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Research Article

Multitarget Linear-Quadratic Control Problem: Semi-Infinite Interval

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We consider multitarget linear-quadratic control problem on semi-infinite interval. We show that the problem can be reduced to a simple convex optimization problem on the simplex.

1. Introduction

Let $(H, \langle , \rangle)$ be a Hilbert space, Z be its closed vector subspace, $h_1, \ldots, h_m$, and c be vectors in H. Consider the following optimization problem:

$$\max_{1 \le i \le m} \|h - h_i\| \longrightarrow \min, \quad h \in c + Z. \tag{1.1}$$

Here $\| \cdot \|$ is the norm in H induced by the scalar product $\langle , \rangle$. In [1], we analyzed (1.1) using duality theory for infinite-dimensional second-order cone programming. We obtained a reduction of this problem to a finite-dimensional second-order cone programming and applied this result to a multitarget linear-quadratic control problem on a finite time interval. In this paper, we consider a reduction (1.1) to even simpler optimization problem of minimization of convex quadratic function on the $(m-1)$ dimensional simplex. We then apply this result to the analysis of a multitarget linear-quadratic control problem on semi-infinite time interval. We show that the coefficients of the quadratic function admit a simple expressions in term of the original data.

2. Reduction to a Simple Quadratic Programming Problem

Let $f_i(h) = \|h - h_i\|^2$, $i = 1,2,\ldots,m$. It is obvious that (1.1) is equivalent to the following optimization problem:

$$\begin{gathered} z \longrightarrow \min, \\ f_i(h) \le z, \quad i = 1,2,\ldots,m, \quad h \in c + Z. \end{gathered} \tag{2.1}$$

Consider the Lagrange function

$$\begin{aligned} \mathcal{L}(\lambda_1,\ldots,\lambda_m,h,z) &= z + \sum_{i=1}^{m}\lambda_i\big(f_i(h) - z\big) \\ &= z\left(1 - \sum_{i=1}^{m}\lambda_i\right) + \sum_{i=1}^{m}\lambda_i f_i(h). \end{aligned} \tag{2.2}$$

Notice that despite the fact that our original problem is infinite dimensional, the usual KKT theorem holds true (see e.g., [2], page 72). It is also clear that Slater conditions are satisfied. Hence, optimality condition for (2.1) takes the form

$$\begin{gathered} \lambda_i \ge 0, \quad \lambda_i\big(f_i(h) - z\big) = 0, \quad i = 0,1,2,\ldots,m, \\ \frac{\partial \mathcal{L}}{\partial z} = 0, \quad \sum_{i=1}^{m}\lambda_i \nabla f_i(h) \in Z^{\perp}, \end{gathered} \tag{2.3}$$

where $\nabla f_i(h) = 2(h - h_i)$, $i = 1,2,\ldots,m$, $Z^{\perp}$ is the orthogonal complement of Z in H. Conditions (2.3) lead to

$$\begin{gathered} \sum_{i=0}^{m}\lambda_i = 1, \quad \lambda_i \ge 0, \quad i = 1,2,\ldots,m, \\ \pi_Z(h) = \sum_{i=1}^{m}\lambda_i(\pi_Z h_i). \end{gathered} \tag{2.4}$$

Here $\pi_Z : H \to Z$ is the orthogonal projection. Let us form the Lagrange dual of (2.1). Consider

$$\varphi(\lambda_1,\lambda_2,\ldots,\lambda_m) = \min\{\mathcal{L}(\lambda_1,\ldots,\lambda_m,h,z) : h \in c + Z,\ z \in Z\}. \tag{2.5}$$

Using (2.4), we obtain that

$$\varphi(\lambda_1,\lambda_2,\ldots,\lambda_m) = \sum_{i=1}^{m}\lambda_i f_i(h(\lambda_1,\ldots,\lambda_m)), \tag{2.6}$$

where

$$h(\lambda_1,\dots,\lambda_m) = \pi_{Z^\perp}(c) + \sum_{i=1}^{m}\lambda_i \pi_Z(h_i). \tag{2.7}$$

