

## Research Article

# Modeling Erlang's Ideal Grading with Multirate BPP Traffic

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This paper presents a complete methodology for modeling gradings (also called non-full-availability groups) servicing single-service and multi-service traffic streams. The methodology worked out by the authors makes it possible to determine traffic characteristics of various types of gradings with state-dependent call arrival processes, including a new proposed structure of the Erlang's Ideal Grading with the multirate links. The elaborated models of the gradings can be used for modeling different systems of modern networks, for example, the radio interfaces of the UMTS system, switching networks carrying a mixture of different multirate traffic streams, and video-on-demand systems. The results of the analytical calculations are compared with the results of the simulation data for selected gradings, which confirm high accuracy of the proposed methodology.

## 1. Introduction

Models of state-dependent systems are one of the most frequently considered models in traffic theory. Two types of dependencies between the processes occurring in communications systems and the occupancy states of the systems can be distinguished. One is the dependence between the call admission process of new calls for service and the occupancy state. This dependence can be the effect of the structure of a system (e.g., a grading [1], a limited-availability group [2]) or, alternatively, can result from an adopted particular admission strategy of new calls (e.g., a system with bandwidth reservation [3] or a threshold system [4, 5]). In the other type of the dependence, a dependence between the call arrival process of new calls to the system and the occupancy state of the system takes place. A dependence of this type is to be found most frequently in systems with a limited number of traffic sources, that is, for instance, in systems with Engset or Pascal traffic streams [6–8].

One of the first multiservice systems with state-dependent call admission process to be investigated was systems with bandwidth reservation [9–11]. The studies carried out at the time dealt both with systems with state-independent call arrival process [9, 11, 12] and with state-dependent call arrival process [3]. Parallely, research studies on a model of a group of links servicing jointly multirate traffic streams were also conducted, that is, on the model of the so-called limited-availability group [2, 3]. Recently, along with the introduction of wireless multi-service systems (e.g., UMTS), works on systems that offer a possibility of dynamic adjustment of allocated resources to calls depending on the occupancy state of the system, that is, threshold systems [3, 4, 13], systems with compression [14, 15], and systems with priorities [16–19], have become particularly significant.

The studies on modeling state-dependent multi-service systems carried out hitherto have not, however, included so far one of the most basic models of telecommunications systems with state-dependent call admission process and call arrival process, that is, the model of a grading with multi-service traffic streams generated by Engset and Pascal traffic sources [8]. In the model, the dependence between the call admission process and the occupancy state of the system results from a particular structure of the system. Gradings, also known as non-full-availability groups [1], with single-rate traffic were in exchanges of telecommunications networks until the end of the 1980s. With the introduction of electronic exchanges, the groups of this type were stopped being used in their direct form though were continued (and still are) to be used in analytical models of more complex systems, such as, for example, multi-service switching networks, 3G/4G cellular systems and Video-on-Demand (VoD) systems. In models of switching networks, calculations of the blocking probability in multirate switching networks come down to calculations of this probability in a single-rate system, that is, in a grading [20]. In the case of the 3G/4G mobile systems the models of non full-availability systems with multi-service traffic can be applied for modeling the so-called soft capacity [21]. In these systems the value of interference can be directly modeled by the appropriate value of the so-called availability parameter of the grading. An example of a non-full-availability systems is also VoD systems. In brief, it is composed of disks containing offered films. The non full-availability of such a systems results from the fact that not every film is stored on each disk [22–24].

One of the basic structures of gradings is the so-called Erlang's Ideal Grading (EIG) with single-service call stream. This structure and its first analytical model were presented by Erlang [25]. The first model of Erlang's Ideal Grading with multi-service traffic and identical value (for all traffic classes serviced by the group) of the availability parameter has been proposed in [20]. In [26], a model of a group with multirate (multiservice) traffic and a variable value of the availability parameter has been proposed. In the latter work, a model of the EIG has been used to model systems with bandwidth reservation (with identical capacity and the structure of offered traffic). Then, the model has been expanded to include a possibility to carry on with calculations for noninteger values of the availability parameter [27]. On the basis of this model two application models are proposed: for calculations of blocking probability in a Video-on-Demand system [28] and for calculations of blocking probability in packet networks implementing DiffServ architecture [29]. The authors are currently investigating use of the generalized model of Erlang's Ideal Grading to model radio interfaces that have to accommodate interference [30].

The present paper aims at summing up the results obtained in the study on gradings and at working out a coherent and uniform methodology for modeling these groups, for different availability parameters and for any streams of offered traffic, both single- and multi-service.

The remaining part of the paper is organized as follows. Section 2 presents the most important information on state-dependent systems. Section 3 discusses the known analytical models of gradings with PCT1 traffic (Pure Chance Traffic of Type 1) and PCT2 (Pure Chance Traffic of Type 2) [7]. (In traffic theory, traffic offered and generated by infinite number of traffic sources (Poisson-type call streams) is defined as PCT1 (*Pure Chance Traffic of Type 1*), whereas the model of a system that services such streams—with the assumption of an exponential service time—is defined as the Erlang model. The term PCT2 (*Pure Chance Traffic of Type 2*) is then given to traffic offered by a finite number of traffic sources (binomial call stream distribution), that is, traffic considered both in the Bernoulli model, in which the capacity of the system is higher than the number of traffic sources, and in the Engset model. In the literature the terms: PCT1 traffic and Erlang traffic stream, PCT2 traffic and Engset traffic stream, as well as the call stream with binomial distribution and the Engset traffic stream, are often used interchangeably. In many publications one can often find the acronym BPP (Binomial-Poisson-Pascal) used to define Erlang, Engset, and Pascal traffic streams. Each letter in the acronym represents the names of the call streams that generate relevant traffic streams, that is, streams with binomial distribution, Poisson and Pascal, in which the number of traffic sources for particular classes of calls is higher than the capacity of the system.) Additionally, a new model for the gradings with different availabilities and a possibility to service calls of the BPP type, that is, calls arriving according to Binomial-Poisson-Pascal distributions, is also proposed. In Section 4 a new structure of Erlang's Ideal Grading with the multirate links is proposed. In Section 5 the results of the analytical modeling are compared with the data obtained in the simulation experiments for the considered types of gradings. Section 6 concludes the paper.

## 2. Properties of State-Dependent Systems

This section presents the idea of modeling systems with state-dependent call arrival process and state-dependent call admission process. For this purpose, we will carry out an analysis of the reversibility property of the Markov process occurring in the system under consideration and we will devise an appropriate formula that makes a determination of the occupancy distribution possible. The conclusions from the analyses carried out in this section will be then used to describe models of gradings, which are one of the first state-dependent systems to be considered in traffic theory.

Let us consider a system with state-dependent call admission process and state-dependent call arrival process. In multirate systems the dependence between the call admission process and the state of the system may result both from the structure of the system and from the adopted admission policy for new calls. A good example of a system with state-dependent call admission process, resulting from the structure, is a limited-availability group that is a model of a group of separated links [2, 31], as well as a grading. In systems in which the dependence of the state of the call admission process results from a particular policy adopted for the arrival process for new calls, the most representative are systems with bandwidth reservation [11, 12, 32] and threshold systems [3–5] in which the actual amount of resources allocated to calls of individual traffic classes can change with a change in the occupancy state of the system. The dependence between the call arrival process and the occupancy state of the system usually occurs in systems with a finite number of traffic sources, that is, in systems with Engset (with binomial call streams) and Pascal (with negative binomial calls streams) traffic streams [8].

Consider a system with the capacity of  $V$  BBUs. (The conventional notion of “link” defined as a unit of capacity of the telecommunications system is rather of historical importance. In this paper, to define the smallest unit of capacity of the system, the notion of basic bandwidth unit (BBU) of a group is used [33].) The system is offered traffic streams of three types:  $m_I$  Erlang streams from the set  $I = \{1, \dots, i, \dots, m_I\}$ ,  $m_J$  Engset streams from the set  $J = \{1, \dots, j, \dots, m_J\}$ , and  $m_K$  Pascal streams from the set  $K = \{1, \dots, k, \dots, m_K\}$ . The call intensity for Erlang traffic of class  $i$  is  $\lambda_i$ . The parameter  $\lambda_j(n_j)$  determines the call intensity for the Engset traffic stream of class  $j$ , whereas the parameter  $\lambda_k(n_k)$  determines the call intensity for Pascal traffic stream of class  $k$ . The arrival rates  $\lambda_j(n_j)$  and  $\lambda_k(n_k)$  depend on the number of  $n_j$  and  $n_k$  of currently serviced calls of class  $j$  and  $k$ . In the case of Engset stream, the arrival rate of class  $j$  stream decreases with the number of serviced traffic sources:

$$\lambda_j(n_j) = (N_j - n_j)\Lambda_j, \quad (2.1)$$

where  $N_j$  is the number of Engset traffic sources of class  $j$ , while  $\Lambda_j$  is the arrival rate of calls generated by a single free source of class  $j$ . In the case of Pascal stream of class  $k$ , the arrival rate increases with the number of serviced traffic sources:

$$\lambda_k(n_k) = (S_k + n_k)\gamma_k, \quad (2.2)$$

where  $S_k$  is the number of Pascal traffic sources of class  $k$ , while  $\gamma_k$  is the arrival rate of calls generated by a single free source of class  $k$ .

The total intensity of Erlang traffic of class  $i$  offered to the group is equal to

$$A_i(n) = A_i = \frac{\lambda_i}{\mu_i}, \quad (2.3)$$

whereas the intensity of Engset traffic  $\alpha_j$  and Pascal traffic  $\beta_k$  of class  $j$  and  $k$ , respectively, offered by one free source is equal to

$$\alpha_j = \frac{\Lambda_j}{\mu_j}, \quad \beta_k = \frac{\gamma_k}{\mu_k}. \quad (2.4)$$

In (2.3) and (2.4) the parameter  $\mu$  is the average service intensity with the exponential distribution. Thus, the mean traffic offered to the system in the state of  $n$  BBUs being busy by idle class  $j$  Engset traffic sources and idle class  $k$  Pascal sources is equal to

$$A_j(n) = (N_j - n_j(n))\alpha_j, \quad A_k(n) = (S_k + n_k(n))\beta_k, \quad (2.5)$$

where  $n_j(n)$  and  $n_k(n)$  denoted the average number of class  $j$  Engset sources and class  $k$  Pascal sources serviced in the occupancy state  $n$ .

The number of BBUs demanded by calls of an arbitrary class  $c$  is denoted by  $t_c$  (in the present paper, the letter “ $i$ ” denotes an Erlang traffic class, the letter “ $j$ ” an Engset traffic class, the letter “ $k$ ” a Pascal traffic class, and the letter “ $c$ ” an arbitrary traffic class).

The occupancy distribution in the state-dependent system servicing multi-service BPP traffic streams can be determined on the basis of the following formula [8]:

$$\begin{aligned}
 nP(n) = & \sum_{i=1}^{m_I} A_i t_i P(n - t_i) \sigma_i(n - t_i) \\
 & + \sum_{j=1}^{m_J} \alpha_j t_j [N_j - n_j(n - t_j)] P(n - t_j) \sigma_j(n - t_j) \\
 & + \sum_{k=1}^{m_K} \beta_k t_k [S_k + n_k(n - t_k)] P(n - t_k) \sigma_k(n - t_k),
 \end{aligned} \tag{2.6}$$

where  $\sigma_c(n)$  is the conditional transition probability between macrostate  $n$  (i.e.,  $n$  BBU being busy) and state  $n + t_c$  (state of  $n + t_c$  BBU being busy),  $P(n)$  denotes the probability of macrostate  $n$ , that is, the probability that the system is in state of  $n$  BBU being busy, and  $P(n) = 0$  for  $n < 0$  and  $n > V$ .

A rigorous derivation of formula (2.6) is presented in Appendix A. This derivation is a generalization of the reasoning presented in [34] for systems with state-independent call arrival and admission process.

