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Research Article

Optimal Maintenance of a Production System with L Intermediate Buffers

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We consider a production-inventory system that consists of an input-generating installation, a production unit and L intermediate buffers. It is assumed that the installation transfers the raw material $j \in \{1, \ldots, L\}$ to buffer B_j, and the production unit pulls the raw material $j \in \{1, ..., L\}$ from buffer B_j. We consider the problem of the optimal preventive maintenance of the installation if the installation deteriorates stochastically with usage and the production unit is always in operative condition. We also consider the problem of the optimal preventive maintenance of the production unit if the production unit deteriorates stochastically with usage and the installation is always in operative condition. Under a suitable cost structure and for given contents of the buffers, it is proved that the average-cost optimal policy for the first (second) problem initiates a preventive maintenance of the installation (production unit) if and only if the degree of deterioration of the installation (production unit) exceeds some critical level. Numerical results are presented for both problems.

1. Introduction

In the present paper, we study two problems, and we generalize the results obtained in two previous papers by Kyriakidis and Dimitrakos [1] and Pavitsos and Kyriakidis [2] that are concerned with the preventive maintenance of a production-inventory system. We consider a manufacturing system (see Figure 1) in which an input-generating installation (I) transfers L raw materials to a subsequent production unit (P). We assume that L buffers $B_1, \ldots, B_L$ have been built between the installation and the production unit. The installation transfers the raw

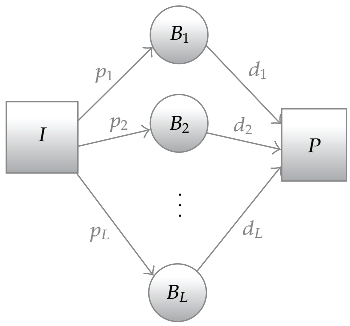

Figure 1: The production-inventory system.

material $j \in \{1, \ldots, L\}$ to the buffer B_j, and the production unit pulls this raw material from the buffer B_j. The buffers have finite capacities.

In the first problem it is assumed that the installation deteriorates stochastically over time, and the production unit is always in operative condition. The deteriorating process for the installation is described by some known transition probabilities between different degrees of deterioration. A discrete-time Markov decision model is considered for the optimal preventive maintenance of the installation. The maintenance times are assumed to be geometrically distributed, and the cost structure includes operating costs of the installation, costs for storing the raw materials in the buffers, maintenance costs and costs due to production delay when the installation does not operate or operate partially and the contents of some or all buffers are below some specific levels. It is proved that for fixed contents of the buffers the policy that minimizes the long-run expected average cost per unit time is of control-limit type, that is, it initiates a preventive maintenance of the installation if and only if its degree of deterioration exceeds some critical level. This result generalizes the structural result that was obtained by Kyriakidis and Dimitrakos [1] for the case in which $L = 1$. In the second problem it is assumed that the production unit deteriorates stochastically over time and the installation is always in operative condition. The deteriorating process for the production unit is described by some known transition probabilities between different degrees of deterioration. A discrete-time Markov decision model is formulated for the optimal preventive maintenance of the production unit. The maintenance times are assumed to be geometrically distributed, and the cost structure includes operating costs of the production unit, costs for the maintenance of the production unit, storage costs, penalty costs, and costs due to the lost production. It is proved that for fixed contents of the buffers the average-cost optimal policy is again of control-limit type, that is, it initiates a preventive maintenance of the production unit if and only if its degree of deterioration exceeds some critical level. This result generalizes the structural result that was obtained by Pavitsos and Kyriakidis [2] for the case in which $L = 1$.

An example of this system could be a production machine that pulls L different parts from L buffers and assembles them in order to produce the final product. These parts are transferred by a feeder to the buffers. Note that in the last twenty years a great number of maintenance models for production-inventory systems have been studied (see Van Der Duyn Schouten and Vanneste [3], Meller and Kim [4], Iravani and Duenyas [5], Sloan [6], Yao et al. [7], Rezg et al. [8], Dimitrakos and Kyriakidis [9], Karamatsoukis and Kyriakidis [10], and Hadidi et al. [11]). In these models, the preventive maintenance depends on the working condition of a machine and the level of a subsequent buffer. The first problem that we study in the present paper has its origin in a model introduced by Van Der Duyn Schouten and Vanneste [3]. The states of that model consist of the age of a machine and the content of a subsequent buffer that is fed by the machine. The cost structure included costs due to lost production that were incurred when a repair was performed on the machine and the buffer was empty. The repair times of the machine were assumed to be geometrically distributed. It was proved that, for fixed buffer content, the average-cost optimal policy initiates a preventive maintenance of the machine if and only if its age is greater than or equal to a critical value.

