set and (X, d) a complete metric space. Let $F : X^3 \to X$ be a mapping having the mixed monotone property on X. Assume that there exist constants $a, b, c \in [0, 1)$ such that $a + b + c < 1$ for which*

$$d(F(x, y, z), F(u, v, w)) \leq ad(x, u) + bd(y, v) + cd(z, w) \tag{1.8}$$

for all $x \geq u$,$y \leq v$ and $z \geq w$. Assume that X has the following properties:

(i) *if a nondecreasing sequence $x_n \to x$, then $x_n \leq x$ for all n,*

(ii) *if a nonincreasing sequence $y_n \to y$, then $y_n \geq y$ for all n.*

If there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq F(x_0, y_0, z_0), \qquad y_0 \geq F(y_0, x_0, y_0), \qquad z_0 \leq F(z_0, y_0, x_0), \tag{1.9}$$

then there exist $x, y, z \in X$ such that

$$F(x, y, z) = x, \qquad F(y, x, y) = y, \qquad F(z, y, x) = z. \tag{1.10}$$

Recently, Theorem 1.8 is extended to cone metric spaces by Rao and Kishore [33]. On the other hand, very recently, Aydi et al. [34] introduced the following concepts.

Definition 1.9 (see [34]). Let $(X, \leq)$ be a partially ordered set and $F : X^3 \to X$. We say that F has the mixed strict monotone property if, for any $x, y, z \in X$,

$$\begin{aligned} &x_1, x_2 \in X, \ x_1 < x_2 \Longrightarrow F(x_1, y, z) < F(x_2, y, z), \\ &y_1, y_2 \in X, \ y_1 < y_2 \Longrightarrow F(x, y_1, z) > F(x, y_2, z), \\ &z_1, z_2 \in X, \ z_1 < z_2 \Longrightarrow F(x, y, z_1) < F(x, y, z_2). \end{aligned} \tag{1.11}$$

Definition 1.10 (see [34]). Let $(X, d, \leq)$ be a partially ordered metric space. A mapping $F : X^3 \to X$ is said to be a generalized Meir-Keeler type contraction if, for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that

$$\varepsilon \leq \max\{d(x,u), d(y,v), d(z,r)\} < \varepsilon + \delta(\varepsilon) \Longrightarrow d(F(x,y,z), F(u,v,r)) < \varepsilon, \tag{1.12}$$

for all $x, y, z, u, v, r \in X$ with $x \leq u$, $y \geq v$ and $z \leq r$.

In the following, we consider the partial case of Definition 1.10 and we introduce the following.

Definition 1.11. Let $(X, p, \leq)$ be a partially ordered partial metric space. A mapping $F : X^3 \to X$ is said to be a generalized p-Meir-Keeler type contraction if, for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that

$$\varepsilon \leq \max\{p(x,u), p(y,v), p(z,r)\} < \varepsilon + \delta(\varepsilon) \Longrightarrow p(F(x,y,z), F(u,v,r)) < \varepsilon, \tag{1.13}$$

for all $x, y, z, u, v, r \in X$ with $x \leq u$, $y \geq v$ and $z \leq r$.

Remark 1.12. It is immediate to show that if $F : X^3 \to X$ is a generalized p-Meir-Keeler type contraction, then

$$p(F(x,y,z), F(u,v,r)) < \max\{p(x,u), p(y,v), p(z,r)\} \tag{1.14}$$

for all $x, u, y, v, z, r, \in X$ with $x < u, y \geq v, z < r$ or $x \leq u, y > v, z \leq r$.

Proposition 1.13. *Let $(X, p, \leq)$ be a partially ordered partial metric space and $F : X^3 \to X$ a given mapping. If (1.8) is satisfied, then F is a generalized p-Meir-Keeler type function.*

Proof. Assume that (1.8) is satisfied. For all $\varepsilon > 0$, one can check that (1.13) is satisfied with $\delta(\varepsilon) = (1/(a+b+c) - 1)\varepsilon$. □

In the sequel, we use the following notations given in [34]. Let $\widetilde{F} : X^3 \to X^3$ be such that, for $a, b, c \in X$,

$$\widetilde{F}(a,b,c) = (F(a,b,c), F(b,a,b), F(c,b,a)). \tag{1.15}$$

Let $x_0, y_0, z_0 \in X$ be such that

$$x_0 < F(x_0, y_0, z_0), \qquad y_0 \geq F(y_0, x_0, y_0), \qquad z_0 < F(z_0, y_0, x_0). \tag{1.16}$$

We consider sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ such that

$$\underbrace{\begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix}}_{A_n} = \underbrace{\begin{bmatrix} F(x_{n-1}, y_{n-1}, z_{n-1}) \\ F(y_{n-1}, x_{n-1}, y_{n-1}) \\ F(z_{n-1}, y_{n-1}, x_{n-1}) \end{bmatrix}}_{\widetilde{F}(A_{n-1})} = \underbrace{\begin{bmatrix} F^n(x_0, y_0, z_0) \\ F^n(y_0, x_0, y_0) \\ F^n(z_0, y_0, x_0) \end{bmatrix}}_{\widetilde{F^n}(A_0)}, \tag{1.17}$$

for $n = 1, 2, 3, \ldots$.

