Research Article

Multiscale Discretization Scheme Based on the Rayleigh Quotient Iterative Method for the Steklov Eigenvalue Problem

Hai Bi and Yidu Yang

School of Mathematics and Computer Science, Guizhou Normal University, Guiyang 550001, China

Correspondence should be addressed to Hai Bi, junglebee@163.com

This paper discusses efficient numerical methods for the Steklov eigenvalue problem and establishes a new multiscale discretization scheme and an adaptive algorithm based on the Rayleigh quotient iterative method. The efficiency of these schemes is analyzed theoretically, and the constants appeared in the error estimates are also analyzed elaborately. Finally, numerical experiments are provided to support the theory.

1. Introduction

Steklov eigenvalue problems have several deep applications both in physical and mechanical fields. For instance, they are found in the study of surface waves (see [1]), in the analysis of stability of mechanical oscillators immersed in a viscous fluid (see [2]), and in the study of the vibration modes of a structure in contact with an incompressible fluid (see, e.g. [3]). Thus, numerical methods for Steklov eigenvalue problems have attracted more and more scholars' attention in recent years, for example, see [4–12].

However, in practical applications, it is a challenging task to adopt efficient numerical methods to reduce the computational costs, such as the CPU time and the storage requirement, without any loss of the accuracy of numerical approximation. A two-grid finite element discretization scheme is considered one of these efficient methods. This discretization technique was first introduced by Xu [13, 14] for nonsymmetric and nonlinear elliptic problems, and due to its outstanding performance in computation, it has been successfully applied and further investigated for many other problems, for example, Poisson eigenvalue equations and integral equations in [15], nonlinear eigenvalue problems in [16], Schrödinger equation in [17, 18], Stokes equations in [19], and so forth. As for the Steklov eigenvalue

problem, Li and Yang [20], Bi and Yang [21] established conforming and nonconforming finite element two-grid discretization scheme, respectively, and obtained satisfactory results. So far, to the best of our knowledge, there have been developed two kinds of two-grid discretization scheme for eigenvalue problems. The first type is contributed by Xu and Zhou [15], as just mentioned, and the other is proposed by Yang and Bi in [22], which has been applied to elliptic eigenvalue problems [22] and eigenvalue problem of electric field [23]. Compared with the first type, the second one is based on the shifted-inverse power method which makes itself more efficient, and starting from it, one can construct multiscale discretization scheme and parallel algorithm. So, this paper aims to establish a multi-scale discretization scheme based on the shifted-inverse power method, and an adaptive algorithm for the Steklov eigenvalue problem. The special work of this paper is as follows.

(1) It is well known that the Rayleigh quotient iteration method is a basic approach for solving matrix eigenvalue problems (see Algorithm 27.3 in [24]). The multi-scale discretization scheme established in this paper is a combination of the finite element method and the Rayleigh quotient iteration method. We prove that this scheme is efficient, and analyze elaborately that the constants appeared in the error estimates are independent of mesh size and the iterative time. It can be seen easily that a two-grid discretization scheme is a special case of the multi-scale algorithm by taking the iterative time equal to 2 (see Scheme 2). From the theoretical analysis (see Theorem 3.2) we know that when taking $H = \mathcal{O}(\sqrt[3]{h})$, the resulted solution can maintain an asymptotically optimal accuracy, while with the schemes in [20, 21], the resulted solution maintains an asymptotically optimal accuracy by taking $H = \mathcal{O}(\sqrt{h})$.

(2) The multi-scale discretization scheme is actually an iterative process. Since the approximate eigenvalue sequence obtained by the scheme converges to the exact eigenvalue (see Theorem 3.1, Corollary 3.4) and the constants appeared in its error estimates are independent of the mesh size and the iterative time, we can stop the iteration when the error between two neighboring approximate eigenvalues is less or equal to a given error tolerance. Thus, we establish an adaptive algorithm (see Scheme 3). With this adaptive scheme, first we solve an eigenvalue problem on a coarse grid, in each step after that we only need to solve a linear algebraic system on a fine grid. Compared with the existing adaptive method (e.g., see Algorithm 1 in [11]) which computes an eigenvalue problem in each step, our approach reduces the computational complexity.

The rest of this paper is organized as follows. In the subsequent section, some preliminaries needed in this paper are presented. In Section 3, a multi-scale and a two-grid discretization scheme are proposed, and the error estimates are also given. In Section 4, an adaptive algorithm is established. Finally, numerical experiments are provided to support our theoretical analysis.

2. Preliminaries

Let $H^s(\Omega)$ and $H^s(\partial\Omega)$ denote Sobolev spaces on Ω and $\partial\Omega$ with real order s, respectively. The norms in $H^s(\Omega)$ and $H^s(\partial\Omega)$ are denoted by $\|\cdot\|_s$ and $\|\cdot\|_{s,\partial\Omega}$, respectively.

Consider the model problem

$$-\Delta u + u = 0 \text{ in}\Omega, \quad \frac{\partial u}{\partial n} = \lambda u \text{ on } \partial\Omega, \tag{2.1}$$

where $\Omega \subset R^2$ is a polygonal domain and $\partial u/\partial n$ is the outward normal derivative on $\partial\Omega$.

The weak form of (2.1) is given by the following: find $\lambda \in R, 0 \neq u \in H^1(\Omega)$, such that

$$a(u, v) = \lambda b(u, v), \quad \forall v \in H^1(\Omega), \tag{2.2}$$

where

$$a(u, v) = \int_\Omega \nabla u \nabla v + uv dx, \quad b(u, v) = \int_{\partial \Omega} uv \, ds. \tag{2.3}$$

It is easy to know that $a(\cdot, \cdot)$ is a symmetric, continuous, and $H^1(\Omega)$-elliptic bilinear form on $H^1(\Omega) \times H^1(\Omega)$. So, we use $a(\cdot, \cdot)$ and $\| \cdot \|_a = \sqrt{a(\cdot, \cdot)} = \| \cdot \|_1$ as the inner product and norm on $H^1(\Omega)$, respectively. We use $b(\cdot, \cdot)$ and $\| \cdot \|_b = \sqrt{b(\cdot, \cdot)}$ as the inner product and norm on $L^2(\partial \Omega)$, respectively.

From [25], we know that (2.2) has a countable sequence of positive eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \longrightarrow +\infty \tag{2.4}$$

(here each eigenvalue occurs as many times as given by its multiplicity), and the corresponding eigenfunctions $u_k \in H^{1+r}(\Omega)(k = 1, 2, \ldots)$, where $r = 1$ if Ω is convex, and $r < (\pi/\omega)$ which can be arbitrarily close to π/ω when Ω is concave (with ω being the largest inner angle of Ω, $\omega < 2\pi$).

Let π_h be a regular triangulation of Ω with the mesh diameter h, and let S^h be a piecewise polynomial space of degree m $(m \geq 1)$ defined on π_h.