As a result of the analysis of formula (2.6) it is noticeable that the determination of the occupancy distribution has to be preceded by a determination of the average number of serviced traffic sources of all traffic classes in particular occupancy states of the system. The number  $n_c(n)$  of serviced traffic sources of class  $c$  in occupancy state  $n$  directly influences the value of offered traffic in systems with the state-dependent call arrival process and can be determined on the basis of multiple iterative runs of (2.6) [8].

In line with [8, 35], the algorithm for a determination of the occupancy distribution in a system with state-dependent call arrival process and state-dependent call admission process can be written in the form of Algorithm 2.1.

*Algorithm 2.1.* Algorithm for a determination of the occupancy distribution in state-dependent systems can be stated in the following steps:

- (1) determination of conditional transition probabilities  $\sigma_c(n)$ ;
- (2) setting of the iteration number  $l = 0$ ;
- (3) determination of initial values of  $n_j^{(l)}(n)$ ,  $n_k^{(l)}(n)$ :

$$\forall_{1 \leq j \leq m_J} \forall_{0 \leq n \leq V} n_j^{(l)}(n) = 0, \quad \forall_{1 \leq k \leq m_K} \forall_{0 \leq n \leq V} n_k^{(l)}(n) = 0; \tag{2.7}$$

- (4) increase of the iteration number:  $l = l + 1$ ;

(5) determination of state probabilities  $P^{(l)}(n)$

$$\begin{aligned} nP^{(l)}(n) = & \sum_{i=1}^{m_i} A_i t_i \sigma_i(n-t_i) P^{(l)}(n-t_i) \\ & + \sum_{j=1}^{m_j} \alpha_j [N_j - n_j^{(l-1)}(n-t_j)] \sigma_j(n-t_j) t_j P^{(l)}(n-t_j) \\ & + \sum_{k=1}^{m_k} \beta_k [S_k + n_k^{(l-1)}(n-t_k)] \sigma_k(n-t_k) t_k P^{(l)}(n-t_k); \end{aligned} \quad (2.8)$$

(6) calculation of average number of serviced calls  $n_j^{(l)}(n)$  and  $n_k^{(l)}(n)$  [12]:

$$n_c^{(l+1)}(n) = \begin{cases} \frac{A_c^{(l+1)}(n-t_c) \sigma_c(n-t_c) P^{(l)}(n-t_c)}{P^{(l)}(n)} & \text{for } 0 \leq n \leq V, \\ 0 & \text{otherwise;} \end{cases} \quad (2.9)$$

(7) repetition of steps 4–6 until the assumed accuracy  $\xi$  of the iterative process is obtained:

$$\forall_{n \in (0, V)} \left( \left| \frac{n_j^{(l-1)}(n) - n_j^{(l)}(n)}{n_j^{(l)}(n)} \right| \leq \xi, \left| \frac{n_k^{(l-1)}(n) - n_k^{(l)}(n)}{n_k^{(l)}(n)} \right| \leq \xi \right). \quad (2.10)$$

Having the occupancy distribution established it is possible to determine basic traffic characteristics of the system with multirate traffic and state-dependent call arrival and call admission processes, that is,

(1) blocking probabilities (time congestion)  $E_c$  for calls of particular traffic classes

$$E_c = \sum_{n=0}^V (1 - \sigma_c(n)) P(n); \quad (2.11)$$

(2) loss probability (call congestion) for class  $j$  Engset traffic stream:

$$B_j = \frac{\sum_{n=0}^V P(n) [1 - \sigma_j(n)] [N_j - n_j(n)] \Lambda_j}{\sum_{n=0}^V P(n) [N_j - n_j(n)] \Lambda_j}, \quad (2.12)$$

where  $[N_j - n_j(n)] [1 - \sigma_j(n)] \Lambda_j$  is the stream of lost calls in macrostate  $n$ ;

(3) loss probability (call congestion) for class  $k$  Pascal traffic stream:

$$B_k = \frac{\sum_{n=0}^V P(n) [1 - \sigma_k(n)] [S_k + n_k(n)] \gamma_k}{\sum_{n=0}^V P(n) [S_k + n_k(n)] \gamma_k}, \quad (2.13)$$

where  $[S_k + n_k(n)] [1 - \sigma_k(n)] \gamma_k$  is the stream of lost calls in macrostate  $n$ .

The presented process of a determination of the average number of serviced traffic sources in particular occupancy states is a convergent process. A proof for the convergence is shown in Appendix B. The proof in Appendix B is a generalization of the proof given in [34] for systems with state-independent call admission process and carrying BPP traffic. The presented generalized algorithm for modeling systems with multi-service traffic and with state-dependent call arrival and admission processes will be applied further on in the paper for modeling gradings that are characterized by different availabilities and different structures of offered traffic.

### 3. Models of Erlang's Ideal Grading

#### 3.1. Characteristics of the Grading

Gradings (non full-availability groups) are one of the "oldest" systems with state-dependent call admission process in telecommunications. In such groups, individual traffic sources have no access to all  $V$  BBUs but only to some of them. The number of BBUs to which traffic sources have access is called availability (accessibility) and is denoted by the symbol  $d$ . Traffic sources that have access to the same BBUs of a group form the so-called load group (incoming group in [1]). The number of load groups will be denoted by the symbol  $g$ . Different traffic sources can have access to the same BBUs of a group. This phenomenon is called partial multiplication of outgoing links. The average number of load groups for one BBU of the group is called the multiplication coefficient.

The occurrence of the phenomenon of partial multiplication results in the availability of a grading to be within the boundaries:

$$d \leq V \leq gd. \quad (3.1)$$

Let us consider now boundary cases of the structure of the grading. In the first case, when  $d = V$  BBU, we obtain a grading that services one load group, that is, the full-availability group. In the second case, when  $gd = V$  BBU, the grading is composed of  $g$  full-availability groups with capacities of  $d$  BBU.

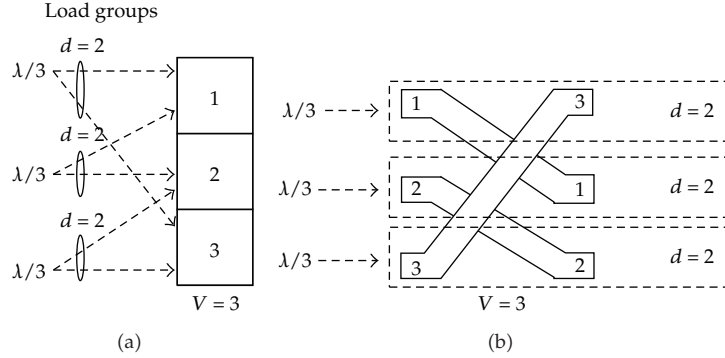
By taking availability of basic bandwidth units to load groups as a criterion, gradings can be divided into the two following groups [1]:

- (i) graded groups—in which, along with an increase in the number of BBU, the number of load groups that have access to this BBU also increases or remains unchanged;
- (ii) uniform groups—in which each BBU is always available to the same number of load groups.

A particular case of a uniform grading is Erlang's Ideal Grading (ideally symmetrical non full-availability group) [25]. This group is characterized by the following properties:

- (i) the number of load groups in the grading is equal to the number of possible choices of  $d$  BBUs from among all  $V$  BBUs (two load groups differ from each other in at least one BBU):

$$g = \binom{V}{d}; \quad (3.2)$$



**Figure 1:** Erlang's Ideal Grading ( $V = 3$ ,  $d = 2$ ,  $g = 3$ ): (a) offered traffic distribution, (b) concept of availability.

- (ii) each load group has access to the same number of BBUs in a group equal to  $d$ ;
- (iii) traffic offered to the grading by all load groups is identical;
- (iv) BBUs are chosen for new calls randomly.

Figure 1 shows an example of Erlang's Ideal Grading described by the parameters:  $V = 3$ ,  $d = 2$ ,  $g = 3$ .

### 3.2. Model of Erlang's Ideal Grading with a Single-Rate Erlang Traffic Stream

Let us consider the simplest model of the grading, that is, Erlang's Ideal Grading, which is offered a single ( $m_I = 1$ ) Poisson call stream with the intensity  $\lambda$  and which demands  $t = 1$  BBU for service [25]. The service time of a call has an exponential distribution with the parameter  $\mu$ . The average traffic intensity offered to the group is equal to

$$A = \frac{\lambda}{\mu}. \quad (3.3)$$

Figure 2 presents a state diagram of the Markov process. This process is one-dimensional and in order to determine the occupancy distribution it is possible to use directly (2.6). In the case of the considered group, due to the fact that  $m_I = 1$ , (2.6) takes on the following form:

$$nP(n) = A\sigma(n-1)P(n-1), \quad (3.4)$$

where  $\sigma(n)$ —the conditional transition probability—determines the dependence between the call admission process in occupancy state  $n$  BBU and the structure of the grading.

Let us determine now the values of the parameters  $\sigma(n)$ . Since traffic offered by all the load groups is identical, whereas the basic bandwidth units in the group are chosen randomly, then the load of each of the BBUs in the group under consideration is identical. Therefore, for any number of busy BBUs in the group  $n$  ( $0 \leq n \leq V$ ), the occupancy probability of  $j$  output BBUs—available for a given load group ( $0 \leq j \leq d$ )—is equal to the occupancy probability of  $j$  BBUs—available in any other load group. The blocking probability in Erlang's Ideal Grading



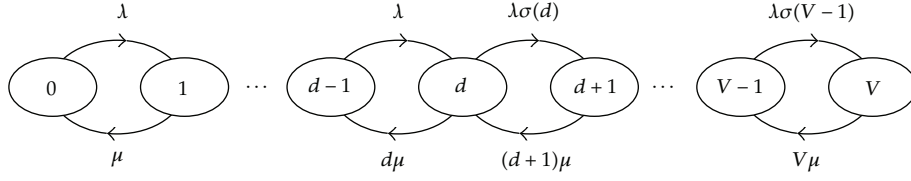


Figure 2: State diagram of the call service process in Erlang's Ideal Grading.

is equal to zero for all occupancy states  $n < d$ , since in such states, for each and every load group, there is at least one BBU available. In the case when  $n \geq d$ , the conditional blocking probability of a single load group is equal to

$$\beta(n) = \frac{\binom{n}{d}}{\binom{V}{d}}. \quad (3.5)$$

The conditional transition probability  $\sigma(n)$  between the states of the service process in the group is then equal to:

$$\sigma(n) = 1 - \beta(n). \quad (3.6)$$

Equation (3.6) indicates the fact that, in the occupancy state of the group  $n$ , only part of the call stream with the intensity  $\lambda(1 - \beta(n)) = \lambda\sigma(n)$  will be admitted for service. After taking into consideration the states in which the blocking state can occur, the total blocking probability in the group can be determined:

$$E = \sum_{n=d}^V (1 - \sigma(n))P(n). \quad (3.7)$$

The recursive notation (3.4) of the occupancy distribution in Erlang's Ideal Grading can be easily transformed into explicit form, proposed by Erlang [25]:

$$P(n) = \frac{(A^n/n!) \prod_{k=d}^{n-1} \sigma(k)}{\sum_{j=0}^V (A^j/j!) \prod_{k=d}^{j-1} \sigma(k)}. \quad (3.8)$$

After taking into consideration all blockable occupancy states of the group—on the basis of (3.7)—we can obtain the total blocking probability in Erlang's Ideal Grading:

$$E = \text{EIF}(A, V, d) = \sum_{n=d}^V \beta(n)P(n). \quad (3.9)$$

Because of the nature of offered traffic, the loss probability  $B$  in the considered group is equal to the blocking probability  $E$ . Equation (3.9), worked out by Erlang as early as 1917 [25], is called *Erlang's Interconnection Formula*—EIF.

### 3.3. Model of Erlang's Ideal Grading with a Single-Rate Engset Traffic Stream

Erlang's interconnection formula enables to determine the blocking probability in a grading with Erlang traffic. To determine distribution in a grading that services only one stream of single-rate Engset traffic ( $t_j = 1, m_j = 1$ ), on the basis of formula (2.6), we get

$$nP(n) = (N - 1)\alpha \sigma(n - 1)P(n - 1), \quad (3.10)$$

where the conditional transition coefficient  $\sigma(n)$  is determined by formula (3.6).