Our first auxiliary result is as follows.

Proposition 1.14. *Let $(X, p, \leq)$be a partially ordered partial metric space, and let$F : X^3 \to X$be a given mapping such that the following hypotheses hold:*

(i) *F has the mixed strict monotone property,*

(ii) *F is a generalized p-Meir-Keeler type function,*

(iii) *$\exists (x, y, z), (u, v, r) \in X^3$ such that $x < u$, $y \geq v$ and $z < r$.*

Then,

$$P\left(\widetilde{F^n}(x, y, z), \widetilde{F^n}(u, v, r)\right) \longrightarrow 0, \quad \text{as } n \longrightarrow +\infty. \tag{1.18}$$

Proof. Let $(x, y, z) = (x_0, y_0, z_0)$ and $(u, v, r) = (u_0, v_0, r_0)$. We show that

$$\begin{aligned} x_n &= F^n(x_0, y_0, z_0,) < F^n(u_0, v_0, r_0) = u_n, \\ y_n &= F^n(y_0, x_0, y_0) > F^n(v_0, u_0, v_0) = v_n, \quad \forall n = 1, 2, \ldots, \\ z_n &= F^n(z_0, y_0, x_0) < F^n(r_0, v_0, u_0) = r_n, \end{aligned} \tag{1.19}$$

with $F = F^1$.

Due to the fact that F has the mixed strict monotone property, together with the assumption that $x < u$, $y \geq v$ and $z < r$, we obtain

$$\begin{aligned} x_1 = F(x, y, z) &= F(x_0, y_0, z_0) < F(u_0, y_0, z_0) \\ &\Longrightarrow F(x_0, y_0, z_0) < F(u_0, v_0, z_0) \\ &\Longrightarrow F(x_0, y_0, z_0) < F(u_0, v_0, r_0) = u_1. \end{aligned} \tag{1.20}$$

Analogously, we have

$$y_1 = F(y_0, x_0, y_0) > F(v_0, u_0, v_0) = v_1, \qquad z_1 = F(z_0, y_0, x_0) < F(r_0, v_0, u_0) = r_1. \tag{1.21}$$

Thus, (1.19) holds for $n = 1$. By using the same arguments, we show that (1.19) holds also for $n = 2$. In fact,

$$\begin{aligned}
x_2 &= F^2(x_0, y_0, z_0) = F(x_1, y_1, z_1) \\
&= F(F(x_0, y_0, z_0), F(y_0, x_0, y_0), F(z_0, y_0, x_0)) \\
&< F(F(u_0, v_0, r_0), F(y_0, x_0, y_0), F(z_0, y_0, x_0)) \\
&< F(F(u_0, v_0, r_0), F(v_0, u_0, v_0), F(z_0, y_0, x_0)) \\
&< F(F(u_0, v_0, r_0), F(v_0, u_0, v_0), F(r_0, v_0, u_0)) \\
&= F^2(u_0, v_0, r_0) = F(u_1, v_1, r_1) = u_2.
\end{aligned} \tag{1.22}$$

Similarly, we find

$$y_2 = F^2(y_0, x_0, y_0) \geq F^2(v_0, u_0, v_0) = v_2, \qquad z_2 = F^2(z_0, y_0, x_0) < F^2(r_0, v_0, u_0) = r_2. \tag{1.23}$$

Inductively, we get that (1.19) holds.

By Remark 1.12, together with (1.19), we have

$$\begin{aligned}
p(x_{n+2}, u_{n+2}) &= p\Big(F^{n+2}(x_0, y_0, z_0), F^{n+2}(u_0, v_0, r_0)\Big) \\
&= p(F(x_{n+1}, y_{n+1}, z_{n+1}), F(u_{n+1}, v_{n+1}, r_{n+1})) \\
&< \max\{p(x_{n+1}, u_{n+1}), p(y_{n+1}, v_{n+1}), p(z_{n+1}, r_{n+1})\},
\end{aligned} \tag{1.24}$$

$$\begin{aligned}
p(z_{n+2}, r_{n+2}) &= p\Big(F^{n+2}(z_0, y_0, x_0), F^{n+2}(r_0, v_0, u_0)\Big) \\
&= p(F(z_{n+1}, y_{n+1}, x_{n+1}), F(r_{n+1}, v_{n+1}, u_{n+1})) \\
&< \max\{p(z_{n+1}, r_{n+1}), p(y_{n+1}, v_{n+1}), p(x_{n+1}, u_{n+1})\},
\end{aligned} \tag{1.25}$$