Notice that for any $h \in c + Z$, $\pi_{Z^\perp}(h) = \pi_{Z^\perp}(c)$. Here $\pi_{Z^\perp} : H \to Z^\perp$ is the orthogonal projection of H onto orthogonal complement $Z^\perp$ of Z. To further simplify (2.6), introduce the notation

$$h(\lambda) = \sum_{i=1}^{m}\lambda_i h_i. \tag{2.8}$$

Then

$$\begin{aligned} f_j(h(\lambda_1,\dots,\lambda_m)) &= \left\|\pi_Z\left(h(\lambda) - h_j\right) + \pi_{Z^\perp}\left(c - h_j\right)\right\|^2 \\ &= \left\|\pi_Z\left(h(\lambda) - \pi_Z\left(h_j\right)\right)\right\|^2 + \left\|\pi_{Z^\perp}\left(c - h_j\right)\right\|^2 \\ &= \|\pi_Z(h(\lambda))\|^2 + \left\|\pi_Z\left(h_j\right)\right\|^2 - 2\left\langle \pi_Z(h(\lambda)),\ \ \pi_Z\left(h_j\right)\right\rangle \\ &\quad + \left\|\pi_{Z^\perp}\left(c - h_j\right)\right\|^2. \end{aligned} \tag{2.9}$$

Hence, according to (2.6), we have the following:

$$\begin{aligned} \varphi(\lambda_1,\dots,\lambda_m) = \|\pi_Z(h(\lambda))\|^2 + \sum_{j=1}^{m}\lambda_j \left\|\pi_Z\left(h_j\right)\right\|^2 \\ - 2\langle \pi_Z(h(\lambda)),\ \ \pi_Z(h(\lambda))\rangle + \sum_{j=1}^{m}\lambda_j \left\|\pi_{Z^\perp}\left(c - h_j\right)\right\|^2. \end{aligned} \tag{2.10}$$

We, hence, arrive at the following expression of φ:

$$\varphi(\lambda_1,\dots,\lambda_m) = -\left\|\pi_Z\left(\sum_{i=1}^{m}\lambda_i h_i\right)\right\|^2 + \sum_{j=1}^{m}\lambda_j\left(\left\|\pi_Z\left(h_j\right)\right\|^2 + \pi_{Z^\perp}\left(c - h_j\right)\right\|^2\right). \tag{2.11}$$

We can simplify (2.11) somewhat. Notice that

$$\left\|\pi_{Z^\perp}\left(c - h_j\right)\right\|^2 = \|\pi_{Z^\perp}(c)\|^2 + \left\|\pi_{Z^\perp}\left(h_j\right)\right\|^2 - 2\left\langle \pi_{Z^\perp}(c),\ \ \pi_{Z^\perp}\left(h_j\right)\right\rangle. \tag{2.12}$$

Consequently,

$$\begin{aligned}\varphi(\lambda_1,\ldots,\lambda_m) &= -\|\pi_Z(h(\lambda))\|^2 + \sum_{j=1}^{m}\lambda_j\|h_j\|^2 \\ &\quad - 2\langle \pi_{Z^\perp}(c),\ \ \pi_{Z^\perp}(h(\lambda))\rangle + \|\pi_{Z^\perp}(c)\|^2 \\ &= -\|h(\lambda)\|^2 + \|\pi_{Z^\perp}(h(\lambda)-c)\|^2 + \sum_{j=1}^{m}\lambda_j\|h_j\|^2.\end{aligned} \tag{2.13}$$

Here,

$$h(\lambda) = \sum_{i=1}^{m}\lambda_i h_i. \tag{2.14}$$

Hence, the Lagrange dual to (2.1) takes the following form:

$$\begin{gathered}\varphi(\lambda_1,\ldots,\lambda_m) \longrightarrow \max, \\ \sum_{i=1}^{m}\lambda_i = 1, \qquad \lambda_i \geq 0, \quad i = 1,2,\ldots,m.\end{gathered} \tag{2.15}$$

If $(\lambda_1^*,\ldots,\lambda_m^*)$ is an optimal solution to (2.15), we can recover the optimal solution of the original problem using the relation (2.7), and $\varphi(\lambda_1^*,\ldots,\lambda_m^*)$ gives the optimal value for the original problem (1.1).