The rest of the paper is organized as follows. In the next section, we describe the problem in which only the installation deteriorates with usage, and we derive the structure of the average-cost optimal policy. In Section 3, we study the case in which only the production unit deteriorates over time, and the structure of the average-cost optimal policy is derived. Numerical results are presented for both problems. In the final section, the main conclusions of the paper are summarized, and we propose topics for future research.

2. The Problem when the Installation Deteriorates Stochastically

We consider a production-inventory system (see Figure 1) which consists of an installation (I) that supplies the buffer B_j with the raw material $j \in \{1,\ldots,L\}$ and a production unit (P) which pulls d_j units of the raw material $j \in \{1,\ldots,L\}$ from buffer B_j during one unit of time. It is assumed that the production unit is always in operative condition, and that the installation may fail as time evolves. The buffer B_j, $j = 1,2,\ldots,L$ has finite capacity which is equal to K_j units of raw material j. As long as the buffer B_j, $j = 1,\ldots,L$ is not full and the installation is in operative condition, the installation may transfer $p_j(> d_j)$ units of the raw material $j \in \{1,\ldots,L\}$ to buffer B_j during one unit of time and the difference $p_j - d_j$ is stored in buffer B_j. As soon as buffer B_j, $j = 1,\ldots,L$ is filled up the installation reduces its speed from p_j to d_j. The numbers p_j, d_j, K_j, $j = 1,\ldots,L$ are assumed to be integers.

We suppose that the installation is inspected at discrete, equidistant time epochs $\tau = 0,1,\ldots$ (say every day), and is classified into one of $m+2$ working conditions $0,1,\ldots,m+1$ which describe increasing levels of deterioration. Working condition 0 denotes a new installation (or functioning as good as new), while working condition $m+1$ means that the installation is in failed (inoperative) condition and it cannot transfer the raw materials to the buffers. The intermediate working conditions $1,\ldots,m$ are operative and are ordered ascendingly to reflect their relative degree of deterioration. The transition probability of moving from working condition i at time epoch τ to working condition r at time epoch $\tau+1$ is equal to p_{ir}. We assume that the probability of eventually reaching the working condition $m+1$ from any initial working condition i is nonzero. If at a time epoch τ the installation is found to be at failure state $m+1$, a corrective maintenance is mandatory. If at a time epoch τ the installation is found to be at any working condition $i \le m$, a preventive maintenance may be initiated. When a preventive maintenance is performed, the installation does not operate and it does not transfer any raw material to its buffer. It is assumed that the preventive and corrective repair times (expressed in time units) are geometrically distributed with probability of success a_I and b_I, that is, the probability that they will last $t \ge 1$ time units are equal to $(1-a_I)^{t-1}a_I$ and $(1-b_I)^{t-1}b_I$, respectively. When a preventive or a corrective maintenance is performed and the buffer B_j, $j \in \{1,\ldots,L\}$ contains x_j units of raw material j, the production unit pulls from buffer B_j during one unit of time $\min(x_j, d_j)$ units of raw material j. Both maintenances bring the installation to its perfect condition 0.

We introduce the state PM to denote the situation that a preventive maintenance is performed on the installation. Then the state space of the system is the set $S = \{0,\ldots,m+1,PM\} \times \{0,\ldots,K_1\} \times \cdots \times \{0,\ldots,K_L\}$, where $(i,x_1,\ldots,x_L) \in S$ is the state in which i is the working condition of the installation and $x_j \in \{0,\ldots,K_j\}$, $j = 1,\ldots,L$, is the content of buffer B_j. A policy is any rule for choosing actions at each time epoch $\tau = 0,1,\ldots$ The possible actions are: action 1 (initiate a preventive maintenance), action 2 (initiate a corrective maintenance), action $J \subseteq \{1,\ldots,L\}$ (transfer raw materials only to those buffers that belong to the nonempty subset J of the set $\{1,\ldots,L\}$). If at a time epoch τ the installation is found to be at state PM or state $m+1$, the action 1 or the action 2 is compulsory, respectively. If at a time epoch τ the installation is found to be at working condition $i \in \{0,\ldots,m\}$, then we may either choose either action 1 or action $J \subseteq \{1,\ldots,L\}$. Hence, the number of possible actions in this case is 2^L since the number of nonempty subsets of $\{1,\ldots,L\}$ is $\sum_{i=1}^{L}\binom{L}{i} = 2^L - 1$. If at a time epoch τ action J is chosen and j belongs to J with $x_j < K_j$, then the content of buffer B_j at next time epoch $\tau+1$ will be $\min(x_j + p_j - d_j, K_j)$. This increase of the buffer content will happen even if the working condition of the installation at next time epoch $\tau+1$ is the failure state $m+1$. A policy is said to be stationary if at each time epoch $\tau = 0,1,\ldots$, it chooses one action which depends only on the current state of the system.