The blocking probability in the considered system can be determined on the basis of formula (3.7), whereas the loss probability can be expressed by formula (2.12), which—for the considered instance—will take on the following form:

$$B = \frac{\sum_{n=0}^V P(n)[1 - \sigma(n)][N - n]\Lambda}{\sum_{n=0}^V P(n)[N - n]\Lambda}, \quad (3.11)$$

where  $\Lambda$  is the call intensity generated by a single free Engset source.

The recursive notation (3.10) can be expressed in explicit form [1]:

$$P(n) = \frac{\binom{N}{n} \alpha^n \prod_{z=1}^{n-1} \sigma(z)}{1 + \sum_{i=1}^V \binom{N}{i} \alpha^i \prod_{z=1}^{i-1} \sigma(z)}. \quad (3.12)$$

In [1], formula (3.12) is considered to be approximate since the author considered a system in which each load group was assigned fixed and identical number of traffic sources. In the case when traffic sources can generate calls for all load groups with identical probability, formulae (3.10) and (3.12) are precise.

### 3.4. Model of Erlang's Ideal Grading with Single-Rate Erlang Traffic Streams

Let us consider now an ideal grading to which  $m_I$ -independent classes of call streams with the intensities  $\lambda_1, \lambda_2, \dots, \lambda_{m_I}$  are offered. The calls, irrespectively of the class of a stream, demand a single BBU to set up a connection, that is,  $\forall_{1 \leq i \leq m_I} t_i = 1$  BBU. Service times of particular classes have exponential distribution with parameters, respectively, equal to:  $\mu_1, \mu_2, \dots, \mu_{m_I}$ . The average traffic intensity offered by a call stream of class  $i$  can be then determined by formula (2.3). Availability  $d_i$  of each of the serviced call classes is identical and equal to  $d$ —thus the number of load groups (for each class of calls) is determined by formula (3.2).

In line with the assumptions of Erlang's Ideal Grading, traffic offered by individual load groups is identical (traffic offered by individual call classes distributes uniformly onto all load groups—Figure 3). Since all call classes are determined by identical parameters (demand one BBU for service, availability for each class is equal to  $d$  BBUs), then, after taking into consideration a random hunting strategy of free BBUs for new calls (irrespectively on their class), the load of each BBU of the group will be the same. Thus, analogously as in the case of the model of a group that services one class of calls, for any number of  $n$  busy BBUs, the occupancy probability  $j$  BBUs in a given load group is equal to the occupancy probability of  $j$  BBUs in any other load group. The value of the probability  $\sigma_i(n)$  does not depend on

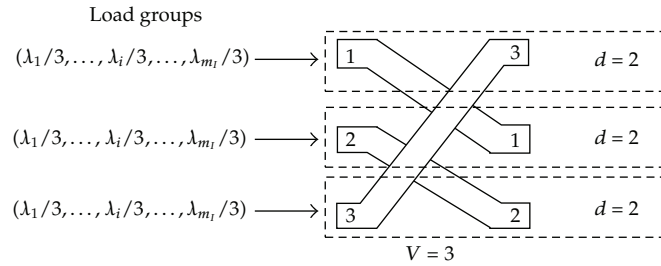


Figure 3: Uniform distribution of offered traffic in the ideal grading.

the class of a call, but it depends exclusively on availability. This means that, in a given occupancy state of the group, the conditional transition probabilities for all classes of calls are equal to one another:

$$\sigma_i(n) = \sigma(n) = 1 - \frac{\binom{n}{d}}{\binom{V}{d}}. \tag{3.13}$$

To determine the characteristics of the system under consideration one can use the recursive dependence (2.6), which can be rewritten in the following way:

$$nP(n) = \sigma(n - 1)P(n - 1) \sum_{i=1}^{m_I} A_i, \tag{3.14}$$

where  $\sigma(n) = 1$  for  $n < d$ , whereas for  $n \geq d$  the parameter  $\sigma(n)$  is determined by formula (3.13).

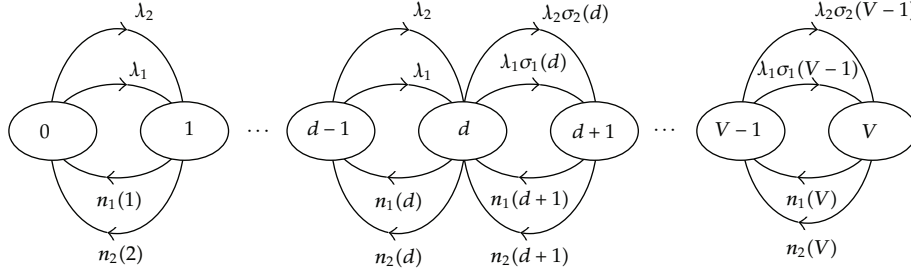
Figure 4 presents a diagram of the one-dimensional Markov process in Erlang's Ideal Grading that corresponds to formula (3.14).

In order to determine the blocking probability in the considered Erlang's Ideal Grading servicing  $m_I$  single-rate traffic streams, notice that for all classes of calls the common value of the availability parameter has been determined. This means that the blocking probability for all classes of calls is identical (irrespective of a class of traffic) and can be determined on the basis of formula (3.7).

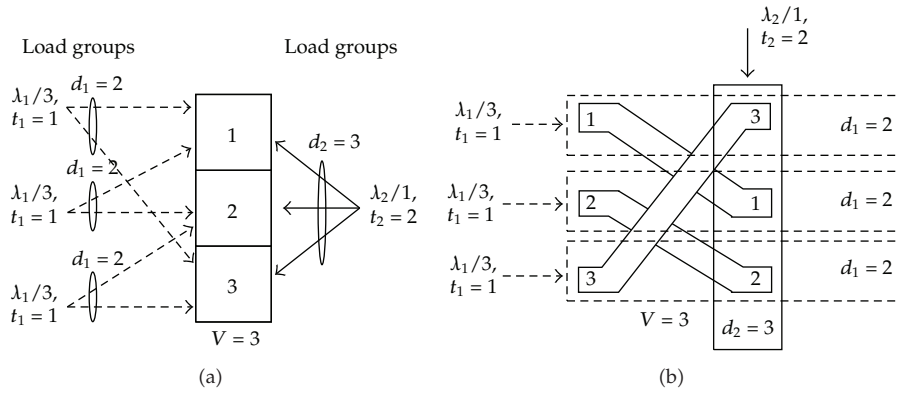
Let us notice too that the values of probabilities  $P(n)$  depend on the sum of traffic offered by all classes of calls, which, in turn, means that the value of the blocking probability of calls of class  $i$  depends on the total traffic offered to the group and does not depend directly on the value of offered traffic of this class. It should be also noted that when  $m_I = 1$ , (3.14) comes down to (3.4).

### 3.5. Model of Erlang's Ideal Grading with Various Availabilities and Single-Rate Erlang Traffic Streams

Let us consider now Erlang's Ideal Grading that services  $m_I$  classes of calls streams. The call streams of all classes are described by identical parameters—exactly as in the previous



**Figure 4:** One-dimensional Markov process in Erlang's Ideal Grading servicing two classes of single-rate calls.



**Figure 5:** Grading with different availabilities ( $d_1 \neq d_2$ ): (a) offered traffic distribution, (b) concept of availability.

section-with an additional assumption that each class of calls has access to  $d_i$  BBUs. This means that each class of calls is related to a different number of load groups:

$$g_i = \binom{V}{d_i}. \quad (3.15)$$

Figure 5 shows a grading with the capacity of 3 BBUs. The group services two classes of calls that have availabilities equal to, respectively,  $d_1 = 2$  and  $d_2 = 3$ . The number of load groups for relevant classes of calls is equal to  $g_1 = 3$  and  $g_2 = 1$ .

The values of the parameter  $\sigma_i(n)$  will depend on a traffic class  $i$ . Due to the properties of the ideal grading, values of these parameters do not depend on the mixture of currently serviced calls of particular classes. For calls of class  $i$  the value of the parameter  $\sigma_i(n)$  can be determined on the basis of the following formula:

$$\sigma_i(n) = 1 - \beta_i(n) = 1 - \frac{\binom{n}{d_i}}{\binom{V}{d_i}}. \quad (3.16)$$

Using the properties of state-dependent systems (Section 2), the occupancy distribution in the considered group can be approximated by (2.6), which, for the considered system with single-rate traffic ( $\forall_{1 \leq i \leq m_i} t_i = 1$ ), will take on the following form:

$$nP(n) = \sum_{i=1}^{m_I} A_i \sigma_i(n-1)P(n-1), \quad (3.17)$$

where  $P(n) = 0$  for  $n < 0$  and  $n > V$ .

Having the occupancy distribution thus determined we are in position to determine the blocking probability of calls of class  $i$ :

$$E_i = \sum_{n=d_i}^V (1 - \sigma_i(n))P(n). \quad (3.18)$$

Also in this case, for  $m_I = 1$ , (3.17) is simplified to (3.4).

It can be proved that the service process in a system with differentiated availability is not reversible. (For the group under consideration, condition (A.3) (Appendix A) takes on the following form:

$$\sigma_i(n)\sigma_{m_i}(n+1) = \sigma_{m_i}(n)\sigma_i(n+1). \quad (3.19)$$

(The parameter  $\sigma_i(n)$  depends thus on the parameter  $d_i$  and hence the condition (3.19) will not be satisfied. Therefore, the process occurring in the group is not a reversible process.) This means that (3.17) and (3.18) are approximate equations. The study carried out by the authors indicates, however, that this approximation achieves high accuracy in all instances.

### **3.6. Model of Erlang's Ideal Grading with Equal Availability and Erlang Multirate Traffic Streams**

Let us consider a grading that is offered  $m_I$ -independent call streams with the intensities  $\lambda_1, \lambda_2, \dots, \lambda_{m_I}$ . The service time of calls of particular classes has an exponential distribution with the parameters, respectively,  $\mu_1, \mu_2, \dots, \mu_{m_I}$ . Therefore, traffic offered by individual call streams can be determined on the basis of (2.3). To set up a connection, the calls demand, respectively,  $t_1, t_2, \dots, t_{m_I}$  BBUs. For all classes of calls serviced by the group, the availability is identical and is equal to  $d$  [20].

Let us consider now the blocking probability of calls for a single load group and the conditional transition probability  $\sigma_i(n)$ . The blocking state for the calls of class  $i$  occurs in a given load group when the number of free BBUs in this group will be lower than  $t_i$  BBUs. Thus, a call of class  $i$  will not be admitted for service if the system is in one of the occupancy states that belongs to the set  $\Psi_i = \{(d - t_i + 1), (d - t_i + 2), \dots, d\}$ . If at this point we assume that there are  $n$  busy BBUs in the whole of the group, then the group under consideration will be blocked only when  $x$  busy BBUs in the group will satisfy the condition  $x \in \Psi_i$ .