3. Linear-Quadratic Case

Denoted by $L_2^n[0,\infty)$, the vector space of square integrable functions $f : [0,\infty) \to R^n$. Let $H = L_2^n[0,\infty) \times L_2^m[0,\infty)$, and

$$Z = \{(\alpha,\beta) \in H : \alpha \text{ is absolutely continuous on } [0,\infty),\ \dot{\alpha} = A\alpha + B\beta,\ \ \alpha(0) = 0\}. \tag{3.1}$$

Here A (respectively B) is an n by n (respectively n by m) matrix. Observe that

$$\begin{gathered}\langle(\alpha_1,\beta_1),\ (\alpha_2,\beta_2)\rangle = \int_0^{\infty}\left[\alpha_1(t)^T\alpha_2(t) + \beta_1(t)^T\beta_2(t)\right]dt, \\ (\alpha_i,\beta_i) \in H, \qquad i = 1,2.\end{gathered} \tag{3.2}$$

In this setting, the problem (1.1) admits a natural interpretation as a linear-quadratic multitarget control problem. An interesting solution for this problem for $m = 2$ is described in [3]. In our approach, we need an explicit computation of the coefficients of the objective function (2.13) which in turn requires an explicit description of orthogonal projection π_Z. Such a description has been found in [4]. We briefly describe it here.

Theorem 3.1. *Let C be an antistable n by n matrix (i.e., real parts of all eigenvalues of C are positive). Consider the following system of linear differential equations:*

$$\dot{x} = Cx + f, \tag{3.3}$$

where $f \in L_2^n[0,\infty)$. Then there exists a unique solution $L(f)$ of (3.3) belonging to $L_2^n[0,\infty)$. Moreover, the map $L : L_2^n[0,\infty) \to L_2^n[0,\infty)$ is linear and bounded. Explicitly,

$$L(f)(t) = -\int_0^{\infty} e^{-C\tau} f(t+\tau)d\tau. \tag{3.4}$$

For the proof, see [4].

Consider the algebraic Riccati equation

$$KBB^T K + A^T K + KA - I = 0. \tag{3.5}$$

We assume that (3.5) has a real symmetric solution K_{st} such that the matrix

$$F = A + BB^T K_{\text{st}} \tag{3.6}$$

is stable (i.e., real parts of all eigenvalues of F are negative). Notice that such a solution exists if and only if the pair (A, B) is stabilizable. See, for example, [5].

Theorem 3.2. *We have the following:*

$$Z^{\perp} = \left\{ \left(\dot{p} + A^T p, B^T p\right);\ p \in L_2^n[0,\infty),\ \ p \text{ is absolutely continuous},\ \ \dot{p} \in L_2^n[0,\infty) \right\}. \tag{3.7}$$

Given that $(\psi, \varphi) \in H$, we have

$$\psi = x - \left(\dot{p} + A^T p\right), \tag{3.8}$$

$$\varphi = u - B^T p, \tag{3.9}$$

where x is the solution of the differential equation

$$\dot{x} = \left(A + BB^T K_{\text{st}}\right)x + BB^T \rho + B\varphi, \quad x(0) = 0, \tag{3.10}$$

$$u = B^T K_{\text{st}} x + B^T \rho + \varphi, \tag{3.11}$$

$$p = K_{\text{st}} x + \rho, \tag{3.12}$$

and ρ is a unique solution to the differential equation

$$\dot{\rho} = -\left(A + BB^T K_{\text{st}}\right)^T \rho - K_{\text{st}} B\varphi - \psi \tag{3.13}$$

belonging to $L_2^n[0,\infty)$.