$$\begin{aligned}
p(y_{n+2}, v_{n+2}) &= p\Big(F^{n+2}(y_0, x_0, y_0), F^{n+2}(v_0, u_0, v_0)\Big) \\
&= p(F(y_{n+1}, x_{n+1}, y_{n+1}), F(v_{n+1}, u_{n+1}, v_{n+1})) \\
&< \max\{p(y_{n+1}, v_{n+1}), p(x_{n+1}, u_{n+1}), p(y_{n+1}, v_{n+1})\} \\
&\leq \max\{p(z_{n+1}, r_{n+1}), p(y_{n+1}, v_{n+1}), p(x_{n+1}, u_{n+1})\}.
\end{aligned} \tag{1.26}$$

Let $\Delta_{n+1} := \max\{p(x_{n+1}, u_{n+1}), p(y_{n+1}, v_{n+1}), p(z_{n+1}, r_{n+1})\}$. Combining (1.24)–(1.26), we get

$$\Delta_{n+2} < \Delta_{n+1}, \quad \forall n = 1, 2 \ldots. \tag{1.27}$$

If we denote $B_n = (u_n, v_n, r_n)$, then, by definition of the partial metric P and (1.27), we have

$$P(A_{n+2}, B_{n+2}) < P(A_{n+1}, B_{n+1}). \tag{1.28}$$

Consequently, the sequence $\{t_n\} = \{P(A_n, B_n)\}$ is decreasing. Hence, $\{t_n\}$ converges, say to $\varepsilon \geq 0$. Clearly, if $\varepsilon = 0$, we have finished. Suppose, on the contrary, $\varepsilon > 0$. Thus, there exists $k \in \{1, 2, \ldots\}$ such that

$$\varepsilon \leq t_n = P(A_n, B_n) < \varepsilon + \delta(\varepsilon) \quad \text{for any } n \geq k. \tag{1.29}$$

In particular, for $n = k$, we have

$$\varepsilon \leq t_k = P(A_k, B_k) < \varepsilon + \delta(\varepsilon), \tag{1.30}$$

that is equal to

$$\varepsilon \leq \Delta_k < \varepsilon + \delta(\varepsilon). \tag{1.31}$$

It follows from (1.19) and the hypothesis (ii) that

$$p\big(F(x_k, y_k, z_k), F(u_k, v_k, r_k)\big) < \varepsilon \tag{1.32}$$

which is equivalent to

$$p(x_{k+1}, u_{k+1}) < \varepsilon. \tag{1.33}$$

Moreover, we have

$$p\big(y_{k+1}, v_{k+1}\big) < \varepsilon, \qquad p(z_{k+1}, r_{k+1}) < \varepsilon. \tag{1.34}$$

Combining (1.33) and (1.34), we have

$$\Delta_{k+1} < \varepsilon. \tag{1.35}$$

Thus, $t_{k+1} = P(A_{k+1}, B_{k+1}) < \varepsilon$ which is a contradiction with respect to (1.29), and so $\varepsilon = 0$. We conclude that

$$P(A_n, B_n) = P\Big(\widetilde{F^n}(x, y, z), \widetilde{F^n}(u, v, r)\Big) \longrightarrow 0, \quad \text{as } n \longrightarrow +\infty. \tag{1.36}$$

□

Remark 1.15. The previous proposition remains true if, in (iii), we change the assumption

$$\exists (x, y, z), (u, v, r) \in X^3 \quad \text{such that } x < u, y \geq v, z < r \tag{1.37}$$

with the following

$$\exists (x, y, z), (u, v, r) \in X^3 \quad \text{such that } x \leq u,\ y > v,\ z \leq r. \tag{1.38}$$

2. Existence of Tripled Fixed Point

The following theorem is our first main result.

Theorem 2.1. *Let $(X, p, \leq)$ be a partially ordered complete partial metric space. Suppose that X has the following properties:*

(a) *if $\{x_n\}$ is a sequence such that $x_{n+1} > x_n$ for each $n = 1, 2, \ldots$ and $x_n \to x$, then $x_n < x$ for each $n = 1, 2, \ldots$,*

(b) *if $\{y_n\}$ is a sequence such that $y_{n+1} < y_n$ for each $n = 1, 2, \ldots$ and $y_n \to y$, then $y_n > y$ for each $n = 1, 2, \ldots$.*

Assume that $F : X^3 \to X$ satisfies the following hypotheses:

(i) *F has the mixed strict monotone property,*

(ii) *F is a generalized p-Meir-Keeler type function,*

(iii) *there exist $x_0, y_0, z_0 \in X$ such that*

$$x_0 < F(x_0, y_0, z_0), \qquad y_0 \geq F(y_0, x_0, y_0), \qquad z_0 < F(z_0, y_0, x_0,). \tag{2.1}$$

Then, F has a tripled fixed point, that is, there exist $x, y, z \in X$ such that

$$F(x, y, z) = x, \qquad F(y, x, y) = y, \qquad F(z, y, x) = z. \tag{2.2}$$

Also, $p(x, x) = p(y, y) = p(z, z) = 0$.