The probability of such an event can be determined on the basis of a hypergeometric distribution [20]:

$$\beta(n, x) = \frac{\binom{d}{x} \binom{V-d}{n-x}}{\binom{V}{n}}. \quad (3.20)$$

Taking into account all possible blocking states of the considered component group, the blocking probability of this group for calls of class  $i$ , in the occupancy state  $n$ , will be equal to

$$\beta_i(n) = \sum_{x=d-t_i+1}^k \beta(n, x), \quad (3.21)$$

where  $k = n$  for  $(d - t_i + 1) \leq n < d$  and  $k = d$  for  $n \geq d$ , whereas the conditional transition probability for calls of class  $i$  is equal to

$$\sigma_i(n) = 1 - \beta_i(n) = 1 - \frac{\sum_{x=d-t_i+1}^k \binom{d}{x} \binom{V-d}{n-x}}{\binom{V}{n}}. \quad (3.22)$$

Using the properties worked out for the system with state-dependent call admission process (Section 2) and taking into consideration the expression (3.22), the occupancy distribution in the considered group can be determined on the basis of an appropriately modified formula (2.6):

$$nP(n) = \sum_{i=1}^{m_1} A_i t_i \sigma_i(n - t_i) P(n - t_i), \quad (3.23)$$

where  $P(n) = 0$  for  $n < 0$  and  $n > V$ . Eventually, the blocking probability for the calls of class  $i$  is equal to

$$E_i = \sum_{n=d-t_i+1}^V [1 - \sigma_i(n - t_i)] P(n). \quad (3.24)$$

If we adopt that calls of all classes demand one BBU for service, then the considered model comes down to the model described in Section 3.5. It can be proved that the service process in the considered system is not reversible which in turn means that the distribution (3.23) and (3.24) are approximate distributions. (Since the values of the parameter  $\sigma_i(n)$  depend on the number of BBUs demanded by particular classes of calls, the condition (A.3) (Appendix A) will never be satisfied. This means that the Markov process occurring in the group under consideration is not a reversible process.) This approximate distribution, however, is characterized by high accuracy, validated by numerous simulation experiments.

When, in turn, the number of serviced classes of calls will be limited to just one and  $t = 1$  BBU, then the considered model will come down to the precise model of a group described in Section 3.4.

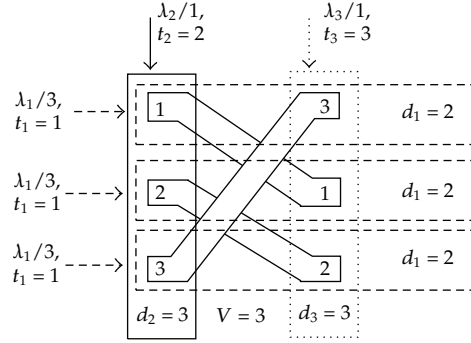


Figure 6: Grading with multirate traffic and different availabilities.

### 3.7. Generalized Model of Erlang's Ideal Grading with Various Availabilities and Multirate Erlang-Engset-Pascal Traffic Streams

Let us consider now further generalizations of the model of grading presented in Section 3.6. Assume that the group is offered three types of call streams (Section 2):  $m_I$  streams from the set  $I = \{1, \dots, i, \dots, m_I\}$ , arriving in accordance with a Poisson distribution,  $m_J$  call streams from the set  $J = \{1, \dots, j, \dots, m_J\}$ , arriving in accordance with a binomial distribution, and  $m_K$  call streams from the set  $K = \{1, \dots, k, \dots, m_K\}$ , arriving in accordance with a Pascal distribution (negative binomial distribution). Our further assumption is that calls of individual classes are characterized by different availability equal to, respectively,  $d_1, d_2, \dots, d_{m_I}$ , and different values of demanded BBUs, equal to, respectively,  $t_1, t_2, \dots, t_m$ . (The grading with various availabilities and Erlang traffic streams was considered in [21].) The adopted assumptions imply that calls that require identical number of BBUs, but differ in availability, constitute two different classes of calls. An example of such a group is presented in Figure 6.

Taking into consideration different values of the parameter  $d_c$  in Formula (3.22), the equation defining the conditional transition probability for calls of class  $c$  (index  $c$  denotes any class of calls) takes on the following form [21]:

$$\sigma_c(n) = 1 - \beta_c(n) = 1 - \frac{\sum_{x=d_c-t_c+1}^k \binom{d_c}{x} \binom{V-d_c}{n-x}}{\binom{V}{n}}, \quad (3.25)$$

where  $k = n - t_c$  for  $(d_c - t_c + 1) \leq (n - t_c) < d_c$  and  $k = d_c$  for  $(n - t_c) \geq d_c$ .

After taking into consideration the dependence between the value of offered traffic of Engset and Pascal class and the occupancy state of the system, as well as the value of the conditional transition probability (3.25), the occupancy distribution in the considered group can be determined on the basis of the modified formula (2.6):

$$nP(n) = \sum_{c=1}^m A_c(n - t_c) t_c \sigma_c(n - t_c) P(n - t_c), \quad (3.26)$$

where  $P(n) = 0$  for  $n < 0$  and  $n > V$  and  $A_c(n)$  is determined on the basis of (2.5) for Engset and Pascal traffic streams.

Having thus determined occupancy distribution, the blocking probability in the considered model of the group can be determined on the basis of formula (3.24).

### 3.8. Model of Erlang's Ideal Grading for Noninteger Values of Availability

Formulae (3.24), (3.25), and (3.26) enable us to determine the values of blocking probabilities in Erlang's Ideal Grading with Erlang-Engset-Pascal traffic only for integer values of the parameter  $d$ . Further on in the paper, a simple approximate method for a determination of the value of the blocking probability in IIG with Erlang-Engset-Pascal traffic for non-integer values of the availability parameter will be proposed. The worked out method is based on the idea presented in [27] for a model of a grading with Erlang traffic. In the proposed method, a given class of calls  $c$ , in which the parameter  $d_c$  takes on non-integer values, is replaced by two fictitious classes with integer values of availability ( $d_{c_1}, d_{c_2}$ ) and offered traffic ( $A_{c_1}(n), A_{c_2}(n)$ ). Values of these parameters are defined in the following way:

$$\begin{aligned} d_{c_1} &= \lfloor d_c \rfloor, \\ d_{c_2} &= \lceil d_c \rceil. \end{aligned} \quad (3.27)$$

Taking into consideration formulae (2.3) and (2.5), traffic offered by the new fictitious classes of calls is equal to, respectively,

$$\begin{aligned} A_{c_1}(n) &= A_c(n)[1 - (d_c - d_{c_1})], \\ A_{c_2}(n) &= A_c(n)(d_c - d_{c_1}), \end{aligned} \quad (3.28)$$

where the difference  $(d_c - d_{c_1})$  defines the fractional part of the parameter  $d_c$ . Such a definition of the parameters  $A_{c_1}(n), A_{c_2}(n), d_{c_1}, d_{c_2}$  means that the value of fictitious traffic  $A_{c_2}$  is directly proportional to the fractional part of the availability parameter, that is, to  $\Delta_c = d_c - d_{c_1}$ , whereas the value of fictitious traffic  $A_{c_1}(n)$  is directly proportional to the complement  $\Delta_c$ , that is, to the value  $1 - \Delta_c = 1 - (d_c - d_{c_1})$ .

Let us consider Erlang's Ideal Grading with the capacity  $V$  and the number of serviced traffic classes equal to  $m_M$ . Let us assume, for convenience, that only the availability parameter of one class, that is, class  $c$  takes on non-integer values. After replacing class  $c$  with two fictitious classes:  $c_1$ , and  $c_2$ , with assigned values of availability and traffic intensity (formulae (3.27) and (3.28)), it is possible to determine, on the basis of formulae (3.24) and (3.25), the blocking probabilities of all classes of calls, including the blocking probability of new classes of calls  $E_{c_1}$  and  $E_{c_2}$ . Then, assuming that the blocking probability of the fictitious traffic class is directly proportional to the value of this traffic, we are in position to evaluate the blocking probability for the calls of class  $c$  for non-integer value of availability  $d_c$ :

$$E_c = \frac{A_{c_1}(0)E_{c_1} + A_{c_2}(0)E_{c_2}}{A_c(0)}. \quad (3.29)$$

In the case of a higher number of classes with non-integer availabilities, each class of calls is replaced by two fictitious classes with the parameters determined by formulae (3.27) and (3.28). Further calculations are carried out exactly as in the case of the two classes of calls.



The results of the simulation experiments conducted by the authors have confirmed the substantial accuracy of the proposed solution [21, 27].

#### 4. Erlang's Ideal Grading with the Multirate Links

Let us consider now an analytical model for a new structure of Erlang's Ideal Grading. The group is composed of  $v$  links with the capacity of  $f$  BBUs. The structure of links forms an Erlang's Ideal Grading (Figure 7). The total capacity of the group is equal to  $V = vf$ . The group services  $m_I$  classes of calls with the intensities  $\lambda_1, \lambda_2, \dots, \lambda_{m_I}$ . The service time of calls of particular classes has the exponential distribution with the parameters, respectively,  $\mu_1, \mu_2, \dots, \mu_{m_I}$ . Thus, traffic offered by each class of calls can be determined on the basis of Formula (2.3). The calls demand, respectively,  $t_1, t_2, \dots, t_{m_I}$  BBUs. The group availability, expressed in BBUs, is equal to

$$D = df, \quad (4.1)$$

where  $d$  is the availability parameter expressed in the number of links. A new call is admitted for service only when it will be serviced by BBUs that belong to one of all available links. Additionally, the group satisfies all the assumptions made for EIG, that is,

- (i) free link for a new call is randomly chosen (free BBUs within the selected free link are also randomly chosen),
- (ii) offered traffic distributes uniformly in all load groups.

Figure 7 presents an example of a group with the multirate links with the capacity of 12 BBUs. The group is composed of  $v = 3$  links with the capacity  $f = 4$  BBUs each. The group services two classes of calls ( $t_1 = 1, t_2 = 4$ ). Since availability to links is fixed ( $d = 2$ ), traffic sources that are related to the serviced classes of calls are divided into three load groups ( $g = \binom{v}{d} = \binom{3}{2} = 3$ ).

Let us determine now the blocking probability  $\beta_i$  for calls of class  $i$  in a single load group. A blocking state for calls of class  $i$  occurs in any randomly chosen load group in a case when none of the available links has at least  $t_i$  free BBUs. This event always occurs when the total number of free BBUs in available links is lower than the demanded number of  $t_i$  BBUs for calls of class  $i$ . Following similar reasoning as with the case of the grading without multirate links (Section 3.6), the blocking state will always occur if the service process in the load group under consideration will be in one of the states belonging to the set  $\Psi_i = \{(D - t_i + 1), (D - t_i + 2), \dots, D\}$ . In the group with the multirate links, the set  $\Psi_i$  does not include, however, all blocking states. Let us then consider such an unfavourable distribution of busy BBUs in available links with the example of the group presented in Figure 7. Our considerations will be carried out for the call of class 2, which demands 4 BBUs. The second load group has two links, numbered as 2 and 1. Assume that the second load group is in the occupancy state 2 BBUs. This means that the group services two calls of the first class ( $t_1 = 1$ ).

A feasible arrangement of busy BBUs in the available links of the second load group is presented in Figure 8 (which link is busy is of no importance here, but rather the number of busy BBUs in particular links). Two possible combinations of the arrangement of busy BBUs are clearly visible. In the first case (Figure 8(a)), all busy BBUs are in one available link (there are two such arrangements at hand). Thus, all BBUs in the other link are free and the blocking

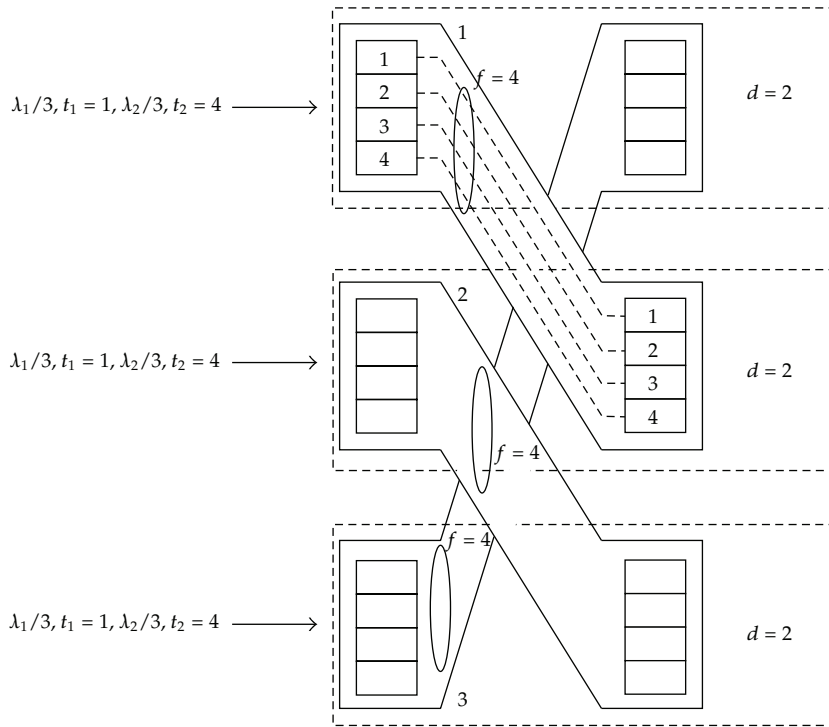


Figure 7: Proposed structure of Erlang's Ideal Grading with multirate links ( $V = 12, v = 3, f = 4$ ).