In particular, $(x,u) \in Z$, $-(\dot{p} + A^T p, B^T p) \in Z^{\perp}$, *and consequently* Z *is a closed subspace in* H *with*

$$\pi_Z(\psi,\varphi) = (x,u), \qquad \pi_{Z^{\perp}}(\psi,\varphi) = -\left(\dot{p} + A^T p, B^T p\right). \tag{3.14}$$

Remark 3.3. The required solution ρ exists and unique by Theorem 3.1, since the matrix $-(A + BB^T K_{\text{st}})$ is antistable.

Sketch of the Proof

Let $p \in L_2^n[0,\infty)$ be absolutely continuous and such that $\dot{p} \in L_2^n[0,\infty)$. Suppose that $(x,u) \in Z$. Then

$$\begin{aligned}
\left\langle (x,u), \left(\dot{p} + A^T p, B^T p\right) \right\rangle &= \int_0^{\infty} \left(x^T \dot{p} + x^T A^T p + u B^T p\right) dt \\
&= \int_0^{\infty} \left[x^T \dot{p} + (Ax + Bu)^T p\right] dt \\
&= \int_0^{\infty} \left(x^T \dot{p} + \dot{x}^T p\right) dt \\
&= \int_0^{\infty} \frac{d}{dt}\left(x^T p\right) dt \\
&= \lim_{\tau \to \infty} x^T(\tau) p(\tau) - x(0)^T p(0).
\end{aligned} \tag{3.15}$$

But $x(\tau)$, $p(\tau) \to 0$, as $\tau \to \infty$ (see e.g., [4] for details) and $x(0) = 0$. Hence,

$$\left\langle (x,u), \left(\dot{p} + A^T p, B^T p\right) \right\rangle = 0. \tag{3.16}$$

Let us now show that the decomposition (3.5) and (3.9) takes place for an arbitrary $(\psi,\varphi) \in H$. Indeed, using (3.12),

$$\dot{p} = K_{\text{st}} \dot{x} + \dot{\rho}. \tag{3.17}$$

Hence by (3.10) and (3.13),

$$\dot{p} = K_{\text{st}}\left(A + BB^T K_{\text{st}}\right) x + K_{\text{st}} BB^T \rho + K_{\text{st}} B\varphi - \left(A + BB^T K_{\text{st}}\right)^T \rho - K_{\text{st}} B\varphi - \psi. \tag{3.18}$$

Combining all terms with x and all terms with ρ in two separate groups, we obtain that

$$\begin{aligned}\dot{p} + A^T p &= \dot{p} + A^T K_{\mathrm{st}} x + A^T \rho \\ &= \left(K_{\mathrm{st}} A + K_{\mathrm{st}} B B^T K_{\mathrm{st}} + A^T K_{\mathrm{st}}\right) x \\ &\quad + \left(K_{\mathrm{st}} B B^T - A^T - K_{\mathrm{st}} B B^T + A^T\right) \rho - \psi.\end{aligned} \tag{3.19}$$

Using now the fact that K_{st} satisfies (3.5), we obtain that

$$\dot{p} + A^T p = x - \psi \tag{3.20}$$

which is (3.8). Using (3.11) and (3.12), we obtain that

$$\begin{aligned}u - B^T p &= B^T K_{\mathrm{st}} x + B^T \rho + \varphi - B^T K_{\mathrm{st}} x - B^T \rho \\ &= \varphi,\end{aligned} \tag{3.21}$$

which is (3.9). Finally, it is clear that for x and u defined by (3.11) and (3.12), we have

$$\dot{x} = Ax + Bu \tag{3.22}$$

and consequently $(x, u) \in Z$. This completes the proof of Theorem 3.2.

Looking at (2.13), we see that the evaluation of coefficients of the quadratic function requires the knowledge of expressions of the type $\|\pi_{Z^\perp}(h)\|^2$, where $h \in H$.