The conforming finite element approximation of (2.2) is the following: find $\lambda_h \in R$, $0 \neq u_h \in S^h \subset H^1(\Omega)$, such that

$$a(u_h, v) = \lambda_h b(u_h, v), \quad \forall v \in S^h. \tag{2.5}$$

It is well known that (2.5) has a finite sequence of eigenvalues

$$0 < \lambda_{1,h} \leq \lambda_{2,h} \leq \cdots \lambda_{N_h,h} \left(N_h = \dim S^h \right), \tag{2.6}$$

and their corresponding eigenfunctions are $u_{k,h}(k = 1, 2, \cdots, N_h)$.

Consider the following source problem (2.7) associated with (2.2) and the approximate source problem (2.8) associated with (2.5), respectively.

Find $u \in H^1(\Omega)$, such that

$$a(u, v) = b(f, v), \quad \forall v \in H^1(\Omega). \tag{2.7}$$

Find $u_h \in S^h$, such that

$$a(u_h, v) = b(f, v), \quad \forall v \in S^h. \tag{2.8}$$

Several regular estimates of the Steklov eigenvalue problem are presented in the following lemma, which will be used in the sequel.

Lemma 2.1. *If $f \in L^2(\partial\Omega)$, then there exists a unique solution $u \in H^{1+(r/2)}(\Omega)$ to (2.7) and*

$$\|u\|_{1+(r/2)} \leq C_p \|f\|_{0,\partial\Omega}. \tag{2.9}$$

If $f \in H^{1/2}(\partial\Omega)$, then there exists a unique solution $u \in H^{1+r}(\Omega)$ to (2.7) and

$$\|u\|_{1+r} \leq C_p \|f\|_{1/2,\partial\Omega}. \tag{2.10}$$

If $f \in L^2(\partial\Omega)$, then there exists a unique solution $u \in H^1(\Omega)$ to (2.7) and

$$\|u\|_a \leq C_p \|f\|_{0,\partial\Omega}, \tag{2.11}$$

where C_p appeared in (2.9), (2.10), and (2.11) stands for a positive constant but may take different values.

Proof. As regards the proof of (2.9) and (2.10), see [26]. Since $a(\cdot,\cdot)$ is $H^1(\Omega)$-elliptic, from (2.7), it is easy to know that (2.11) is valid. □

Thus, from (2.7), we can define the operator $A : L^2(\partial\Omega) \to H^1(\Omega)$ by

$$a(Af,v) = b(f,v), \quad \forall v \in H^1(\Omega). \tag{2.12}$$

Similarly, from (2.8), we define the operator $A_h : L^2(\partial\Omega) \to S^h \subset H^1(\Omega)$ by

$$a(A_h f,v) = b(f,v), \quad \forall v \in S^h. \tag{2.13}$$

It is obvious that $A : H^1(\Omega) \to H^1(\Omega)$ is a self-adjoint operator. In fact, for any $u,v \in H^1(\Omega)$, $a(Au,v) = b(u,v) = b(v,u) = a(Av,u) = a(u,Av)$. Analogously, A_h is also a self-adjoint operator. Observe that Af and $A_h f$ are the exact solution and the finite element solution of (2.7), respectively, and $a(Af - A_h f,v) = 0, \forall v \in S^h \subset H^1(\Omega)$. Define the Ritz-Galerkin projection operator $P_h : H^1(\Omega) \to S^h$ by

$$a(u - P_h u,v) = 0, \quad \forall u \in H^1(\Omega), \ \forall v \in S^h. \tag{2.14}$$

Thus, $\forall f \in H^1(\Omega)$,

$$a(A_h f - P_h(Af),v) = a(A_h f - Af + Af - P_h(Af),v) = 0, \quad \forall v \in S^h. \tag{2.15}$$

Therefore, $A_h f = P_h A f, \forall f \in H^1(\Omega)$, then $A_h = P_h A$. From Lemma 2.1 and the interpolation error estimate, we have

$$
\begin{aligned}
\|A_h - A\|_a &= \sup_{g \in H^1(\Omega)} \frac{\|(A_h - A)g\|_a}{\|g\|_a} \\
&= \sup_{g \in H^1(\Omega)} \frac{\|P_h A g - A g\|_a}{\|g\|_a} \\
&\leq \sup_{g \in H^1(\Omega)} \frac{C_I h^r \|A g\|_{1+r}}{\|g\|_a} \\
&\leq \sup_{g \in H^1(\Omega)} \frac{C_I h^r \|g\|_a}{\|g\|_a} = C_I h^r \to 0 (h \to 0),
\end{aligned}
\tag{2.16}
$$

where C_I is the interpolation constant. It is clear that A_h is a finite rank operator, then A is a completely continuous operator. From [25, 27], we know that (2.2) and (2.5) have the following equivalent operator forms, respectively:

$$
\begin{aligned}
A u &= \mu u, \\
A_h u_h &= \mu_h u_h,
\end{aligned}
\tag{2.17}
$$

where $\mu = 1/\lambda, \mu_h = 1/\lambda_h$. In this paper, μ_k, and $\mu_{k,h}$, λ_k and $\lambda_{k,h}$ are all called eigenvalues.

Suppose that the algebraic multiplicity of μ_k is equal to q, $\mu_k = \mu_{k+1} = \cdots = \mu_{k+q-1}$. Let $M(\mu_k)$ be the space spanned by all eigenfunctions corresponding to μ_k of A, and let $M_h(\mu_k)$ be the direct sum of eigenspaces corresponding to all eigenvalues of A_h that converge to μ_k. Let $\widehat{M}(\mu_k) = \{v : v \in M(\mu_k), \|v\|_a = 1\}$, $\widehat{M}_h(\mu_k) = \{v : v \in M_h(\mu_k), \|v\|_a = 1\}$. We also write $M(\lambda_k) = M(\mu_k)$, $M_h(\lambda_k) = M_h(\mu_k)$, $\widehat{M}(\lambda_k) = \widehat{M}(\mu_k)$, and $\widehat{M}_h(\lambda_k) = \widehat{M}_h(\mu_k)$.