Proof. Let $x_0, y_0, z_0 \in X$ be as in (iii). We construct sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ according to (1.17).

We claim that, for all $n \geq 2$, we have

$$\begin{gathered} \cdots > x_n > x_{n-1} > \cdots > x_1 > x_0, \\ \cdots < y_n < y_{n-1} < \cdots < y_1 \leq y_0, \\ \cdots > z_n > z_{n-1} > \cdots > z_1 > z_0. \end{gathered} \tag{2.3}$$

Indeed, we will use a mathematical induction to prove (2.3). Clearly, we have

$$x_0 < F(x_0, y_0, z_0) = x_1, \qquad y_0 \geq F(y_0, x_0, y_0) = y_1, \qquad z_0 < F(z_0, y_0, x_0) = z_1. \tag{2.4}$$

Suppose now that the inequalities in (2.3) hold for some $n \geq 2$. By the mixed strict monotone property of F, together with (1.17), we have

$$\begin{aligned} x_n &= F(x_{n-1}, y_{n-1}, z_{n-1}) < F(x_n, y_n, z_n) = x_{n+1}, \\ y_n &= F(y_{n-1}, x_{n-1}, y_{n-1}) > F(y_n, x_n, y_n) = y_{n+1}, \\ z_n &= F(z_{n-1}, y_{n-1}, x_{n-1}) < F(z_n, y_n, x_n) = z_{n+1}. \end{aligned} \tag{2.5}$$

Thus, (2.3) holds for all $n \geq 2$.

Putting $(x,y,z)=A_0$ and $(u,v,r)=A_1$ and by Proposition 1.14, we get

$$P\left(\widetilde{F^n}(A_0),\widetilde{F^n}(A_1)\right)\longrightarrow 0,\quad \text{as } n\longrightarrow+\infty, \tag{2.6}$$

which is equivalent to

$$P(A_n,A_{n+1})\longrightarrow 0,\quad \text{as } n\longrightarrow+\infty. \tag{2.7}$$

Take an arbitrary $\varepsilon>0$. It follows from (2.7) that there exists $k\in\mathbb{N}$ such that

$$P(A_k,A_{k+1})<\delta(\varepsilon). \tag{2.8}$$

Without loss of the generality, assume that $\delta(\varepsilon)\leq\varepsilon$ and define the following set:

$$\Pi:=\left\{A=(x,y,z)\in X^3:P\left(\widetilde{F}(A_k),\widetilde{F}(A)\right)<\varepsilon+\delta(\varepsilon),\ x>x_k,y\leq y_k,z>z_k\right\}. \tag{2.9}$$

We claim that

$$\widetilde{F}(A)\in\Pi\quad\forall A\in\Pi. \tag{2.10}$$

Take $A\in\Pi$. Then, by (2.8) and the triangle inequality (which still holds for partial metrics), we have

$$\begin{aligned}
P\left(A_k,\widetilde{F}(A)\right)&=\max\{p(x_k,F(x,y,z)),p(y_k,F(y,x,y)),p(z_k,F(z,y,x))\}\\
&\leq\max\{p(x_k,x_{k+1})+p(x_{k+1},F(x,y,z)),p(y_k,y_{k+1})\\
&\qquad+p(y_{k+1},F(y,x,y)),p(z_k,z_{k+1})+p(z_{k+1},F(z,y,x))\}\\
&=\max\{p(x_k,x_{k+1})+p(F(x_k,y_k,z_k),F(x,y,z)),p(y_k,y_{k+1})\\
&\qquad+p(F(y_k,x_k,y_k),F(y,x,y)),p(z_k,z_{k+1})+p(F(z_k,y_k,x_k),F(z,y,x))\}\\
&\leq P(A_k,A_{k+1})+P\left(\widetilde{F}(A_k),\widetilde{F}(A)\right)\\
&<\delta(\varepsilon)+P\left(\widetilde{F}(A_k),\widetilde{F}(A)\right).
\end{aligned} \tag{2.11}$$

We consider the following two cases.