Theorem 3.4. *Let* $h = (\psi, \varphi) \in H$, *and* $\rho \in L_2^n[0, \infty)$ *is the function entering the decomposition* (3.8) *and* (3.9) *and described in* (3.13). *Then*

$$\|\pi_Z(h)\|^2 = \left\|B^T \rho + \varphi\right\|^2, \tag{3.23}$$

$$\|\pi_{Z^\perp}(h)\|^2 = \|h\|^2 - \left\|B^T \rho + \varphi\right\|^2. \tag{3.24}$$

Proof. Let $(y, v) \in Z$. Let, further,

$$\Delta(y, v) = \left(v - B^T K_{\mathrm{st}} y - B^T \rho - \varphi\right)^T \left(v - B^T K_{\mathrm{st}} y - B^T \rho - \varphi\right). \tag{3.25}$$

Here for simplicity of notations, we suppressed the dependence on t. Then

$$\Delta(y, v) = \Delta_1 + \Delta_2 + \Delta_3, \tag{3.26}$$

where

$$\Delta_1 = (\nu - \varphi)^T (\nu - \varphi), \ \Delta_2 = (K_{st} y + \rho)^T BB^T (K_{st} y + \rho), \ \text{and} \ \Delta_3 = -2(\nu - \varphi)^T \left(B^T K_{st} y + \rho\right). \tag{3.27}$$

Since $(y, \nu) \in Z$, we have

$$\dot{y} = Ay + B\nu, \qquad y(0) = 0. \tag{3.28}$$

Hence,

$$\begin{aligned}
\Delta_2 &= y^T (K_{st} BB^T K_{st}) y + \rho^T BB^T \rho + 2\rho^T BB^T K_{st} y, \\
\Delta_3 &= -2(B\nu - B\varphi)^T (K_{st} y + \rho) \\
&= -2(\dot{y} - Ay - B\varphi)^T (K_{st} y + \rho) \\
&= -2\dot{y} K_{st} y + y^T \left(A^T K_{st} + K_{st} A\right) y + 2(B\varphi)^T K_{st} y \\
&\quad - 2\dot{y}^T \rho + 2(Ay)^T \rho + 2(B\varphi)^T \rho.
\end{aligned} \tag{3.29}$$

Notice that $\dot{y}^T \rho + y^T \dot{\rho} = (d/dt)(y^T \rho)$. Hence,

$$\begin{aligned}
\Delta(y, \nu) = (\nu - \varphi)^T (\nu - \varphi) &+ y^T \left(K_{st} BB^T K_{st} + A^T K_{st} + K_{st} A\right) y \\
&+ 2y^T \left(\dot{\rho} + K_{st} B\varphi + K_{st} BB^T \rho + A^T \rho\right) + \left(B^T \rho\right)^T \left(B^T \rho\right) \\
&+ 2\varphi^T \left(B^T \rho\right) - \frac{d}{dt}\left(y^T \rho\right) - \frac{d}{dt}\left(y^T K_{st} y\right).
\end{aligned} \tag{3.30}$$

Using the fact that K_{st} is a solution to (3.5) and (3.13), we obtain that

$$\begin{aligned}
\Delta(y, \nu) &= (\nu - \varphi)^T (\nu - \varphi) + y^T y - 2y^T \psi + \left(B^T \rho + \varphi\right)^T \left(B^T \rho + \varphi\right) \\
&\quad - \varphi^T \varphi - \frac{d}{dt}\left(y^T \rho\right) - \frac{d}{dt}\left(y^T K_{st} y\right) \\
&= (\nu - \varphi)^T (\nu - \varphi) + (y - \psi)^T (y - \psi) + \left(B^T \rho + \varphi\right)^T \left(B^T \rho + \varphi\right) \\
&\quad - \varphi^T \varphi - \psi^T \psi - \frac{d}{dt}\left(y^T \rho\right) - \frac{d}{dt}\left(y^T K_{st} y\right).
\end{aligned} \tag{3.31}$$