Let

$$
\delta_h(\lambda_k) = \sup_{w \in \widehat{M}(\lambda_k)} \inf_{v \in S^h} \|w - v\|_a.
\tag{2.18}
$$

From Lemma 2.1, we know that $\widehat{M}(\lambda_k) \subset H^{1+r}(\Omega)$, then by the interpolation error estimate, we have

$$
\delta_h(\lambda_k) \leq C_I h^r \sup_{w \in \widehat{M}(\lambda_k)} \|w\|_{1+r}.
\tag{2.19}
$$

Lemma 2.2. *Let $\lambda_{k,h}$ and λ_k be the kth eigenvalue of* (2.5) *and* (2.2), *respectively. Then*

$$
\lambda_k \leq \lambda_{k,h} \leq \lambda_k + C_1 \delta_h^2(\lambda_k),
\tag{2.20}
$$

for any eigenfunction $u_{k,h}$ corresponding to $\lambda_{k,h}$, satisfying $\|u_{k,h}\|_a = 1$, there exists $u_k \in \widehat{M}(\lambda_k)$ such that

$$\|u_{k,h} - u_k\|_a \le C_2 \delta_h(\lambda_k), \tag{2.21}$$

and for any $u_k \in \widehat{M}(\lambda_k)$, there exists $u_h \in M_h(\lambda_k)$ such that

$$\|u_h - u_k\|_a \le C_3 \delta_h(\lambda_k), \tag{2.22}$$

where $C_i, i = 1, 2, 3$ are constants independent of h.

 Proof. By the argument in [25, 27], we can obtain the desired results.   □

The following lemma states a crucial property (but straightforward) of eigenvalue and eigenfunction approximation.

Lemma 2.3. *Let (λ, u) be an eigenpair of (2.2), then for any $w \in H^1(\Omega), \|w\|_b \ne 0$, the Rayleigh quotient $a(w, w)/\|w\|_b^2$ satisfies*

$$\frac{a(w, w)}{\|w\|_b^2} - \lambda = \frac{\|w - u\|_a^2}{\|w\|_b^2} - \lambda \frac{\|w - u\|_b^2}{\|w\|_b^2}. \tag{2.23}$$

Proof. See, for instance, Lemma 9.1 in [27] for details.   □

We also need the following basic estimate of shifted-inverse power method (see Theorem 3.2 in [22]).

Lemma 2.4. *Let (μ_0, u_0) be an approximation for (μ_k, u_k), where μ_0 is not an eigenvalue of A_h, and $u_0 \in S^h$ with $\|u_0\|_a = 1$. Suppose that $\max_{k \le j \le k+q-1} |(\mu_{j,h} - \mu_{k,h})/(\mu_0 - \mu_{j,h})| \le 1/2$, $\text{dist}(u_0, M_h(\mu_k)) \le 1/2$, $|\mu_0 - \mu_{j,h}| \ge \rho/2$ for $j \ne k, k + 1, \ldots, k + q - 1$, and $u \in S^h, u_k^h \in S^h$ satisfy*

$$(\mu_0 - A_h)u = u_0, u_k^h = \frac{u}{\|u\|_a}. \tag{2.24}$$

Then

$$\text{dist}\left(u_k^h, \widehat{M}_h(\mu_k)\right) \le \frac{16}{\rho} |\mu_0 - \mu_{k,h}| \, \text{dist}(u_0, M_h(\mu_k)), \tag{2.25}$$

where $\rho = \min_{\mu_j \ne \mu_k} |\mu_j - \mu_k|$ be the separation constant of the eigenvalue μ_k.

For the multi-scale discretization scheme established in this paper, the conditions of Lemma 2.4 are satisfied which will be verified in the proof of Theorem 3.1.

3. Multi-Scale Discretization Scheme

In this section, we combine the finite element method with the Rayleigh quotient iteration method and establish a multi-scale discretization scheme. Let $\{\pi_{h_i}\}_1^l$ be a family of regular meshes, $h_{i-1} \gg h_i$, and let $\{S^{h_i}\}_1^l$ be the conforming finite element spaces defined on $\{\pi_{h_i}\}_1^l$, and let $\pi_H = \pi_{h_1}$, $S^H = S^{h_1}$, $\pi_h = \pi_{h_l}$, and $S^h = S^{h_l}$.

Scheme 1 (multi-scale discretization scheme).

Step 1. Solve (2.2) on the π_H: find $\lambda_{k,H} \in R$, $u_{k,H} \in S^H$ such that $\|u_{k,H}\|_a = 1$ and

$$a(u_{k,H}, v) = \lambda_{k,H} b(u_{k,H}, v), \quad \forall v \in S^H. \tag{3.1}$$

Step 2. Execute the assignments: $u_k^{h_1} \Leftarrow u_{k,H}$, $\lambda_k^{h_1} \Leftarrow \lambda_{k,H}$, $i \Leftarrow 2$.

Step 3. Solve a linear system on the π_{h_i}: find $\tilde{u} \in S^{h_i}$ such that

$$a(\tilde{u}, v) - \lambda_k^{h_{i-1}} b(\tilde{u}, v) = b\left(u_k^{h_{i-1}}, v\right), \quad \forall v \in S^{h_i}. \tag{3.2}$$

And set $u_k^{h_i} = \tilde{u} / (\|\tilde{u}\|_a)$.

Step 4. Compute the Rayleigh quotient

$$\lambda_k^{h_i} = \frac{a\left(u_k^{h_i}, u_k^{h_i}\right)}{b\left(u_k^{h_i}, u_k^{h_i}\right)}. \tag{3.3}$$

Step 5. If $i = l$, then output $(\lambda_k^{h_l}, u_k^{h_l})$, that is, (λ_k^h, u_k^h), stop. Else, $i \Leftarrow i + 1$, and return to Step 3.

Theorem 3.1. *Let $(\lambda_k^{h_l}, u_k^{h_l})$ be obtained by Scheme 1. Assume that $(\lambda_k^{h_{l-1}}, u_k^{h_{l-1}})$ approximate an eigenpair, $(\lambda_k, \overline{u})$, of (2.2), $\overline{u} \in \widehat{M}(\lambda_k)$, and $\lambda_k^{h_{l-1}} - \lambda_k$ is a small quantity of lower order than $\lambda_{k,h_l} - \lambda_k$. Then there exists $u_k \in M(\lambda_k)$ such that*

$$\|u_k^{h_l} - u_k\|_a \leq \frac{48}{\rho} C_4 C_5 \lambda_k \left|\lambda_k^{h_{l-1}} - \lambda_k\right| \operatorname{dist}\left(u_k^{h_{l-1}}, \widehat{M}(\lambda_k)\right) + 3C_2 q \delta_{h_l}(\lambda_k), \tag{3.4}$$

$$\left|\lambda_k^{h_l} - \lambda_k\right| \leq 2\lambda_k (1 + C_6 \lambda_k) \left\|u_k^{h_l} - u_k\right\|_a^2, \quad l \geq 2, \tag{3.5}$$

where the constants C_4, C_5, and C_6 are stated in (3.6), (3.11), and (3.29), respectively.

Proof. We use Lemma 2.4 to complete the proof. First, we will verify that the conditions of Lemma 2.4 are satisfied.