Case 1 ($P(A_k,A)\leq\varepsilon$). By Remark 1.12 and the definition of Π, the inequality (2.11) turns into

$$\begin{aligned}
P\left(A_k,\widetilde{F}(A)\right)&<\delta(\varepsilon)+P\left(\widetilde{F}(A_k),\widetilde{F}(A)\right)\\
&<\delta(\varepsilon)+P(A_k,A)\\
&<\delta(\varepsilon)+\varepsilon.
\end{aligned} \tag{2.12}$$

Case 2 ($\varepsilon < P(A_k, A) < \delta(\varepsilon) + \varepsilon$). That is,

$$\varepsilon < \max\{p(x, x_k), p(y, y_k), p(z, z_k)\} < \delta(\varepsilon) + \varepsilon. \tag{2.13}$$

Since $x > x_k, z > z_k, y \leq y_k$, then, by (ii), we have

$$\begin{aligned}
&p(F(x, y, z), F(x_k, y_k, z_k)) < \varepsilon, \\
&p(F(y, x, y), F(y_k, x_k, y_k)) < \varepsilon, \\
&p(F(z, y, x), F(z_k, y_k, x_k)) < \varepsilon.
\end{aligned} \tag{2.14}$$

Hence, combining (2.14) and (2.11), we get

$$P\Big(A_k, \widetilde{F}(A)\Big) < \delta(\varepsilon) + \varepsilon. \tag{2.15}$$

On the other hand, using (i), one can easily check that

$$F(x, y, z) > x_k, \qquad F(y, x, y) \leq y_k, \qquad F(z, y, x) > z_k. \tag{2.16}$$

Hence, we conclude that (2.10) holds. By (2.8), we have that $A_{k+1} \in \Pi$, and so, by (2.10) we get

$$\begin{aligned}
A_{k+1} \in \Pi &\Longrightarrow \widetilde{F}(A_{k+1}) = A_{k+2} \in \Pi \\
&\Longrightarrow \widetilde{F}(A_{k+2}) = A_{k+3} \in \Pi \\
&\cdots \\
&\Longrightarrow A_n \in \Pi \quad \forall n > k.
\end{aligned} \tag{2.17}$$

Then, for all $n, m > k$, we have

$$P(A_n, A_m) \leq P(A_n, A_k) + P(A_k, A_m) < 2(\varepsilon + \delta(\varepsilon)) \leq 4\varepsilon. \tag{2.18}$$

By definition of P, we have

$$\lim_{n,m \to \infty} p(x_n, x_m) = \lim_{n,m \to \infty} p(y_n, y_m) = \lim_{n,m \to \infty} p(z_n, z_m) = 0. \tag{2.19}$$

Consequently, by definition of the metric d_p, $d_p(x, y) \leq 2p(x, y)$, so we get

$$\lim_{n,m \to \infty} d_p(x_n, x_m) = \lim_{n,m \to \infty} d_p(y_n, y_m) = \lim_{n,m \to \infty} d_p(z_n, z_m) = 0. \tag{2.20}$$

Therefore, $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are Cauchy sequences in the metric space (X, d_p). Since (X, p) is a complete partial metric space, then, by Lemma 1.2, (X, d_p) is also a complete metric space. Hence, there exists a point $(x, y, z) \in X^3$ such that

$$d_p(x_n, x), d_p(y_n, y), d_p(z_n, z) \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty. \tag{2.21}$$

Again, by Lemma 1.2 and (2.19), we obtain

$$\begin{aligned} p(x, x) &= \lim_{n \to \infty} p(x_n, x) = \lim_{n,m \to \infty} p(x_n, x_m) = 0, \\ p(y, y) &= \lim_{n \to \infty} p(y_n, y) = \lim_{n,m \to \infty} p(y_n, y_m) = 0, \\ p(z, z) &= \lim_{n \to \infty} p(z_n, z) = \lim_{n,m \to \infty} p(z_n, z_m) = 0. \end{aligned} \tag{2.22}$$

We will prove that

$$F(x, y, z) = x, \qquad F(y, x, y) = y, \qquad F(z, y, x) = z. \tag{2.23}$$

To this aim, take an arbitrary $\varepsilon > 0$. Since

$$x_n = F^n(x_0, y_0, z_0) \longrightarrow x, \qquad y_n = F^n(y_0, x_0, y_0) \longrightarrow y, \qquad z_n = F^n(z_0, y_0, x_0) \longrightarrow z, \tag{2.24}$$

then there exist $n_1, n_2, n_3 \in \mathbb{N}$ such that by (2.22)

$$\begin{aligned} p(x_l, x) &= p\left(F^l(x_0, y_0, z_0), x\right) < p(x, x) + \varepsilon = \varepsilon, \\ p(y_q, y) &= p\left(F^q(y_0, x_0, y_0), y\right) < p(y, y) + \varepsilon = \varepsilon, \\ p(z_r, z) &= p\left(F^r(z_0, y_0, x_0), z\right) < p(z, z) + \varepsilon = \varepsilon, \end{aligned} \tag{2.25}$$

for all $l \geq n_1$, $q \geq n_2$, $r \geq n_3$. Now, taking $n = \max\{n_1, n_2, n_3\}$ and using Remark 1.12 with the assumption