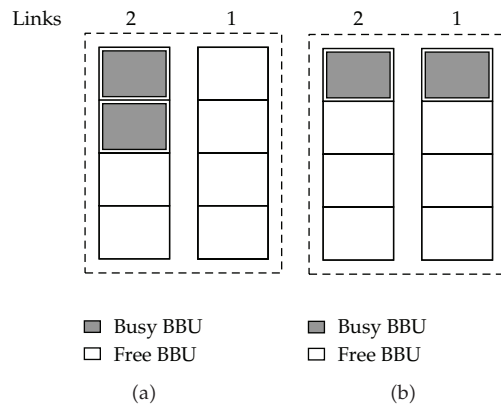


Figure 8: Arrangement of busy (free) BBUs in the second load group.

state does not exist. In the other case (Figure 8(b)), there is one busy BBU in each available link of the second load group (there is only one such arrangement). Such an arrangement of busy BBUs causes the group to be in the blocking state for calls that demand 4 BBUs. In the considered case, the blocking state can occur in one of three possible arrangements of busy BBUs in the available links. Let us generalize the above considerations. The blocking state in any randomly selected load group composed of  $d$  links can occur for calls of class  $i$ -demanding  $t_i$  BBUs—if the number of busy BBUs in each available link is equal or higher

than  $f - t_i + 1$ . This means that the blocking state can occur if in all  $d$  available links the number of busy BBUs is equal or higher than:

$$x = (f - t_i + 1)d. \quad (4.2)$$

On the basis of the considerations, we can complement now the set of states  $\Psi_i$  with such states in which the number of busy BBUs in the links of the load group is contained within  $(f - t_i + 1)d_i \leq x < D - t_i + 1$ . Therefore, the set  $\Psi_i$  for the grading with the multirate links can be rewritten as follows:  $\Psi_i = \{((f - t_i + 1)d), ((f - t_i + 1)d) + 1, \dots, (D - t_i), (D - t_i + 1), (D - t_i + 2), \dots, D\}$ .

Assume now that the number of all busy BBUs in the group is equal to  $n$ . The load group under consideration will be blocked if  $x$  busy BBUs in this group ( $x \leq n$ ) will satisfy the condition  $x \in \Psi_i$ . The probability of such an event can be written as follows:

$$\beta_i(n, x) = P(x \in \Psi_i | n)P_A(x), \quad (4.3)$$

where

- (i)  $P(x \in \Psi_i | n)$  is the probability of such an event that the number of busy BBUs in the load group satisfies the condition  $x \in \Psi_i$ , under the assumption that there are  $n$  busy BBUs in the whole group,
- (ii)  $P_A(x)$  is the probability of the unfavourable arrangement of  $x$  busy BBUs in the group which leads to blocking event. This probability is equivalent to the probability of unfavourable arrangement of  $D - x$  free BBUs in a given load group.

Due to the properties of Erlang's Ideal Grading, the probability  $P(x \in \Psi_i | n)$  can be approximated on the basis of the hyper-geometrical distribution that, in the considered case, will take on the following form:

$$P(x \in \Psi_i | n) = \frac{\binom{D}{x} \binom{V-D}{n-x}}{\binom{V}{n}}. \quad (4.4)$$

The blocking probability  $P_A(x)$  of calls of class  $i$  in a given load group can be defined as the ratio of unfavourable arrangement of free BBUs to the number of all possible arrangements of free BBUs in the load group. The number of arrangements of  $z$  elements in  $v$  boxes, with the capacity of  $f$  elements each, can be defined by the following combinatorial formula [2]:

$$F(z, k, f) = \sum_{i=0}^{\lfloor z/(f+1) \rfloor} (-1)^i \binom{v}{i} \binom{z+v-1-i(f+1)}{v-1}. \quad (4.5)$$

Using (4.5), the formula determining the blocking probability  $P_A(x)$  of the load group in which there are  $x$  busy BBUs can be written in the following form:

$$P_A(x) = \frac{F(D-x, d, t_i - 1)}{F(D-x, d, f)}, \quad (4.6)$$

where  $F(D-x, d, t_i-1)$  denotes the number of unfavourable distributions  $D-x$  of free BBUs in  $d$  links, under the assumption that each of them has  $t_i - 1$  free BBUs at the maximum (which is the necessary condition for the blocking state to occur), while  $F(D-x, d, f)$  denotes the number of all possible distributions of free BBUs.

Substituting (4.4) and (4.6) to (4.3), we get

$$\beta_i(n, x) = \left[ \frac{\binom{D}{x} \binom{V-D}{n-x}}{\binom{V}{n}} \right] \frac{F(D-x, d, t_i-1)}{F(D-x, d, f)}. \quad (4.7)$$

Taking all possible blocking states in the load group into consideration, the blocking probability for this group for calls of class  $i$ , in the occupancy state  $n$ , will be equal to

$$\beta_i(n) = \sum_{x=(f-t_i+1)d}^k \beta_i(n, x), \quad (4.8)$$

where  $k = n$  for  $(f-t_i+1)d \leq n \leq D$  and  $k = D$  for  $n > D$ .

It should be stressed that, due to the properties of Erlang's Ideal Grading, formula (4.8) determines the conditional blocking probability in the discussed group. We determine now the conditional transition probability  $\sigma_i(n)$  for calls of class  $i$ . The conditional transition probability for calls of class  $i$  complements the conditional blocking probability ( $\beta_i(n)$ ), determined by formula (4.8). Therefore, we can write

$$\sigma_i(n) = 1 - \beta_i(n) = 1 - \sum_{x=(f-t_i+1)d}^k \beta_i(n, x). \quad (4.9)$$

After determining the values of conditional transition probabilities  $\sigma_i(n)$  for calls of different classes, it is possible to determine the occupancy distribution in the group on the basis of an appropriately modified Formula (2.6) (i.e., after taking into account (4.7), (4.8), and (4.9)), will take on the following form:

$$nP(n) = \sum_{i=1}^{m_i} A_i t_i \sigma_i(n-t_i) P(n-t_i). \quad (4.10)$$

The blocking probability in the considered group for calls of class  $i$  can be written as follows:

$$E_i = \sum_{n=(f-t_i+1)d}^V P(n) \beta_i(n). \quad (4.11)$$

Distribution (4.10) is an approximate distribution since the service process of calls in Erlang's Ideal Grading with the multirate links is not a reversible process (due to the fact that equality (3.19) is not satisfied, just as in the case of distribution (3.17) determined in Section 3.5 for the model of EIG with various availabilities and single-rate Erlang traffic streams.

## 5. Numerical Examples

The accuracy of particular models of gradings discussed in earlier sections has been verified in simulations. For this purpose, an appropriate simulator has been designed that makes it possible to study Erlang's Ideal Gradings with large capacities servicing any number of classes of calls. The appropriate simulation program also enables us to simulate groups with non-integer values of availability servicing multirate BPP traffic streams. The simulation program has been constructed according to event-scheduling approach and has been implemented in C++. The simulation results are shown in the form of appropriately denoted points with 95 percent confidence interval, calculated according to the *t*-Student distribution for 5 series, with 100000 calls (of the class generating the least number of calls) in each series.

The results obtained in the study are presented in Figures 9–16. The results presented in Figures 9–16 are presented in relation to traffic offered to one BBU of the group.

Figure 9 shows the results obtained for Erlang's Ideal Grading with the capacity of  $V = 32$  BBUs that services single-service traffic (Section 3.2). The group services  $m_I = 3$  classes of single-rate calls with different availability for individual classes ( $d_1 = 5$  BBUs,  $d_2 = 10$  BBUs,  $d_3 = 25$  BBUs). Traffic offered by particular classes of calls satisfies the condition  $A_1 : A_2 : A_3 = 1 : 1 : 1$ .

Figures 10 and 11 show the results obtained for the grading servicing multirate traffic. Similarly as in the previous examples, the capacity of the group was equal to 32 BBUs. In both presented examples, the groups service three classes of calls with the following demands:  $t_1 = 1$  BBU,  $t_2 = 2$  BBUs,  $t_3 = 6$  BBUs. Figure 10 shows the results for the group in which all classes of calls are described by an identical availability parameter  $d = 20$  BBU. Figure 11 shows the results for the group in which each class of calls has a different availability parameter ( $d_1 = 5$  BBUs,  $d_2 = 10$  BBUs,  $d_3 = 25$  BBUs).

The results of modeling gradings with Erlang-Engset-Pascal multirate traffic are presented in Figures 12–14. Figure 12 shows the results obtained for Erlang's Ideal Grading with the capacity of  $V = 32$  BBUs that services three traffic classes:  $m_I = 1$ ,  $m_J = 1$ ,  $m_K = 1$ . First class (Erlang) demands  $t_1 = 1$  BBU, second class (Engset) demands  $t_2 = 3$  BBUs, and third class (Pascal) demands  $t_3 = 5$  BBUs. All classes of calls are described by an identical availability parameter  $d = 20$  BBUs. Figure 13 presents the results of modeling full-availability groups with bandwidth reservation by a grading. The presented results indicate that it is possible to find such an availability value for each class of calls in which equalisation of probabilities of serviced call classes at the level of blocking in a group with reservation ensues. The subsequent graph, presented in Figure 14, also includes those values of availability in which equalisation of blocking takes place. It is easily noticeable that equalisation at the level obtained in the group with reservation is possible when availability of the class demanding the highest number of BBUs is equal to the capacity of the group. If availability for this class is lower, then equalisation of blocking probabilities is still possible though at a higher level.

The next step was to evaluate accuracy of the model of a grading with multirate links. The study was carried out for groups carrying a mixture of different multirate traffic that had a different structure (number of links, capacity of links, and availability). Figure 15 shows the results for the group with the following structure:  $V = 500$  BBUs,  $v = 10$  links,  $f = 50$  BBUs,  $d = 7$  link, and  $D = 350$  BBUs. The group services three classes of calls that demand, respectively,  $t_1 = 1$  BBU,  $t_2 = 3$  BBUs, and  $t_3 = 7$  BBUs. It was adopted that traffic offered by all the classes of calls satisfied the condition:  $A_1 : A_2 : A_3 = 1 : 1 : 1$ . Figure 16, in turn, shows the results of the investigations for the group:  $V = 448$  BBUs,  $v = 7$  links,  $f = 64$  BBUs,

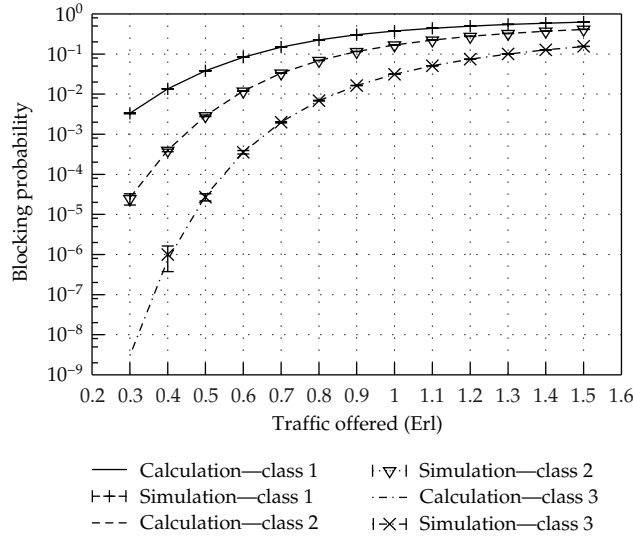


Figure 9: Blocking probability in Erlang’s Ideal Grading servicing 3 classes of calls ( $V = 32$  BBUs,  $t_1 = t_2 = t_3 = 1$  BBU,  $d_1 = 5$  BBUs,  $d_2 = 10$  BBUs,  $d_3 = 25$  BBUs).