Select $\mu_0 = 1/\lambda_k^{h_{l-1}}$, and $u_0 = \lambda_k^{h_{l-1}} A_{h_l} u_k^{h_{l-1}} / \|\lambda_k^{h_{l-1}} A_{h_l} u_k^{h_{l-1}}\|_a$. From (2.16), we know that $\|A_{h_l} - A\|_a \to 0(h_l \to 0)$, then there exists a constant C_4 independent of h_l and l such that

$$\left\|A_{h_l} f\right\|_a \leq C_4 \|f\|_a, \quad \forall f \in H^1(\Omega). \tag{3.6}$$

Thus, by the assumption, we deduce that

$$
\begin{aligned}
\left\| \lambda_k^{h_{l-1}} A_{h_l} u_k^{h_{l-1}} - \overline{u} \right\|_a &\leq \left\| \lambda_k^{h_{l-1}} A_{h_l} u_k^{h_{l-1}} - \lambda_k^{h_{l-1}} A_{h_l} \overline{u} + \lambda_k^{h_{l-1}} A_{h_l} \overline{u} - \lambda_k A_{h_l} \overline{u} + \lambda_k A_{h_l} \overline{u} - \lambda_k A \overline{u} \right\|_a \\
&\leq C_4 \lambda_k^{h_{l-1}} \left\| u_k^{h_{l-1}} - \overline{u} \right\|_a + C_4 \left| \lambda_k^{h_{l-1}} - \lambda_k \right| \cdot \|\overline{u}\|_a + \lambda_k \|(A_{h_l} - A)\overline{u}\|_a.
\end{aligned}
\tag{3.7}
$$

Note that in any normed space, for any nonzero $u, v \in S^h$, there holds

$$
\left\| \frac{u}{\|u\|} - \frac{v}{\|v\|} \right\| \leq 2 \frac{\|u - v\|}{\|u\|}, \quad \left\| \frac{u}{\|u\|} - \frac{v}{\|v\|} \right\| \leq 2 \frac{\|u - v\|}{\|v\|}.
\tag{3.8}
$$

Hence, we have

$$
\begin{aligned}
\mathrm{dist}\left(u_0, \widehat{M}(\lambda_k)\right) &\leq \left\| u_0 - \frac{\overline{u}}{\|\overline{u}\|_a} \right\|_a \leq 2\|\lambda_k^{h_{l-1}} A_{h_l} u_k^{h_{l-1}} - \overline{u}\|_a \\
&\leq 2\left\{ C_4 \lambda_k^{h_{l-1}} \|u_k^{h_{l-1}} - \overline{u}\|_a + C_4 \left| \lambda_k^{h_{l-1}} - \lambda_k \right| \cdot \|\overline{u}\|_a + \lambda_k \|(A_{h_l} - A)\overline{u}\|_a \right\}.
\end{aligned}
\tag{3.9}
$$

Using the triangle inequality and (2.22), we get

$$
\mathrm{dist}(u_0, M_{h_l}(\lambda_k)) \leq \mathrm{dist}\left(u_0, \widehat{M}(\lambda_k)\right) + C_3 \delta_{h_l}(\lambda_k).
\tag{3.10}
$$

It follows from (2.20) that $\lambda_{k,h_l} \to \lambda_k (h_l \to 0)$, then by the assumption, we have

$$
\left| \mu_0 - \mu_{k,h_l} \right| = \left| \frac{\lambda_k^{h_{l-1}} - \lambda_k + \lambda_k - \lambda_{k,h_l}}{\lambda_{k,h_l} \lambda_k^{h_{l-1}}} \right| \leq C_5 \left| \lambda_k^{h_{l-1}} - \lambda_k \right|,
\tag{3.11}
$$

where C_5 is a constant independent of h_l. By (2.13), we see that Step 3 in Scheme 1 is equivalent to the following:

$$
a(\widetilde{u}, v) - \lambda_k^{h_{l-1}} a(A_{h_l} \widetilde{u}, v) = a\left(A_{h_l} u_k^{h_{l-1}}, v \right), \quad \forall v \in S^h,
\tag{3.12}
$$

$u_k^{h_l} = \widetilde{u}/\|\widetilde{u}\|_a$, that is,

$$
\left(\frac{1}{\lambda_k^{h_{l-1}}} - A_{h_l} \right) \widetilde{u} = \frac{1}{\lambda_k^{h_{l-1}}} A_{h_l} u_k^{h_{l-1}}, \quad u_k^{h_l} = \frac{\widetilde{u}}{\|\widetilde{u}\|_a}.
\tag{3.13}
$$

Notice that $(1/\lambda_k^{h_{l-1}}) A_{h_l} u_k^{h_{l-1}} = \|\left(1/\lambda_k^{h_{l-1}}\right) A_{h_l} u_k^{h_{l-1}}\|_a u_0$ differs from u_0 by only a constant, then Step 3 is equivalent to

$$
\left(\frac{1}{\lambda_k^{h_{l-1}}} - A_{h_l} \right) \widetilde{u} = u_0, \quad u_k^{h_l} = \frac{\widetilde{u}}{\|\widetilde{u}\|_a}.
\tag{3.14}
$$

In fact, it is obvious that $u_k^{h_l}$ obtained by the above two formulae are the same. When h_{l-1} is small enough, noting that $h_l \ll h_{l-1}$, from (3.10) and (3.9), we get

$$\text{dist}(u_0, M_{h_l}(\mu_k)) \le \frac{1}{2}. \tag{3.15}$$

From (2.20), having in mind that $\lambda_k = \lambda_{k+1} = \cdots = \lambda_{k+q-1}$, we know that

$$\lambda_{j,h_l} - \lambda_k \le C_1 \delta_{h_l}^2(\lambda_k) \quad (j = k, k+1, \ldots, k+q-1), \tag{3.16}$$

then

$$\left| \mu_{j,h_l} - \mu_{k,h_l} \right| = \left| \frac{\lambda_{k,h_l} - \lambda_{j,h_l}}{\lambda_{k,h_l} \lambda_{j,h_l}} \right| = \left| \frac{\lambda_{k,h_l} - \lambda_k + \lambda_j - \lambda_{j,h_l}}{\lambda_{k,h_l} \lambda_{j,h_l}} \right|. \tag{3.17}$$

Combining (3.17) with (3.11) and noting that the quantity on the right-hand side of (3.17) is an infinitesimal of higher order comparing with $\lambda_k^{h_{l-1}} - \lambda_k$, we derive

$$\max_{k \le j \le k+q-1} \left| \frac{\mu_{j,h_l} - \mu_{k,h_l}}{\mu_0 - \mu_{j,h_l}} \right| \le \frac{1}{2}. \tag{3.18}$$

Since ρ is the separation constant, h_{l-1} is small enough, and $h_l \ll h_{l-1}$, there holds

$$\left| \mu_0 - \mu_{j,h_l} \right| \ge \frac{\rho}{2}, \quad j \ne k, k+1, \ldots, k+q-1. \tag{3.19}$$

From the above arguments, we see that the conditions of Lemma 2.4 hold.