$$x_n = F^n(x_0, y_0, z_0) < x, \qquad y_n = F^n(y_0, x_0, y_0) > y, \qquad z_n = F^n(z_0, y_0, x_0) < z, \tag{2.26}$$

by (2.25), we get

$$\begin{aligned} p(x, F(x, y, z)) &\leq p(x, x_{n+1}) + p(x_{n+1}, F(x, y, z)) \\ &= p(x, x_{n+1}) + p\left(F^{n+1}(x_0, y_0, z_0), F(x, y, z)\right) \\ &= p(x, x_{n+1}) + p(F(x_n, y_n, z_n), F(x, y, z)) \\ &< p(x, x_{n+1}) + \max\{p(x_n, x), p(y_n, y), p(z_n, z)\} \\ &< 2\varepsilon. \end{aligned} \tag{2.27}$$

Analogously, we get that

$$p(y, F(y, x, y)) < 2\varepsilon, \qquad p(z, F(z, y, x)) < 2\varepsilon, \tag{2.28}$$

which yield that

$$F(x, y, z) = x, \qquad F(y, x, y) = y, \qquad F(z, y, x) = z. \tag{2.29}$$

□

Remark 2.2. Theorem 2.1 remains true if we replace (iv) with one of the following statements.

There exist $x_0, y_0, z_0 \in X$ such that

$$\begin{aligned}
&(1) \begin{cases} x_0 \leq F(x_0, y_0, z_0), \\ y_0 > F(y_0, x_0, y_0), \\ z_0 < F(z_0, y_0, x_0), \end{cases} \\
&(2) \begin{cases} x_0 \leq F(x_0, y_0, z_0), \\ y_0 > F(y_0, x_0, y_0), \\ z_0 \leq F(z_0, y_0, x_0), \end{cases} \\
&(3) \begin{cases} x_0 \leq F(x_0, y_0, z_0), \\ y_0 \geq F(y_0, x_0, y_0), \\ z_0 < F(z_0, y_0, x_0), \end{cases} \\
&(4) \begin{cases} x_0 < F(x_0, y_0, z_0), \\ y_0 \geq F(y_0, x_0, y_0), \\ z_0 \leq F(z_0, y_0, x_0), \end{cases} \\
&(5) \begin{cases} x_0 < F(x_0, y_0, z_0), \\ y_0 > F(y_0, x_0, y_0), \\ z_0 \leq F(z_0, y_0, x_0). \end{cases}
\end{aligned} \tag{2.30}$$

3. Uniqueness of Tripled Fixed Point

In this section, we will prove the uniqueness of the tripled fixed point.

Theorem 3.1. *In addition to hypotheses of Theorem 2.1, assume that, for all $(x, y, z), (u, v, r) \in X^3$, there exists $(a, b, c) \in X^3$ that is comparable to (x, y, z) and (u, v, r). Then, F has a unique tripled fixed point.*

Proof. The set of tripled fixed points of F is not empty due to Theorem 2.1. We suppose that $A = (x, y, z), A^* = (x^*, y^*, z^*) \in X^3$ are two tripled fixed points of F. We distinguish the following two cases.

Case 1. (x, y, z) is comparable to (x^*, y^*, z^*) with respect to the ordering in X^3, where

$$\lim_{n \to +\infty} p(F^n(x_0, y_0, z_0), x) = p(x, x) = 0,$$
$$\lim_{n \to +\infty} P(F^n(y_0, x_0, y_0), y) = p(y, y) = 0, \tag{3.1}$$
$$\lim_{n \to +\infty} p(F^n(z_0, y_0, x_0), z) = p(z, z) = 0.$$

Without loss of the generality, we may assume that

$$x = F(x, y, z) < F(x^*, y^*, z^*) = x^*,$$
$$y = F(y, x, y) \geq F(y^*, x^*, y^*) = y^*, \tag{3.2}$$
$$z = F(z, y, x) < F(z^*, y^*, x^*) = z^*.$$

By this, definition of P, Lemma 1.3, and Remark 1.12, we have

$$\begin{aligned} 0 < P(A, A^*) &= P((x, y, z), (x^*, y^*, z^*)) \\ &= \max\{p(x, x^*), p(y, y^*), p(z, z^*)\} \\ &= \max\{p(F(x, y, z), F(x^*, y^*, z^*)), p(F(y, x, y), F(y^*, x^*, y^*)), \\ &\qquad p(F(z, y, x), F(z^*, y^*, x^*))\} \\ &< \max\{p(x, x^*), p(y, y^*), p(z, z^*)\} = P(A, A^*), \end{aligned} \tag{3.3}$$

which is a contradiction and therefore must be $A = A^*$.