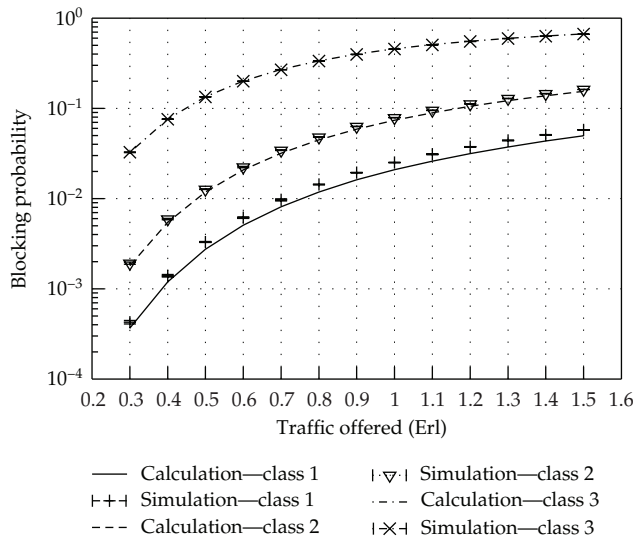


Figure 10: Blocking probability in Erlang’s Ideal Grading servicing 3 classes of calls ( $V = 32$  BBU,  $t_1 = 1$  BBU,  $t_2 = 2$  BBUs,  $t_3 = 6$  BBUs,  $d_1 = d_2 = d_3 = 20$  BBUs).

$d = 5$  links, and  $D = 320$  BBUs. In the latter case it was assumed that offered traffic satisfied the condition:  $A_1 t_1 : A_2 t_2 : A_3 t_3 = 1 : 1 : 1$ .

The comparison of the simulation data with the analytical calculations for the all considered gradings’ structures clearly indicates satisfactory accuracy of the proposed methodology.

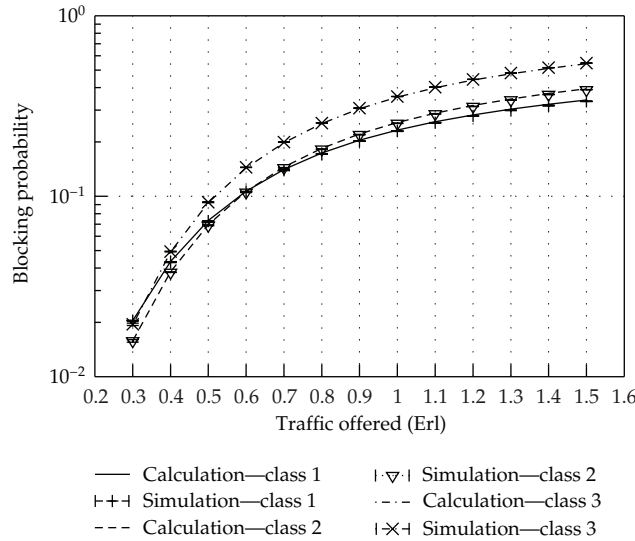


Figure 11: Blocking probability in Erlang’s Ideal Grading servicing 3 classes of calls ( $V = 32$  BBU,  $t_1 = 1$  BBU,  $t_2 = 2$  BBU,  $t_3 = 6$  BBU,  $d_1 = 5$  BBU,  $d_2 = 10$  BBU,  $d_3 = 25$  BBU).

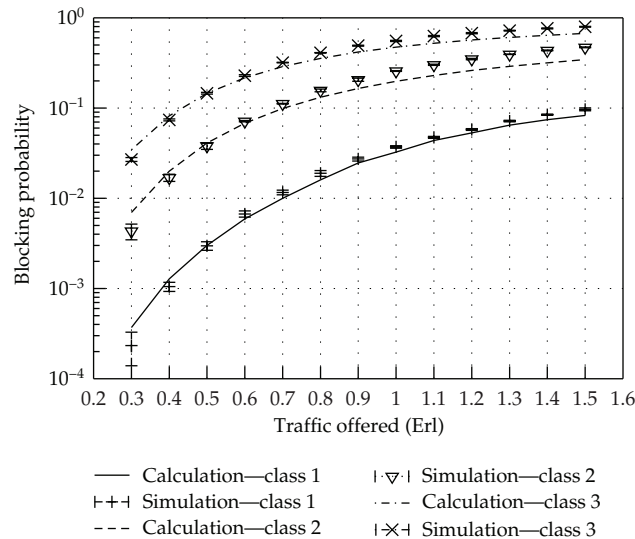
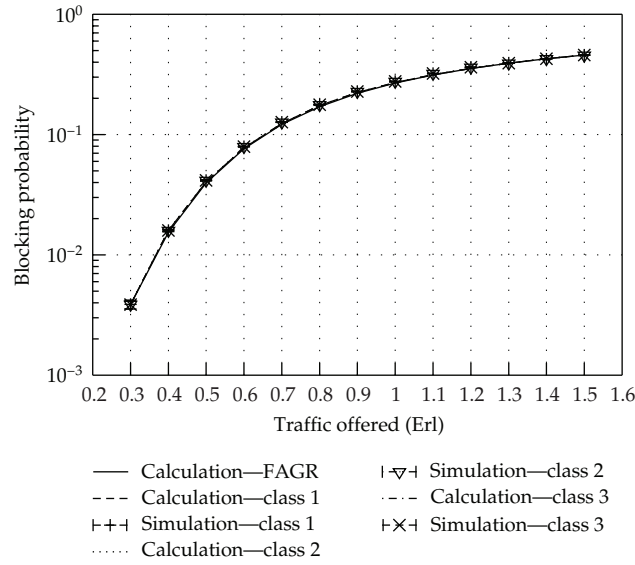


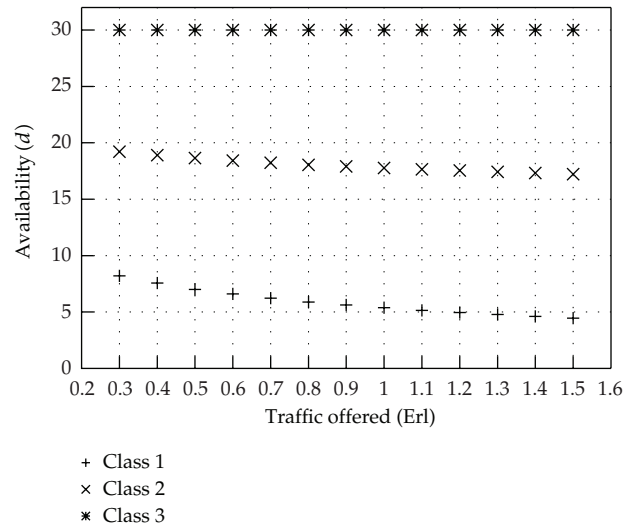
Figure 12: Blocking probability in Erlang’s Ideal Grading servicing 3 classes of calls ( $V = 32$  BBU,  $t_1 = 1$  BBU (Erlang),  $t_2 = 3$  BBU (Engset),  $t_3 = 6$  BBU (Pascal),  $d = 15$ ).

## 6. Conclusions

The paper presents a new complete methodology for modeling Erlang’s Ideal Gradings. In the paper the various structures of EIGs are considered, both with identical and various values of availability parameter for particular traffic streams offered to the groups. The models proposed allow us to determine traffic characteristics of the groups servicing multirate BPP traffic streams, also for the systems with non-integer values of the availability parameter.



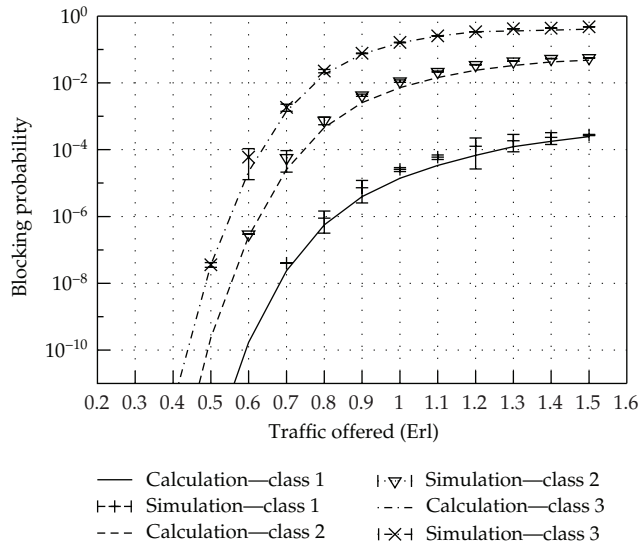
**Figure 13:** Equalised blocking probability in Erlang’s Ideal Grading servicing 3 classes of calls ( $V = 32$  BBU,  $t_1 = 1$  BBU (Erlang),  $t_2 = 3$  BBU (Pascal),  $t_3 = 5$  BBU (Engset)).



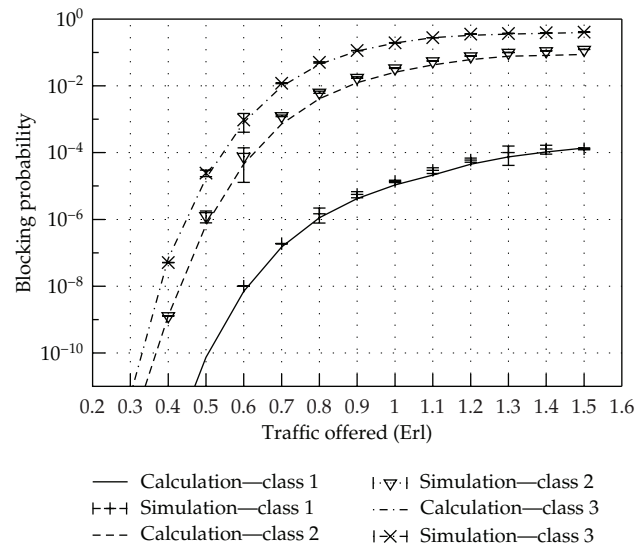
**Figure 14:** Values of availability parameters required for blocking probability equalisation in Erlang’s Ideal Grading servicing 3 classes of calls ( $V = 32$  BBU,  $t_1 = 1$  BBU (Erlang),  $t_2 = 3$  BBU (Pascal),  $t_3 = 5$  BBU (Engset)).

The introduction of a possibility of making calculations for non-integer values has much improved and broadened the scope of the application of models of gradings. As a result of the study carried out by the authors of the paper it turned out that models of gradings could be used to determine traffic characteristics of other state-dependent systems, for example, the systems with bandwidth reservation and the systems with limited-availability.





**Figure 15:** Blocking probability in Erlang’s Ideal Grading with the multirate links:  $V = 500$  BBUs,  $v = 10$  links,  $f = 50$  BBUs,  $d = 7$  links, and  $D = 350$  BBUs.



**Figure 16:** Blocking probability in Erlang’s Ideal Grading with the multirate links:  $V = 448$  BBUs,  $v = 7$  links,  $f = 64$  BBUs,  $d = 5$  links, and  $D = 320$  BBUs.

The models presented in the paper can prove to be very useful in engineering practice. This is best testified by the first applications of the generalized EIG model in modeling overflow systems [29] and VoD systems [28]. VoD systems and overflow systems create directly non full-availability systems. The approximation of such systems by the ideal grading with different availabilities seems to be promising [28, 29]. The models can be also used for modeling the radio interface of the UMTS (WCDMA) system, which is in line with

the current trends in modeling wireless systems by non full-availability models [30]. In this case, interference has a direct influence on the values of availability in a WCDMA cell.