Next, we will prove that (3.4) and (3.5) are valid.

Substituting (3.10) and (3.11) into (2.25), we obtain

$$\text{dist}\left(u_k^{h_l}, \widehat{M}_{h_l}(\mu_k)\right) \le \frac{16}{\rho} C_5 \left| \lambda_k^{h_{l-1}} - \lambda_k \right| \left(\text{dist}\left(u_0, \widehat{M}(\lambda_k)\right) + C_3 \delta_{h_l}(\lambda_k) \right). \tag{3.20}$$

Let the eigenvectors $\{u_{j,h_l}\}_k^{k+q-1}$ be an orthonormal basis of $M_{h_l}(\lambda_k)$. Note that

$$\text{dist}\left(u_k^{h_l}, M_{h_l}(\lambda_k)\right) = \left\| u_k^{h_l} - \sum_{j=k}^{k+q-1} a\left(u_k^{h_l}, u_{j,h_l}\right) u_{j,h_l} \right\|_a. \tag{3.21}$$

Let

$$u^* = \sum_{j=k}^{k+q-1} a\left(u_k^{h_l}, u_{j,h_l}\right) u_{j,h_l}, \tag{3.22}$$

it follows directly from (3.20) that

$$\|u_k^{h_l} - u^*\|_a \leq \frac{16}{\rho} C_5 \left| \lambda_k^{h_{l-1}} - \lambda_k \right| \left(\text{dist}\left(u_0, \widehat{M}(\lambda_k)\right) + C_3 \delta_{h_l}(\lambda_k) \right). \tag{3.23}$$

By Lemma 2.2, there exists $\{u_j^0\}_k^{k+q-1} \subset \widehat{M}(\lambda_k)$ making $u_{j,h_l} - u_j^0$ satisfy (2.21). Let

$$u_k = \sum_{j=k}^{k+q-1} a\left(u_k^{h_l}, u_{j,h_l}\right) u_j^0, \tag{3.24}$$

then $u_k \in M(\lambda_k)$. Using (2.21), we deduce that

$$
\begin{aligned}
\|u^* - u_k\|_a &= \left\| \sum_{j=k}^{k+q-1} a\left(u_k^{h_l}, u_{j,h_l}\right) \left(u_{j,h_l} - u_j^0\right) \right\|_a \\
&\leq \left(\sum_{j=k}^{k+q-1} \left\| u_{j,h_l} - u_j^0 \right\|_a^2 \right)^{1/2} \\
&\leq C_2 \sum_{j=k}^{k+q-1} \delta_{h_l}(\lambda_j) \leq C_2 q \delta_{h_l}(\lambda_k).
\end{aligned}
\tag{3.25}
$$

Combining (3.23) with the above inequality, we have

$$
\begin{aligned}
\|u_k^{h_l} - u_k\|_a &\leq \frac{16}{\rho} C_5 \left| \lambda_k^{h_{l-1}} - \lambda_k \right| \left(\text{dist}\left(u_0, \widehat{M}(\lambda_k)\right) + C_3 \delta_{h_l}(\lambda_k) \right) + C_2 q \delta_{h_l}(\lambda_k) \\
&\leq \frac{16}{\rho} C_5 \left| \lambda_k^{h_{l-1}} - \lambda_k \right| \text{dist}\left(u_0, \widehat{M}(\lambda_k)\right) + 2 C_2 q \delta_{h_l}(\lambda_k).
\end{aligned}
\tag{3.26}
$$

It is obvious that there exists $u_k' \in \widehat{M}(\lambda_k)$ such that

$$\|u_k^{h_{l-1}} - u_k'\|_a = \text{dist}\left(u_k^{h_{l-1}}, \widehat{M}(\lambda_k)\right), \tag{3.27}$$

$\lambda_k A u_k' = u_k' \in \widehat{M}(\lambda_k)$, and $\|(A_{h_l} - A)u_k'\|_a \leq 1/\lambda_k \delta_{h_l}(\lambda_k)$.

Since $u_0 = \lambda_k^{h_{l-1}} A_{h_l} u_k^{h_{l-1}} / \|\lambda_k^{h_{l-1}} A_{h_l} u_k^{h_{l-1}}\|_a$, from (2.23) and (3.8), we derive

$$
\begin{aligned}
\operatorname{dist}\left(u_0, \widehat{M}(\lambda_k)\right) &\le 2\operatorname{dist}\left(\lambda_k^{h_{l-1}} A_{h_l} u_k^{h_{l-1}}, \widehat{M}(\lambda_k)\right) \\
&\le 2\|\lambda_k^{h_{l-1}} A_{h_l} u_k^{h_{l-1}} - \lambda_k A u_k'\|_a \\
&\le 2\|\left(\lambda_k^{h_{l-1}} - \lambda_k\right) A_{h_l} u_k^{h_{l-1}}\|_a + 2\|\lambda_k A_{h_l}\left(u_k^{h_{l-1}} - u_k'\right)\|_a \\
&\quad + 2\|\lambda_k (A_{h_l} - A) u_k'\|_a \\
&\le 2C_4\left|\lambda_k^{h_{l-1}} - \lambda_k\right| + 2C_4 \lambda_k \operatorname{dist}\left(u_k^{h_{l-1}}, \widehat{M}(\lambda_k)\right) + 2\delta_{h_l}(\lambda_k) \\
&\le 3C_4 \lambda_k \operatorname{dist}\left(u_k^{h_{l-1}}, \widehat{M}(\lambda_k)\right) + 2\delta_{h_l}(\lambda_k).
\end{aligned}
\tag{3.28}
$$

Substituting (3.28) into (3.26), we obtain (3.4). Equation (3.4) indicates that $u_k^{h_l}$ converges to u_k in the sense of norm $\|\cdot\|_a$, then from the trace theorem, we know that $u_k^{h_l}$ converges to u_k in the sense of norm $\|\cdot\|_b$; thus, $1/\|u_k^{h_l}\|_b^2 \to 1/\|u_k\|_b^2 = \lambda_k$. Therefore, when h_l is small enough, we have $1/\|u_k^{h_l}\|_b^2 \le 2\lambda_k$. Picking $w = u_k^{h_l}$ in (2.23), we get

$$
\begin{aligned}
\left|\lambda_k^{h_l} - \lambda_k\right| &\le \frac{\|u_k^{h_l} - u_k\|_a^2}{\|u_k^{h_l}\|_b^2} + \lambda_k \frac{\|u_k^{h_l} - u_k\|_b^2}{\|u_k^{h_l}\|_b^2} \\
&\le 2\lambda_k \|u_k^{h_l} - u_k\|_a^2 + 2\lambda_k^2 \|u_k^{h_l} - u_k\|_b^2 \\
&\le 2\lambda_k (1 + C_6 \lambda_k) \|u_k^{h_l} - u_k\|_a^2,
\end{aligned}
\tag{3.29}
$$

where the last inequality in the above holds due to the trace theorem, and the constant C_6 is independent of h_l. $\qquad\square$

Set $l = 2$ and denote $H = h_1, h = h_2$, then we immediately get the following two-grid discretization scheme based on the shifted-inverse power method.