Case 2. (x, y, z) is not comparable to (x^*, y^*, z^*). By assumption, there exists $B = (a, b, c) \in X^3$ which is comparable to both A and A^*. Without loss of the generality, we may assume that

$$\begin{aligned} x = F(x, y, z) &< a, & F(x^*, y^*, z^*) &= x^* < a, \\ y = F(y, x, y) &\geq b, & F(y^*, x^*, y^*) &= y^* \geq b, \\ z = F(z, y, x) &< c, & F(z^*, y^*, x^*) &= z^* < c. \end{aligned} \tag{3.4}$$

From Proposition 1.14 and (3.4), we have

$$\begin{aligned} \lim_{n \to +\infty} P\left(\widetilde{F^n}(A), \widetilde{F^n}(B)\right) &= 0, \\ \lim_{n \to +\infty} P\left(\widetilde{F^n}(A^*), \widetilde{F^n}(B)\right) &= 0. \end{aligned} \tag{3.5}$$

By triangle inequality, we derive

$$\begin{aligned} P(A, A^*) &= \lim_{n\to+\infty} P\left(\widetilde{F^n}(A), \widetilde{F^n}(A^*)\right) \\ &\le \lim_{n\to+\infty} P\left(\widetilde{F^n}(A), \widetilde{F^n}(B)\right) + \lim_{n\to+\infty} P\left(\widetilde{F^n}(B), \widetilde{F^n}(A^*)\right) = 0. \end{aligned} \tag{3.6}$$

By Lemma 1.3, we get $A = A^*$. □

4. Results of Integral Type

Motivated by Suzuki [35] and on the same lines of [31, Theorem 3.1], one can prove the following result.

Theorem 4.1. *Let $(X, p, \le)$ be a partially ordered complete partial metric space, and let $F : X^3 \to X$ be a given mapping. Assume that there exists a function θ from $[0, +\infty)$ into itself satisfying the following:*

(I) $\theta(0) = 0$ *and* $\theta(t) > 0$ *for every* $t > 0$,

(II) θ *is nondecreasing and right continuous,*

(III) *for every* $\varepsilon > 0$, *there exists* $\delta(\varepsilon) > 0$ *such that*

$$\varepsilon \le \theta(\max\{p(x,u), p(y,v), p(z,r)\}) < \varepsilon + \delta(\varepsilon) \Longrightarrow \theta(p(F(x,y,z), F(u,v,r))) < \varepsilon, \tag{4.1}$$

for all $x \ge u, y \le v$ *and* $z \ge r$.

Then, F is a generalized p-Meir-Keeler type function.

The following result is an immediate consequence of Theorems 2.1 and 4.1.

Corollary 4.2. *Let $(X, p, \le)$ be a partially ordered complete partial metric space $F : X^3 \to X$ be a mapping satisfying the following hypotheses:*

(i) *F has the mixed strict monotone property,*

(ii) *for every* $\varepsilon > 0$, *there exists* $\delta(\varepsilon) > 0$ *such that*

$$\varepsilon \le \int_0^{\max\{p(x,u),p(y,v),p(z,r)\}} \phi(t)dt < \varepsilon + \delta(\varepsilon) \Longrightarrow \int_0^{p(F(x,y,z),F(u,v,r))} \phi(t)dt < \varepsilon, \tag{4.2}$$

for all $x \ge u, y \le v$ *and* $z \ge r$, *where* $\phi : [0, +\infty) \to [0, +\infty)$ *is a locally integrable function satisfying* $\int_0^s \phi(t)dt > 0$ *for all* $s > 0$,

(iii) *there exist* $x_0, y_0, z_0 \in X$ *such that*

$$x_0 < F(x_0, y_0, z_0), \qquad y_0 \ge F(y_0, x_0, y_0), \qquad z_0 < F(z_0, y_0, x_0). \tag{4.3}$$

Assume that the hypotheses (a) and (b) given in Theorem 2.1 hold. Then, F has a tripled fixed point.

To end this paper, we give the following corollary.

Corollary 4.3. *Let $(X, d, \leq)$ be a partially ordered complete partial metric space $F : X^3 \to X$ be a mapping satisfying the following hypotheses:*

(i) *F has the mixed strict monotone property,*

(ii) *for all, $x \geq u$, $y \leq v$ and $z \geq r$,*

$$\int_0^{p(F(x,y,z),F(u,v,r))} \phi(t)dt \leq k \int_0^{\max\{p(x,u),p(y,v),p(z,r)\}} \phi(t)dt, \tag{4.4}$$

where $k \in (0,1)$ and ϕ is a locally integrable function from $[0, +\infty)$ into itself satisfying $\int_0^s \phi(t)dt > 0$ for all $s > 0$,

(iii) *there exist $x_0, y_0, z_0 \in X$ such that*

$$x_0 < F(x_0, y_0, z_0), \qquad y_0 \geq F(y_0, x_0, y_0), \qquad z_0 < F(z_0, y_0, x_0). \tag{4.5}$$

Assume that the hypotheses (a) and (b) of Theorem 2.1 hold. Then, F has a tripled fixed point.