The cost structure of the problem includes operating costs of the installation, storage costs, costs due to the lost production, and maintenance costs. If the working condition of the installation is $i \in \{0,\ldots,m\}$ and the buffer B_j, $j \in \{1,\ldots,L\}$ is not full (or full) the cost of transferring p_j (or d_j) units of raw material j to buffer B_j during one unit of time is equal to $c_j(i)$ (or $\tilde{c}_j(i)$). Therefore, if at a time epoch τ the working condition of the installation is found to be $i \in \{0,\ldots,m\}$ and action $J \subseteq \{1,\ldots,L\}$ is chosen, then the operating cost until the next time epoch $\tau+1$ is equal to $\sum_{j\in J_1} c_j(i) + \sum_{j\in J_2} \tilde{c}_j(i)$, where $J_1 \cup J_2 = J$, J_1 corresponds to the buffers that are not full and J_2 corresponds to the buffers that are full. We assume that the cost of holding a unit of raw material $j \in \{1,\ldots,L\}$ in buffer B_j for one unit of time is equal to h_j. The cost rates during a preventive and a corrective maintenance of the installation are equal to c_p and c_f, respectively. When a preventive or a corrective maintenance is performed on the installation and all buffers $B_1,\ldots,B_L$ are empty (i.e., $x_j = 0$, $j = 1,\ldots,L$), the production unit does not pull any raw material $j \in \{1,\ldots,L\}$ from the buffers. In this case we incur a cost due to production delay that is equal to $C > 0$ per unit of time. When $x_j \geq d_j$, $j = 1,\ldots,L$, we do not incur any such cost since all buffers contain enough raw materials to satisfy the demands of the production unit for one unit of time. When the inequality $x_j \geq d_j$ is satisfied for some (but not for all) $j = 1,\ldots,L$, the demands for raw materials of the production unit for one unit of time are partially satisfied. Therefore, the productivity of the production unit is reduced in the sense that the time for the production of the final products increases, since for one unit of time, some of the raw materials that are needed for the production of the final products are not available. In this case it seems reasonable to assume that the cost rate due to production delay is equal to $C\sum_{j=1}^{L}(d_j - x_j)^+ / \sum_{j=1}^{L} d_j$, where $(d_j - x_j)^+ = \max(d_j - x_j, 0)$ is the unavailable quantity of the raw material j during one unit of time. Similarly, if at a time epoch τ the working condition of the installation is found to be $i \in \{1,\ldots,m\}$ and the action $J \subseteq \{1,\ldots,L\}$ is chosen, then the cost due production delay until the time epoch $\tau+1$ is equal to $C\sum_{j\notin J}(d_j - x_j)^+ / \sum_{j=1}^{L} d_j$. The following conditions on the cost structure and on the transition probabilities are assumed to be valid.

Condition 1. For $j \in \{1,\ldots,L\}$ the sequences $\{c_j(i)\}$ and $\{\tilde{c}_j(i)\}$, $0 \leq i \leq m$, are nondecreasing with respect to i. That is, as the working condition of the installation deteriorates, the cost of transferring the raw material $j \in \{1,\ldots,L\}$ to buffer B_j increases.

Condition 2. For $j \in \{1, \ldots, L\}$, $\tilde{c}_j(i) \leq c_j(i)$, $0 \leq i \leq m$. That is, the cost of transferring p_j units of raw material $j \in \{1, \ldots, L\}$ to buffer B_j during one unit of time is greater than or equal to the cost of transferring d_j units of raw material j to buffer B_j during one unit of time.

Condition 3. $0 < b_I < a_I \leq 1$. That is, the expected time required for a preventive maintenance is smaller than the expected time required for a corrective maintenance.

Condition 4. $c_p \leq c_f$. That is, the cost rate of a preventive maintenance does not exceed the cost rate of a corrective maintenance.