Scheme 2 (two-grid discretization scheme).

Step 1. Solve (2.2) on a coarse grid π_H: find $\lambda_{k,H} \in R, u_{k,H} \in S^H$, such that $\|u_{k,H}\|_a = 1$ and

$$
a(u_{k,H}, v) = \lambda_{k,H} b(u_{k,H}, v), \quad \forall v \in S^H.
\tag{3.30}
$$

Step 2. Solve a linear system on a fine grid π_h: find $\tilde{u} \in S^h$, such that

$$
a(\tilde{u}, v) - \lambda_{k,H} b(\tilde{u}, v) = b(u_{k,H}, v), \quad \forall v \in S^h.
\tag{3.31}
$$

Set $u_k^h = \tilde{u}/\|\tilde{u}\|_a$.

Step 3. Compute the Rayleigh quotient

$$\lambda_k^h = \frac{a\left(u_k^h, u_k^h\right)}{b\left(u_k^h, u_k^h\right)}.$$
(3.32)

Then from Theorem 3.1, we have the following error estimates for Scheme 2.

Theorem 3.2. *Let (λ_k^h, u_k^h) be obtained by Scheme 2. Then when H is properly small and $h \ll H$, there exists $u_k \in M(\lambda_k)$ such that*

$$\|u_k^h - u_k\|_a \le \frac{48}{\rho} C_4 C_5 \lambda_k (\lambda_{k,H} - \lambda_k) \operatorname{dist}\left(u_{k,H}, \widehat{M}(\lambda_k)\right) + 3C_2 q \delta_h(\lambda_k),$$
(3.33)

$$\left|\lambda_k^h - \lambda_k\right| \le 2\lambda_k(1 + C_6\lambda_k) \left\|u_k^h - u_k\right\|_a^2.$$
(3.34)

Proof. Since $(\lambda_{k,H}, u_{k,H})$ is obtained by Step 1 in Scheme 2, that is, $(\lambda_{k,H}, u_{k,H})$ is a solution of (2.5) on the coarse grid π_H, it certainly approximates an eigenpair, $(\lambda_k, \overline{u})$, of (2.2), $\overline{u} \in \widehat{M}(\lambda_k)$. Note that $\lambda_{k,h}$ is an eigenvalue of (2.5) on the fine grid π_h, so when H is properly small and $h \ll H$, it is valid that $\lambda_{k,H} - \lambda_k$ is a small quantity of lower order than $\lambda_{k,h} - \lambda_k$. The above arguments imply that the conditions of Theorem 3.1 hold for $l = 2$. Therefore, the desired results follow from Theorem 3.1. □

Remark 3.3. It follows directly from (2.20) that

$$\lambda_{k,H} - \lambda_k \le C_1 \delta_H^2(\lambda_k).$$
(3.35)

And from (2.21), we know that $\operatorname{dist}(u_{k,H}, \widehat{M}(\lambda_k)) \le C_2 \delta_H(\lambda_k)$, then from Theorem 3.2, we get

$$\|u_k^h - u_k\|_a \le \frac{48}{\rho} C_1 C_2 C_4 C_5 \lambda_k \delta_H^3(\lambda_k) + 3C_2 q \delta_h(\lambda_k).$$
(3.36)

Substituting (2.19) into (3.36), we obtain

$$\|u_k^h - u_k\|_a \le \frac{48}{\rho} C_2 C_4 C_5 \lambda_k C_1 C_I^3 H^{3r} \left(\sup_{w \in \widehat{M}(\lambda_k)} \|w\|_{1+r}\right)^3$$
$$+ 3C_2 q C_I h^r \sup_{w \in \widehat{M}(\lambda_k)} \|w\|_{1+r}.$$
(3.37)

From (3.37) and (3.34), we can see that when we take $H = \mathcal{O}(\sqrt[3]{h})$, the resulted solution, (λ_k^h, u_k^h), can maintain an asymptotically optimal accuracy.

Corollary 3.4. *Suppose that $h_i = h_{i-1}^{t_i}, t_i \in (1, 3), i = 2, 3, \ldots, \sup_i t_i < 3$. By using the triangle linear conforming element with regular meshes, when h_1, that is, H is properly small, there exists $u_k \in M(\lambda_k)$ such that the following error estimates for Scheme 1 hold:*

$$\|u_k^{h_l} - u_k\|_a \leq 6qC_2C_I h_l^r \sup_{w \in \widehat{M}(\lambda_k)} \|w\|_{1+r}, \tag{3.38}$$

$$\left|\lambda_k^{h_l} - \lambda_k\right| \leq 2\lambda_k(1 + C_6\lambda_k)\|u_k^{h_l} - u_k\|_a^2, \quad l \geq 2. \tag{3.39}$$

Proof. We prove the results by using induction.

For $l = 2$, we estimate $\lambda_{k,H} - \lambda_k$ and $\text{dist}(u_{k,H}, \widehat{M}(\lambda_k))$ in (3.33), respectively.

Combining (3.35) and (2.19) yields

$$\lambda_{k,H} - \lambda_k \leq C_1 C_I^2 H^{2r} \left(\sup_{w \in \widehat{M}(\lambda_k)} \|w\|_{1+r} \right)^2. \tag{3.40}$$

From Lemma 2.2, and (3.8) we have

$$\text{dist}\left(u_{k,H}, \widehat{M}(\lambda_k)\right) \leq C_2 \delta_H(\lambda_k) \leq C_2 C_I H^r \sup_{w \in \widehat{M}(\lambda_k)} \|w\|_{1+r}. \tag{3.41}$$

Substituting the above estimates into (3.33), we get

$$\begin{aligned}
\|u_k^h - u_k\|_a &\leq \frac{48}{\rho} C_1 C_2 C_4 C_5 \lambda_k C_I^3 H^{3r} \left(\sup_{w \in \widehat{M}(\lambda_k)} \|w\|_{1+r} \right)^3 \\
&\quad + 3qC_2C_I h^r \sup_{w \in \widehat{M}(\lambda_k)} \|w\|_{1+r}.
\end{aligned} \tag{3.42}$$

By the assumption, $h = H^{t_2}, t_2 \in (1, 3)$, we can see that the first term on the right-hand side of (3.42) is a small quantity of higher order than the second term, hence,

$$\|u_k^h - u_k\|_a \leq 6qC_2C_I h^r \sup_{w \in \widehat{M}(\lambda_k)} \|w\|_{1+r}. \tag{3.43}$$