The proposed model also makes it possible to apply directly the effective availability method [36] in modeling broadband switching networks with multirate traffic. It seems feasible to reduce calculations of the blocking probability in multi-stage switching networks servicing a mixture of multirate traffic to calculations of this probability in the single-stage system: in Erlang's Ideal Grading with the multirate links. This means that, after determining the effective availability in the switching network, one distribution will suffice to determine the blocking probability. Such an approach will greatly simplify modeling such complicated systems as multi-stage switching networks servicing multirate traffic.

The analytical calculations were made for many groups servicing multirate BPP traffic and were differentiated by their link capacity and the values of availability parameter. To verify the accuracy of the proposed models, the results of the calculations were compared with the results of the simulation. The simulation experiments indicate good accuracy of the calculations for any randomly selected link capacities. This proves all the adopted theoretical assumptions for the proposed model right. Furthermore, it should be stressed that the calculations made according to the proposed formulae are not complicated and are easily programmable.

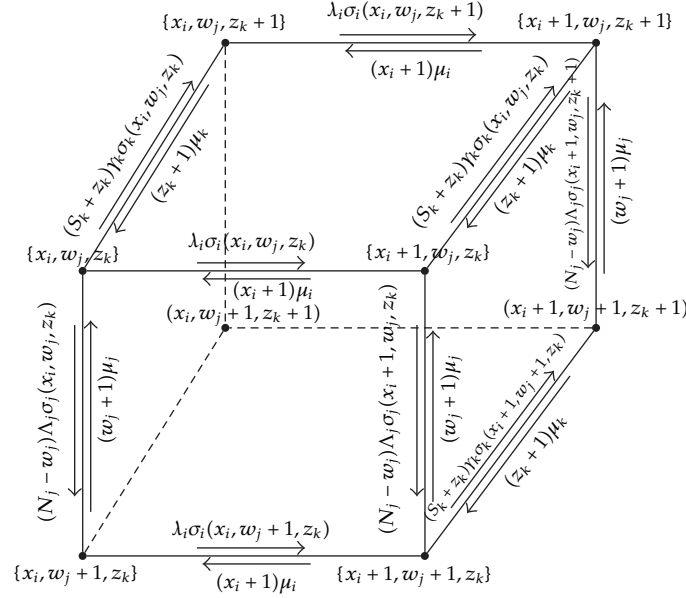
## Appendices

### A. Determination of the Occupancy Distribution in State-Dependent Multirate Systems

Figure 17 shows a fragment of a multidimensional Markov process occurring in the system under consideration. Each state of the process, the so-called microstate, is described by an ordered set of integer numbers  $\{x_1, \dots, x_i, \dots, x_{m_I}, w_1, \dots, w_j, \dots, w_{m_J}, z_1, \dots, z_k, \dots, z_{m_K}\}$ , where  $x_i$  denotes the number of serviced calls of a Erlang traffic stream of class  $i$ ,  $w_j$  denotes the number of serviced calls of an Engset traffic stream of class  $j$ , whereas  $z_k$  denotes the number of serviced calls of a Pascal traffic stream of class  $k$ . For convenience, to simplify the description, the probability of the microstate will be denoted by the symbol  $p(\dots, x_i, w_j, z_k, \dots)$ .

The total number of busy BBUs in the system is limited by the capacity of a group, that is,  $\sum_{c=1}^{m_M} y_c t_c \leq V$ , where  $m_M = m_I + m_J + m_K$  is the total number of traffic classes offered to the system,  $c$  is the index indicating any traffic class (Erlang, Engset, or Pascal), whereas  $y_c$  is the number of serviced calls of class  $c$ . The dependence between the acceptance of new calls and the occupancy state of the system is taken into account in the considered Markov process through the introduction of the conditional transition probability  $\sigma_c(y_1, \dots, y_c, \dots, y_{m_M})$  between neighbouring microstates. This coefficient determines what part of the input call stream will be transferred between the microstates  $\{y_1, \dots, y_c, \dots, y_{m_M}\}$  and  $\{y_1, \dots, y_c + 1, \dots, y_{m_M}\}$ .

One of the fundamental properties of Markov processes is their reversibility [37, 38]. In order to demonstrate and confirm the reversibility of the process it should be checked whether the so-called Kolmogorov criterion is satisfied [38]. According to this criterion, in the case of the Markov process in a system servicing  $m_M$  call streams, a necessary and sufficient condition for the reversibility of the process is the summation, equal to zero, of the products of the intensities of call streams and service streams between two randomly selected microstates for any cycle linking these microstates [7].



**Figure 17:** A fragment of the diagram of the Markov process in a system with state-dependent call admission process servicing BPP traffic.

To determine the reversibility of the process, consider the intensities of the transitions (call and service streams) in a cycle between the following exemplary microstates of the process:  $\{\dots, x_i, w_j, z_k, \dots\}$ ,  $\{\dots, x_i, w_{j+1}, z_k, \dots\}$ ,  $\{\dots, x_{i+1}, w_{j+1}, z_k, \dots\}$ ,  $\{\dots, x_{i+1}, w_{j+1}, z_{k+1}, \dots\}$ ,  $\{\dots, x_{i+1}, w_j, z_{k+1}, \dots\}$ ,  $\{\dots, x_i, w_j, z_{k+1}, \dots\}$ .

The summation of the products of the intensities of call and service streams will be equal to zero if the products of these parameters, determined for the considered states in the cycles “to the right” and “to the left,” will be equal to each other. Let us compare then the products of the intensities of transitions for the considered states of the process for cycles “to the right” and “to the left”:

cycle “to the right”

$$\begin{aligned} & (S_k + z_k) \sigma_z(\dots, x_i, w_j, z_k, \dots) \gamma_k \lambda_i \sigma_i(\dots, x_i, w_j, z_k + 1, \dots) \\ & \times (N_j - w_j) \sigma_j(\dots, x_i + 1, w_j, z_k + 1, \dots) \Lambda_j (z_k + 1) \mu_k (x_i + 1) \mu_i (w_j + 1) \mu_j, \end{aligned} \quad (\text{A.1})$$

cycle “to the left”

$$\begin{aligned} & (N_j - w_j) \sigma_j(\dots, x_i, w_j, z_k, \dots) \Lambda_j \lambda_i \sigma_i(\dots, x_i, w_j + 1, z_k, \dots) \\ & \times (S_k + z_k) \sigma_z(\dots, x_i + 1, w_j + 1, z_k, \dots) \gamma_k (w_j + 1) \mu_k (x_i + 1) \mu_i (z_k + 1) \mu_k. \end{aligned} \quad (\text{A.2})$$

It is easily noticeable that the products of the intensities (A.1) and (A.2) will be equal to each other only when the following condition is met:

$$\begin{aligned} & \sigma_z(\dots, x_i, w_j, z_k, \dots) \sigma_i(\dots, x_i, w_j, z_k + 1, \dots) \sigma_j(\dots, x_i + 1, w_j, z_k + 1, \dots) \\ & = \sigma_j(\dots, x_i, w_j, z_k, \dots) \sigma_i(\dots, x_i, w_j + 1, z_k, \dots) \sigma_z(\dots, x_i + 1, w_j + 1, z_k, \dots). \end{aligned} \quad (\text{A.3})$$

In most systems considered in traffic theory, (A.3) is not satisfied, which means that the Markov process is not a reversible process. However, a non reversible Markov process can be though approximated by a reversible process if the following assumptions are satisfied [2, 39].

*Assumption A.1.* One has

$$\sigma_c(y_1, \dots, y_c, \dots, y_{m_M}) = \sigma_c(n). \quad (\text{A.4})$$

Assumption A.1 implies that for microstates that satisfy the condition

$$\sum_{c=1}^{m_M} y_c = n, \quad (\text{A.5})$$

the dependence between the parameters  $\sigma_c(y_1, \dots, y_c, \dots, y_{m_M})$  and the distribution of busy BBUs among serviced calls can be omitted. A set of microstates that satisfy the condition (A.5) is called a macrostate.

*Assumption A.2.* One has

$$f(n) = \left| \frac{\sigma_c(n) - \sigma_c(n-1)}{\sigma_c(n-1)} \right| \ll 1. \quad (\text{A.6})$$

Assumption A.2 implies that Assumption A.1 is satisfied if  $\sigma_c$  is a slowly changing function of  $n$ . In other words, it is assumed that the differences between the values of parameters  $\sigma_c$  within a given macrostate will not exceed the values of these parameters within neighbouring macrostates.

The approximation of the considered Markov process by a reversible process makes it possible to consider all streams independently and to analyse the process on the basis of the local equilibrium equations. Such equations for an Erlang stream of class  $i$ , Engset stream of

class  $j$ , and Pascal stream of class  $k$  can be written as follows:

$$\begin{aligned} x_i \mu_i p(\dots, x_i, w_j, z_k, \dots) \\ = \lambda_i \sigma_i(\dots, x_i - 1, w_j, z_k, \dots) p(\dots, x_i - 1, w_j, z_k, \dots), \end{aligned} \quad (\text{A.7})$$

$$\begin{aligned} w_j \mu_j p(\dots, x_i, w_j, z_k, \dots) \\ = [N_j - (w_j - 1)] \Lambda_j \sigma_j(\dots, x_i, w_j - 1, z_k, \dots) p(\dots, x_i, w_j - 1, z_k, \dots), \end{aligned} \quad (\text{A.8})$$

$$\begin{aligned} z_k \mu_k p(\dots, x_i, w_j, z_k, \dots) \\ = [S_k + (z_k - 1)] \Upsilon_k \sigma_k(\dots, x_i, w_j, z_k - 1, \dots) p(\dots, x_i, w_j, z_k - 1, \dots). \end{aligned} \quad (\text{A.9})$$

Independence of call streams offered to the group makes it possible to sum up all  $m_I$  equations of the type (A.7) for Erlang streams,  $m_J$  equations of the type (A.8) for Engset streams, and  $m_K$  equations for Pascal streams of the type (A.9) for the microstate  $\{\dots, x_i, w_j, z_k, \dots\}$ . Taking additionally into account the definition of traffic intensity, we get

$$\begin{aligned} p(\dots, x_i, w_j, z_k, \dots) & \left[ \sum_{i=1}^{m_I} x_i t_i + \sum_{j=1}^{m_J} w_j t_j + \sum_{k=1}^{m_K} z_k t_k \right] \\ & = \sum_{i=1}^{m_I} A_i t_i \sigma_i(\dots, x_i - 1, w_j, z_k, \dots) p(\dots, x_i - 1, w_j, z_k, \dots) \\ & + \sum_{j=1}^{m_J} [N_j - (w_j - 1)] \alpha_j t_j \sigma_j(\dots, x_i, w_j - 1, z_k, \dots) p(\dots, x_i, w_j - 1, z_k, \dots) \\ & + \sum_{k=1}^{m_K} [S_k + (z_k - 1)] \beta_k t_k \sigma_k(\dots, x_i, w_j, z_k - 1, \dots) p(\dots, x_i, w_j, z_k - 1, \dots). \end{aligned} \quad (\text{A.10})$$

From this point, the analysis of the process occurring in the system under consideration is carried out at the level of macrostates. The macrostate probability  $P(n)$  is the occupancy probability of  $n$  BBUs in the group and can be expressed by the probabilities of appropriate microstates:

$$P(n) = \sum_{\Omega(n)} p(\dots, x_i, w_j, z_k, \dots), \quad (\text{A.11})$$

where  $\Omega(n)$  is a set of all such microstates  $\{\dots, x_i, w_j, z_k, \dots\}$  that satisfy the following equation:

$$n = \sum_{i=1}^{m_I} x_i t_i + \sum_{j=1}^{m_J} w_j t_j + \sum_{k=1}^{m_K} z_k t_k. \quad (\text{A.12})$$