Integrating (3.31) from 0 to $+\infty$ and using the fact that $y(0) = 0, \ \ y(t), \ \ \rho(t) \to 0$ as $t \to \infty$, we obtain that

$$\int_0^\infty \Delta(y, \nu) dt = \left\| (y - \psi, \nu - \varphi) \right\|^2 - \left\| (\psi, \varphi) \right\|^2 + \left\| B^T \rho + \varphi \right\|^2. \tag{3.32}$$

Notice that $\Delta(y,\nu) \geq 0$ and $\Delta(y,\nu) = 0$ provided $(y,\nu) = \pi_Z(\psi,\varphi)$. See (3.11). Consequently, (3.32) implies that

$$\|(\psi,\varphi)\|^2 = \left\|B^T\rho + \varphi\right\|^2 + \|\pi_{Z^\perp}(\psi,\varphi)\|^2. \tag{3.33}$$

Hence,

$$\|\pi_Z(\psi,\varphi)\|^2 = \left\|B^T\rho + \varphi\right\|^2. \tag{3.34}$$

This completes the proof of Theorem 3.4. □

We can now easily compute the coefficients of the objective function (2.11). Assuming that $h_i = (\psi_i, \varphi_i) \in L_2^n[0,\infty) \times L_2^m[0,\infty)$, $i = 1,2,\ldots,m$, $c = (\alpha,\beta) \in L_2^n[0,\infty) \times L_2^m[0,\infty)$ and noticing that by Theorem 3.4

$$\|\pi_Z(h(\lambda) - c)\|^2 = \int_0^\infty \left[B^T\rho(\lambda) + \varphi(\lambda)\right]^T \left[B^T\rho(\lambda) + \varphi(\lambda)\right] dt, \tag{3.35}$$

where $\rho(\lambda)$ is the solution of the differential equation

$$\frac{d}{dt}\rho(\lambda) = -\left(A + BB^T K_{\mathrm{st}}\right)^T \rho(\lambda) - K_{\mathrm{st}} B(\varphi(\lambda) - \psi(\lambda)), \tag{3.36}$$

belonging to $L_2^n[0,\infty)$ and

$$\varphi(\lambda) = \sum_{i=1}^m \lambda_i(\varphi_i - \beta), \qquad \psi(\lambda) = \sum_{i=1}^m \lambda_i(\psi_i - \alpha). \tag{3.37}$$

Consequently,

$$\rho(\lambda) = \sum_{i=1}^m \lambda_i(\rho_i - \rho_c), \tag{3.38}$$

where ρ_i and ρ_c are $L_2^n[0,\infty)$ solutions of differential equations

$$\begin{gathered} \dot{\rho}_i = -\left(A + BB^T K_{\mathrm{st}}\right)\rho_i - K_{\mathrm{st}} B\varphi_i - \psi_i, \qquad i = 1,2,\ldots,m, \\ \dot{\rho}_c = -\left(A + BB^T K_{\mathrm{st}}\right)\rho_c - K_{\mathrm{st}} B\beta - \alpha, \end{gathered} \tag{3.39}$$

respectively.

Hence,

$$\|\pi_Z(h(\lambda) - c)\|^2 = \int_0^\infty \Gamma(\lambda)^T \Gamma(\lambda) dt, \tag{3.40}$$

where

$$\Gamma(\lambda) = \sum_{i=1}^{m} \lambda_i \left[B^T (\rho_i - \rho_c) + (\varphi_i - \beta) \right], \tag{3.41}$$

which allows us to easily express the objective function (2.13) in terms of integrals of ρ_i and ρ_c.

4. Concluding Remarks

In this paper, we have shown that multitarget linear-quadratic control problem on semi-infinite interval can be reduced to solving a simple convex optimization on the simplex. The reduction involves solving one standard algebraic Riccati equation and $m + 1$ linear differential equations, where m is the number of targets. Notice that our results can be easily extended to discrete-time systems.

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