Suppose that (3.38), (3.39) hold for $l - 1$, that is, there holds

$$\|u_k^{h_{l-1}} - u_k\|_a \leq 6qC_2C_I h_{l-1}^r \sup_{w \in \widehat{M}(\lambda_k)} \|w\|_{1+r}, \tag{3.44}$$

$$\left|\lambda_k^{h_{l-1}} - \lambda_k\right| \leq 2\lambda_k(1 + C_6\lambda_k)\|u_k^{h_{l-1}} - u_k\|_a^2, \quad l \geq 2. \tag{3.45}$$

By (3.8), we get

$$\text{dist}\left(u_k^{h_{l-1}}, \widehat{M}(\lambda_k)\right) \leq 2\|u_k^{h_{l-1}} - u_k\|_a. \tag{3.46}$$

Substituting (3.45) and (3.46) into (3.4), and noting (3.44), we deduce,

$$
\begin{aligned}
\left\| u_k^{h_l} - u_k \right\|_a &\leq \frac{192}{\rho} C_4 C_5 \lambda_k^2 (1 + C_6 \lambda_k) \left(6q C_2 C_I \sup_{w \in \widehat{M}(\lambda_k)} \|w\|_{1+r} \right)^3 h_{l-1}^{3r} + 3 C_2 q \delta_{h_l}(\lambda_k) \\
&\leq \frac{192}{\rho} C_4 C_5 \lambda_k^2 (1 + C_6 \lambda_k) (6q C_2 C_I \sup_{w \in \widehat{M}(\lambda_k)} \|w\|_{1+r})^3 h_{l-1}^{3r} + 3q C_2 C_I h_l^r \sup_{w \in \widehat{M}(\lambda_k)} \|w\|_{1+r} \\
&\leq 6q C_2 C_I h_l^r \sup_{w \in \widehat{M}(\lambda_k)} \|w\|_{1+r},
\end{aligned}
$$

(3.47)

that is, (3.38) holds. The inequality (3.39) follows directly from (3.29). □

From Theorem 3.1 and Corollary 3.4, we can see that the error is independent of the mesh diameters and the iteration time, and Corollary 3.4 also shows that with Scheme 1, by using the linear element, the approximate eigenvalue sequence $\{\lambda_k^{h_l}\}$ converges to $\{\lambda_k\}$ as $l \to \infty$, and the convergence is quick. Hence, we have the reason to believe that $|\lambda_k^{h_l} - \lambda_k^{h_{l+1}}| < |\lambda_k^{h_l} - \lambda_k^{h_{l-1}}|$. A simple calculation shows that

$$
\left| \lambda_k^{h_l} - \lambda_k \right| \leq \left| \lambda_k^{h_l} - \lambda_k^{h_{l+1}} \right| + \left| \lambda_k^{h_{l+1}} - \lambda_k \right| \leq \left| \lambda_k^{h_l} - \lambda_k^{h_{l-1}} \right| + \left| \lambda_k^{h_{l+1}} - \lambda_k \right|.
$$

(3.48)

Note that in (3.48) $|\lambda_k^{h_{l+1}} - \lambda_k|$ is a small quantity of higher order. So we can use $|\lambda_k^{h_l} - \lambda_k^{h_{l-1}}| < \varepsilon$ (ε is a given tolerate error) as a criteria to stop the iteration. Scheme 1 is actually an iterative process. It is natural to establish an adaptive algorithm based on Scheme 1.

Scheme 3 (adaptive algorithm). Give an error tolerance ε, an initial coarse grid π_H, and a grid π_{h_2} derived from π_H.

Step 1. Solve (2.2) on the π_H: find $\lambda_{k,H} \in R, u_{k,H} \in S^H$, such that $\|u_{k,H}\|_a = 1$ and

$$
a(u_{k,H}, v) = \lambda_{k,H} b(u_{k,H}, v), \quad \forall v \in S^H.
$$

(3.49)

Step 2. Execute the assignments: $u_k^{h_1} \Leftarrow u_{k,H}, \lambda_k^{h_1} \Leftarrow \lambda_{k,H}, i \Leftarrow 2$.

Step 3. Solve a linear system on the π_{h_i}: find $\widetilde{u} \in S^{h_i}$, such that

$$
a(\widetilde{u}, v) - \lambda_k^{h_{i-1}} b(\widetilde{u}, v) = b\left(u_k^{h_{i-1}}, v \right), \quad \forall v \in S^{h_i}.
$$

(3.50)

And set $u_k^{h_i} = \widetilde{u} / \|\widetilde{u}\|_a$.

Table 1: Numerical eigenvalues on the square domain $[0,1] \times [0,1]$ by Scheme 2: set $h = \sqrt{2}/512, h = \mathcal{O}(H^t), t \in (1,3]$.

H	h	λ_1^h	λ_2^h	λ_3^h	λ_4^h
$\sqrt{2}/8$	$\sqrt{2}/512$	0.2400791223	1.4923055029	1.4923059934	2.0826626453
$\sqrt{2}/16$	$\sqrt{2}/512$	0.2400791223	1.4923054995	1.4923059872	2.0826625905
$\sqrt{2}/32$	$\sqrt{2}/512$	0.2400791223	1.4923054994	1.4923059871	2.0826625901

Step 4. Compute the Rayleigh quotient

$$\lambda_k^{h_i} = \frac{a\left(u_k^{h_i}, u_k^{h_i}\right)}{b\left(u_k^{h_i}, u_k^{h_i}\right)}. \tag{3.51}$$

Step 5. If $|\lambda_k^{h_i} - \lambda_k^{h_{i-1}}| < \varepsilon$, then output $(\lambda_k^{h_i}, u_k^{h_i})$, stop. Else, return to Step 6.

Step 6. Determine t_{i+1} and $h_{i+1} = h_i^{t_{i+1}}$, then refine π_{h_i} to get $\pi_{h_{i+1}}$. Set $i \Leftarrow i + 1$, and return to Step 3.

4. Numerical Experiments

Example 4.1. We compute the first four approximate eigenvalues of (2.1) with the triangle linear finite element by using Scheme 2 on $\overline{\Omega} = [0,1] \times [0,1]$ and $[0,1] \times [0,1/2] \cup [0,1/2] \times [1/2,1]$, respectively, [9] and [28] have proved that the non-conforming EQ_1^{rot} element can provide the lower bound for the exact eigenvalues, and the minimum-maximum principle ensures that conforming finite elements can give the upper bound for the exact eigenvalues. So we compute the following ranges for the first four exact eigenvalues of (2.1) by the linear triangle element and the EQ_1^{rot} element.