After taking the definition of macrostate (A.12) into consideration, formula (A.10) can be transformed as follows:

$$\begin{aligned}
& np(\dots, x_i, w_j, z_k, \dots) \\
&= \sum_{i=1}^{m_I} A_i t_i p(\dots, x_i - 1, w_j, z_k, \dots) \sigma_i(\dots, x_i - 1, w_j, z_k, \dots) \\
&\quad + \sum_{j=1}^{m_J} [N_j - (w_j - 1)] \alpha_j t_j p(\dots, x_i, w_j - 1, z_k, \dots) \sigma_j(\dots, x_i, w_j - 1, z_k, \dots) \\
&\quad + \sum_{k=1}^{m_K} [S_k + (z_k - 1)] \beta_k t_k p(\dots, x_i, w_j, z_k - 1, \dots) \sigma_k(\dots, x_i, w_j, z_k - 1, \dots).
\end{aligned} \tag{A.13}$$

Summing up on both sides, all the microstates that are elements of the set  $\Omega(n)$ , we get

$$\begin{aligned}
& n \sum_{\Omega(n)} p(\dots, x_i, w_j, z_k, \dots) \\
&= \sum_{i=1}^{m_I} A_i t_i \sum_{\Omega(n)} p(\dots, x_i - 1, w_j, z_k, \dots) \sigma_i(\dots, x_i - 1, w_j, z_k, \dots) \\
&\quad + \sum_{j=1}^{m_J} [N_j - (w_j - 1)] \alpha_j t_j \sum_{\Omega(n)} p(\dots, x_i, w_j - 1, z_k, \dots) \sigma_j(\dots, x_i, w_j - 1, z_k, \dots) \\
&\quad + \sum_{k=1}^{m_K} [S_k + (z_k - 1)] \beta_k t_k \sum_{\Omega(n)} p(\dots, x_i, w_j, z_k - 1, \dots) \sigma_k(\dots, x_i, w_j, z_k - 1, \dots).
\end{aligned} \tag{A.14}$$

After the application of the definition of the macrostate probability (formula (A.11)), we can transform formula (A.14) in the following way:

$$\begin{aligned}
P(n) &= \sum_{i=1}^{m_I} A_i t_i P(n - t_i) \sigma_i(\dots, x_i - 1, w_j, z_k, \dots) \\
&\quad + \sum_{j=1}^{m_J} \alpha_j t_j \sum_{\Omega(n)} [N_j - (w_j - 1)] \frac{p(\dots, x_i, w_j - 1, z_k, \dots)}{\sum_{\Omega(n)} p(\dots, x_i, w_j - 1, z_k, \dots)} \\
&\quad \times \sum_{\Omega(n)} p(\dots, x_i, w_j - 1, z_k, \dots) \sigma_j(\dots, x_i, w_j - 1, z_k, \dots) \\
&\quad + \sum_{k=1}^{m_K} \beta_k t_k \sum_{\Omega(n)} [S_k + (z_k - 1)] \frac{p(\dots, x_i, w_j, z_k - 1, \dots)}{\sum_{\Omega(n)} p(\dots, x_i, w_j, z_k - 1, \dots)} \\
&\quad \times \sum_{\Omega(n)} p(\dots, x_i, w_j, z_k - 1, \dots) \sigma_k(\dots, x_i, w_j, z_k - 1, \dots),
\end{aligned} \tag{A.15}$$

where  $P(n - t_c) = 0$ , if  $n < t_c$ , and the value  $P(0)$  results from a normative condition  $\sum_{n=0}^V P(n) = 1$ . In formula (A.15) the sum:

$$\begin{aligned} \sum_{\Omega(n)} [w_j - 1] \frac{p(\dots, x_i, w_j - 1, z_k, \dots)}{\sum_{\Omega(n)} p(\dots, x_i, w_j - 1, z_k, \dots)} &= n_j(n - t_j), \\ \sum_{\Omega(n)} [z_k - 1] \frac{p(\dots, x_i, w_j, z_k - 1, \dots)}{\sum_{\Omega(n)} p(\dots, x_i, w_j, z_k - 1, \dots)} &= n_k(n - t_k) \end{aligned} \quad (\text{A.16})$$

determine the mean value of the number of calls of, respectively, class  $j$  and class  $k$  in occupancy states  $n - t_j$  and  $n - t_k$ .

After taking formula (A.4) and formulae (A.16) into consideration, formula (A.15) can be written in the form of formula (2.6):

$$\begin{aligned} nP(n) &= \sum_{i=1}^{m_I} A_i t_i P(n - t_i) \sigma_i(n - t_i) \\ &+ \sum_{j=1}^{m_J} \alpha_j t_j [N_j - n_j(n - t_j)] P(n - t_j) \sigma_j(n - t_j) \\ &+ \sum_{k=1}^{m_K} \beta_k t_k [S_k + n_k(n - t_k)] P(n - t_k) \sigma_k(n - t_k), \end{aligned} \quad (\text{A.17})$$

where  $P(n)$  denotes the probability of macrostate  $n$ , that is, the probability that the system is in the occupancy state  $n$  BBU's, and  $P(n) = 0$  for  $n < 0$  and  $n > V$ .

## B. Proof of Convergence of the Method for a Determination of the Average Number of Calls Serviced in State-Dependent Multirate Systems

The presented Algorithm 2.1 of a determination of the average number of serviced traffic sources in particular occupancy states is a convergent process. In order to prove the convergence of the algorithm in question, the further part of the paper includes a proof of two theorems that refer to a determination of the average number of serviced Engset and Pascal calls, respectively.

**Theorem B.1.** *The sequence  $(n_j^{(l)}(n))_{l=0}^{\infty}$  of the average number of serviced Engset calls of class  $j$  in a grading (in a state-dependent systems), where*

$$n_j^{(l)}(n) = \frac{[N_j - n_j^{(l-1)}(n - t_j)] \alpha_j \sigma_j(n - t_j) P^{(l-1)}(n - t_j)}{P^{(l-1)}(n)}, \quad (\text{B.1})$$

$$n_j^{(0)}(n) = 0 \quad \text{for } \forall_{1 \leq j \leq m_J}, \forall_{0 \leq n \leq V}, \quad (\text{B.2})$$

is convergent.

*Proof.* To prove Theorem B.1 we first demonstrate that the subsequent elements of sequence (B.1), starting from the first one, can be presented in the form of a finite sum:

$$n_j^{(l)}(n) = \sum_{s=1}^l (-1)^{s+1} N_j \alpha_j^s \prod_{i=1}^j \sigma_j(n - it_j) \frac{P^{(l-i)}(n - it_j)}{P^{(l-i)}(n - (i-1)t_j)}. \quad (\text{B.3})$$

Since  $n_j^{(0)}(n - t_j) = 0$ , then on the basis of formula (B.1) for  $l = 1$  we get

$$n_j^{(1)}(n) = N_j \alpha_j \sigma_j(n - t_j) \frac{P^{(0)}(n - t_j)}{P^{(0)}(n)}. \quad (\text{B.4})$$

Determining the value  $n_j^{(1)}(n - t_j)$  on the basis of formula (B.4) and using the dependence (B.1) for  $l = 2$ , we get

$$n_j^{(2)}(n) = \frac{[N_j - (N_j \alpha_j \sigma_j(n - 2t_j) P^{(0)}(n - 2t_j) / P^{(0)}(n - t_j))] \alpha_j \sigma_j(n - t_j) P^{(1)}(n - t_j)}{P^{(1)}(n)}. \quad (\text{B.5})$$

Ultimately, formula (B.5) can be rewritten in the following way:

$$\begin{aligned} n_j^{(2)}(n) &= N_j \alpha_j \sigma_j(n - t_j) \frac{P^{(1)}(n - t_j)}{P^{(1)}(n)} \\ &+ -N_j \alpha_j^2 \sigma_j(n - t_j) \sigma_j(n - 2t_j) \frac{P^{(0)}(n - 2t_j) P^{(1)}(n - t_j)}{P^{(0)}(n - t_j) P^{(1)}(n)}. \end{aligned} \quad (\text{B.6})$$

Similarly, for  $l = 3$ , we get

$$\begin{aligned} n_j^{(3)}(n) &= N_j \alpha_j \sigma_j(n - t_j) \frac{P^{(2)}(n - t_j)}{P^{(2)}(n)} \\ &+ -N_j \alpha_j^2 \sigma_j(n - t_j) \sigma_j(n - 2t_j) \frac{P^{(1)}(n - 2t_j) P^{(2)}(n - t_j)}{P^{(1)}(n - t_j) P^{(2)}(n)} \\ &+ N_j \alpha_j^3 \sigma_j(n - t_j) \sigma_j(n - 2t_j) \sigma_j(n - 3t_j) \times \frac{P^{(0)}(n - 3t_j) P^{(1)}(n - 2t_j) P^{(2)}(n - t_j)}{P^{(0)}(n - 2t_j) P^{(1)}(n - t_j) P^{(2)}(n)}. \end{aligned} \quad (\text{B.7})$$

In general, the value of a subsequent element of the sequence ( $n_j^{(l)}(n)$ ) in  $l$ th step can be written by formula (B.3). In passing to the boundary in infinity ( $l \rightarrow \infty$ ), we get

$$\lim_{l \rightarrow \infty} n_j^{(l)}(n) = \sum_{j=1}^{\infty} (-1)^{j+1} N_j \alpha_j^j \prod_{i=1}^j \sigma_j(n - it_j) \frac{P^{(l-i)}(n - it_j)}{P^{(l-i)}(n - (i-1)t_j)}. \quad (\text{B.8})$$



Since for any  $s < 0$ , irrespective of a step in the iteration, the probability that the system is in state  $s$  is equal to 0 (i.e.,  $P^{(l)}(s) = 0$ ), formula (B.8) can be rewritten in the following way:

$$\lim_{l \rightarrow \infty} n_j^{(l)}(n) = \sum_{s=1}^{\lfloor n/t_j \rfloor} (-1)^{s+1} N_j \alpha_j^s \prod_{i=1}^s \sigma_j(n - it_j) \frac{P^{(l-i)}(n - it_j)}{P^{(l-i)}(n - (i-1)t_j)}. \quad (\text{B.9})$$

The sequence on the right side of (B.9) is summable; therefore there is a finite boundary for the sequence  $(n_j^{(l)}(n))_{l=0}^{\infty}$ , which proves its convergence.  $\square$

**Theorem B.2.** *The sequence  $(n_k^{(l)}(n))_{l=0}^{\infty}$  of the average number of serviced Pascal calls of class  $k$  in a grading (in a state-dependent systems), where*

$$n_k^{(l)}(n) = \frac{\left[ S_k + n_k^{(l-1)}(n - t_k) \right] \alpha_k \sigma_k(n - t_k) P^{(l-1)}(n - t_k)}{P^{(l-1)}(n)}, \quad (\text{B.10})$$

$$n_k^{(0)}(n) = 0 \quad \text{for } \forall_{1 \leq k \leq m_k}, \forall_{0 \leq n \leq V},$$

is convergent.

*Proof.* Proceeding analogously as in the case of sequence (B.1), it is possible to prove that the elements of the sequence  $(n_k^{(l)}(n))_{l=0}^{\infty}$  can be expressed by the following dependence:

$$n_k^{(l)}(n) = \sum_{j=1}^l S_k \beta_k^j \prod_{i=1}^j \sigma_k(n - it_k) \frac{P^{(l-i)}(n - it_k)}{P^{(l-i)}(n - (i-1)t_k)}. \quad (\text{B.11})$$

Similarly as in Theorem B.1 and referring to the fact that, irrespectively of a step of the iteration, for any  $s < 0$  the probability that the system is in state  $s$  is equal to 0 (i.e.,  $P^{(l)}(s) = 0$ ), we can write

$$\lim_{l \rightarrow \infty} n_k^{(l)}(n) = \sum_{j=0}^{\lfloor n/t_k \rfloor} S_k \beta_k^j \prod_{i=1}^j \sigma_k(n - it_k) \frac{P^{(l-i)}(n - it_k)}{P^{(l-i)}(n - (i-1)t_k)}. \quad (\text{B.12})$$

The right side of (B.12) is a summable sequence, which proves the convergence of the sequence  $(n_k^{(l)}(n))_{l=0}^{\infty}$ .  $\square$

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