Proof. For all $\varepsilon > 0$, we take $\delta(\varepsilon) = (1/k - 1)\varepsilon$ and we apply Corollary 4.2. □

Remark 4.4. By taking $\phi(t) = 1$, we retrieve the analogous of Theorem 1.8 of Berinde and Borcut on ordered partial metric spaces (with $a = b = c = k/3$). In fact, assume that (1.8) holds for $a = b = c = k/3$, that is,

$$p(F(x,y,z), F(u,v,w)) \leq \frac{k}{3}(p(x,u) + p(y,v) + p(z,w)) \tag{4.6}$$

for all $x \geq u$, $y \leq v$, $z \geq w$. From this inequality, we get that

$$p(F(x,y,z), F(u,v,w)) \leq k \max\{p(x,u), p(y,v), p(z,w)\}, \tag{4.7}$$

which corresponds to (4.4) with $\phi(t) = 1$. Then, we may apply Corollary 4.3.

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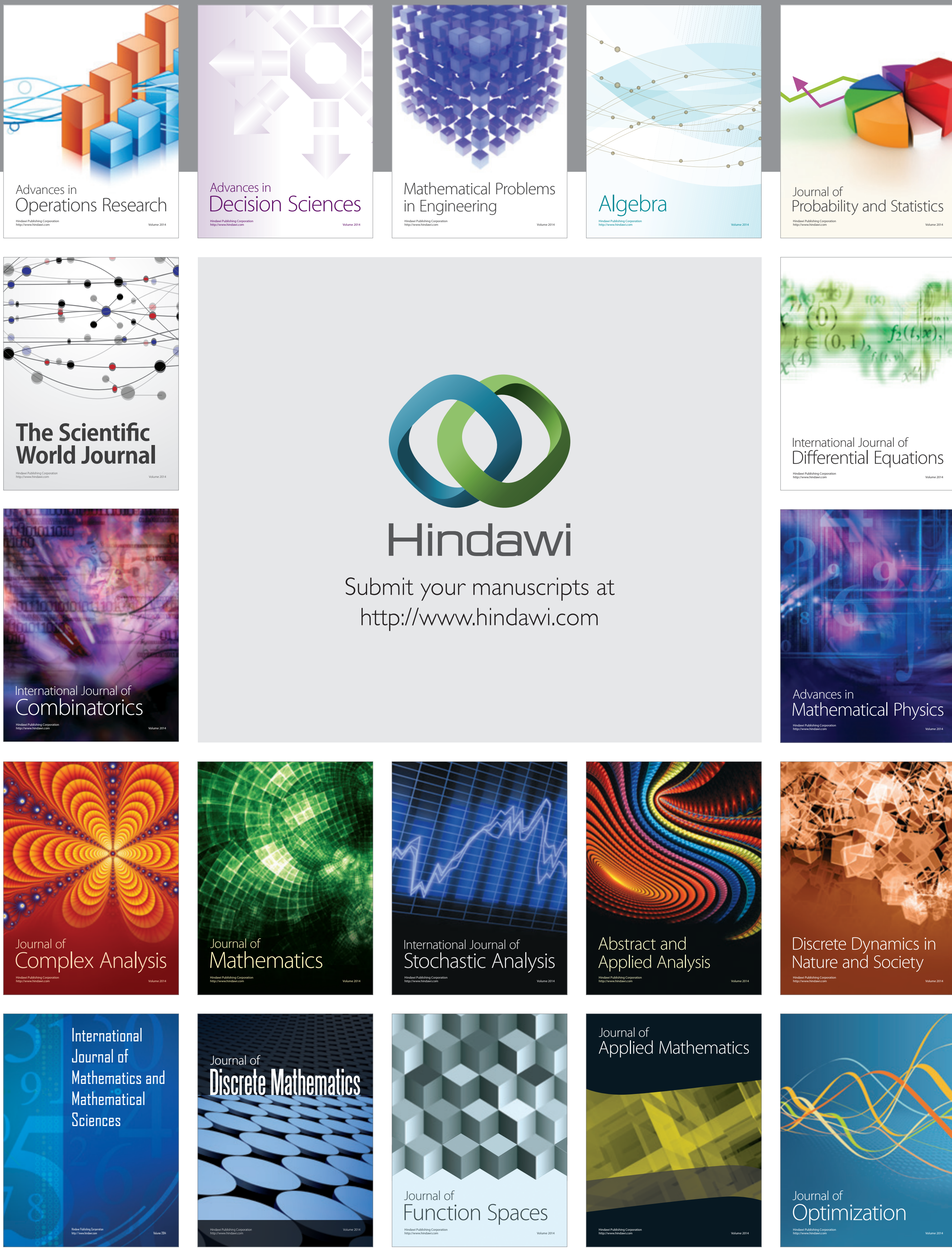
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