When $\overline{\Omega} = [0,1] \times [0,1]$ the first four exact eigenvalues

$$\lambda_1 \approx 0.24007909, \quad \lambda_2 \approx 1.492303, \quad \lambda_3 \approx 1.492303, \quad \lambda_4 \approx 2.08265; \tag{4.1}$$

when $\overline{\Omega} = [0,1] \times [0,1/2] \cup [0,1/2] \times [1/2,1]$ the first four exact eigenvalues

$$\lambda_1 \approx 0.18296424, \quad \lambda_2 \approx 0.8936, \quad \lambda_3 \approx 1.68860, \quad \lambda_4 \approx 3.2179. \tag{4.2}$$

When compute with Scheme 2, we adopt a uniform isosceles right triangulation along three directions to obtain the coarse grid π_H, and refine the coarse grid in a uniform way (each triangle is divided into four congruent subtriangles) repeatedly to obtain the fine grid π_h. Then we compute the approximate eigenvalues with MATLAB7.1. The numerical results are shown in Tables 1 and 2.

Remark 4.2. In Example 4.1, we set the diameter of fine grid $h = \sqrt{2}/512$ and $H = \mathcal{O}(\sqrt[t]{h}), t \in (1,3]$. From Table 1 we can see that the approximate eigenvalues obtain an asymptotically

Table 2: Numerical eigenvalues on the L-shaped domain $[0,1] \times [0,1/2] \cup [0,1/2] \times [1/2,1]$ by Scheme 2: set $h = \sqrt{2}/512, h = \mathcal{O}(H^t), t \in (1,3]$.

H	h	λ_1^h	λ_2^h	λ_3^h	λ_4^h
$\sqrt{2}/8$	$\sqrt{2}/512$	0.1829642799	0.8937364938	1.6886068115	3.2179250611
$\sqrt{2}/16$	$\sqrt{2}/512$	0.1829642799	0.8937364004	1.6886067424	3.2179003466
$\sqrt{2}/32$	$\sqrt{2}/512$	0.1829642799	0.8937363984	1.6886067421	3.2179002027

Table 3: Numerical eigenvalues on the square domain $\overline{\Omega} = [0,1] \times [0,1]$ by Scheme 3: set $h_1 = H = 1/8, h_2 = h_1^2, h_i = h_{i-1}/2, i = 3,4,\ldots$.

	$\varepsilon = 5 \times 10^{-7}$	$\varepsilon = 5 \times 10^{-5}$	$\varepsilon = 5 \times 10^{-5}$	$\varepsilon = 5 \times 10^{-5}$
l	$\lambda_1^{h_l}$	$\lambda_2^{h_l}$	$\lambda_3^{h_l}$	$\lambda_4^{h_l}$
1	0.2402262809	1.5014059516	1.5032096484	2.1452661239
2	0.2400814379	1.4924542699	1.4924853962	2.0836406391
3	0.2400796738	1.4923409581	1.4923487554	2.0828955752
4	0.2400792326	1.4923125932	1.4923145438	2.0827091950
5	—	—	—	2.0826625901
6	—	—	—	—

optimal accuracy. While with the two-grid discretization scheme in [20], to achieve the same accuracy the relationship between the diameter of coarse grid and that of fine grid should satisfy $H = \mathcal{O}(\sqrt[2]{h})$. For example, we carry out the computation on the same grids with $h = \sqrt{2}/512$ and $H = \sqrt{2}/8$, by our scheme the resulted solution $\lambda_1^h = 0.2400791223$ which has 7 significant digits, and by the scheme in [20] the resulted solution $\widetilde{\lambda}_1^h = 0.2400791543$ with 6 significant digits. Note that the computational complexities of these two methods are almost the same which indicates that our method is also efficient.

Remark 4.3. We program by using the finite element package of Chen [29] to solve Example 4.1. It costs 19.745s by using Scheme 2, with $H = \sqrt{2}/8$ and $h = \sqrt{2}/512$, to get the first approximate eigenvalue $\lambda_1^h = 0.2400791223$ of (2.1) on the square $[0,1] \times [0,1]$. To illustrate the efficiency of our scheme we compute the first approximate eigenvalue on the fine grid with mesh diameter $h = \sqrt{2}/512$ directly, and it costs 75.267s to get the same approximation. We also compare the computation times of our approach and direct calculation on the L-shaped domain $[0,1] \times [0,1/2] \cup [0,1/2] \times [1/2,1]$. By using Scheme 2 with $H = \sqrt{2}/8$ and $h = \sqrt{2}/512$, we spend 10.779s to obtain $\lambda_1^h = 0.1829642799$; while computing on the fine grid with mesh size $h = \sqrt{2}/512$ directly it costs 31.986s. From these comparisons we can see that our scheme is very efficient.

Example 4.4. We compute the first four approximate eigenvalues of (2.1) with the triangle linear finite element by using Scheme 3 on $[0,1] \times [0,1]$ and $[0,1] \times [0,1/2] \cup [0,1/2] \times [1/2,1]$, respectively. The results are listed in Tables 3 and 4, respectively. And in Tables 3 and 4, the symbol "-" means that we have obtained the approximate eigenvalues which meet the accuracy requirements and the iteration stops here.

Remark 4.5. In the research of numerical methods for Steklov eigenvalue problems, the boundary element method and the finite element method have been studied. However, we have not seen any literatures on the finite difference method, a major numerical method, for

Table 4: Numerical eigenvalues on the L-shaped domain $\overline{\Omega} = [0,1] \times [0,1/2] \cup [0,1/2] \times [1/2,1]$ by Scheme 3: set $h_1 = H = 1/8, h_2 = h_1^2, h_i = h_{i-1}/2, i = 3,4,\ldots$.

l	$\varepsilon = 5 \times 10^{-7}$ $\lambda_1^{h_l}$	$\varepsilon = 5 \times 10^{-4}$ $\lambda_2^{h_l}$	$\varepsilon = 5 \times 10^{-5}$ $\lambda_3^{h_l}$	$\varepsilon = 5 \times 10^{-4}$ $\lambda_4^{h_l}$
1	0.1831328879	0.9211580591	1.7114443522	3.3645126465
2	0.1829669801	0.8951626591	1.6889966003	3.2204599675
3	0.1829649244	0.8942351070	1.6887001313	3.2185057817
4	0.1829644089	0.8938763111	1.6886254813	3.2180214091
5	0.1829642799	—	1.6886067421	—
6	—	—	—	—

Steklov eigenvalue problems. To apply the finite difference method to Steklov eigenvalue problems would be a new and interesting issue which is our next work.

5. Concluding Remarks

This paper discusses the Steklov eigenvalue problem, and establishes a new multi-scale discretization scheme and an adaptive algorithm. We prove that our approach is efficient. With the adaptive scheme, first we solve an eigenvalue problem on a coarse grid, in each step after that we only need to solve a linear algebraic system on a fine grid. Compared with the existing adaptive method which computes an eigenvalue problem in each step, our approach reduces the computational complexity. However, the a posteriori error indicator we used is a global estimate, thus, to establish a local a posteriori error estimate and improve our adaptive algorithm is our next goal.

Acknowledgments

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