Condition 5 (An Increasing Failure Rate Assumption). For each $k = 0, \ldots, m$, the function $D_k(i) = \sum_{r=k}^{m+1} p_{ir}$ is nondecreasing in i, $0 \leq i \leq m$.

A consequence of this condition is that $I_i \leq_{st} I_{i+1}$, $0 \leq i \leq m$, where I_i is a random variable representing the next working condition of the installation if its present working condition is i. It can be shown (see pages 122-123 in Derman [12]) that this condition is equivalent to the following one:

Condition 6. For any nondecreasing function $h(r)$, $0 \leq r \leq m+1$, the quantity $\sum_{r=0}^{m+1} p_{ir} h(r)$, $0 \leq i \leq m$, is nondecreasing in i.

We consider a discrete-time Markov decision process in which we aim to find a stationary policy which minimizes the long-run expected average cost per unit time. Note that for $L = 1$ this problem was studied in Kyriakidis and Dimitrakos [1].

2.1. The Structure of the Optimal Policy

Let α $(0 < \alpha < 1)$ be a discount factor. The minimum expected n-step discounted cost $V_n^\alpha(i, \overline{x})$, where $(i, \overline{x}) = (i, x_1, \ldots, x_L)$ is the initial state, can be found for all $n = 1, 2, \ldots$, recursively, from the following equations (see chapter 1 in Ross [13]):

$$V_n^\alpha(i,\overline{x}) = \min\left\{ \min_{J \subseteq \{1,\ldots,L\}} \left[\sum_{j\in J} c_j(i) + \sum_{j=1}^{L} h_j x_j + \frac{\sum_{j\notin J} (d_j - x_j)^+}{\sum_{j=1}^{L} d_j} C + \alpha \sum_{r=0}^{m+1} p_{ir} V_{n-1}^\alpha(r, \overline{x}') \right], \right.$$
$$\left. V_n^\alpha(PM, \overline{x}) \right\}, \quad 0 \leq i \leq m,\ 0 \leq x_j < K_j,\ j = 1, \ldots, L, \tag{2.1}$$

$$V_n^\alpha(PM, \overline{x}) = c_p + \sum_{j=1}^{L} h_j x_j + \frac{\sum_{j=1}^{L} (d_j - x_j)^+}{\sum_{j=1}^{L} d_j} C + \alpha a_I V_{n-1}^\alpha\left(0, \left(\overline{x} - \overline{d}\right)^+\right)$$
$$+ \alpha(1 - a_I) V_{n-1}^\alpha\left(PM, \left(\overline{x} - \overline{d}\right)^+\right), \quad 0 \leq x_j \leq K_j,\ j = 1, \ldots, L, \tag{2.2}$$

$$V_n^\alpha(m+1, \overline{x}) = c_f + \sum_{j=1}^{L} h_j x_j + \frac{\sum_{j=1}^{L} (d_j - x_j)^+}{\sum_{j=1}^{L} d_j} C + \alpha b_I V_{n-1}^\alpha\left(0, \left(\overline{x} - \overline{d}\right)^+\right)$$
$$+ \alpha(1 - b_I) V_{n-1}^\alpha\left(m+1, \left(\overline{x} - \overline{d}\right)^+\right), \quad 0 \leq x_j \leq K_j,\ j = 1, \ldots, L, \tag{2.3}$$

where $\overline{x}'$ in (2.1) is a vector with L components in which the jth component equals to $\min(x_j+p_j-d_j, K_j)$, if $j \in J$, while, if $j \notin J$, it is equal to $(x_j-d_j)^+ = \max(x_j-d_j, 0)$ and $(\overline{x}-\overline{d})^+$ in (2.2), and (2.3) is the vector $((x_1-d_1)^+, \ldots, (x_L-d_L)^+)$. The initial condition is $V_0^\alpha(s) = 0$, $s \in S$. Note that, if $x_j = K_j$ for some values of $j \in \{1, \ldots, L\}$, (2.1) is valid if $c_j(i)$ is changed to $\tilde{c}_j(i)$ for these values of j. Note that the first term in the curly brackets in (2.1) corresponds to the best action among all actions $J \subseteq \{1, \ldots, L\}$, while the second term corresponds to action 1 (i.e., initiate a preventive maintenance of the installation). The first part of the following lemma is needed to prove that the average-cost optimal policy is of control-limit type for fixed levels of the buffers.

Lemma 2.1. *For each $n = 0, 1, \ldots$, we have that*

(i) $V_n^\alpha(i, \overline{x}) \le V_n^\alpha(i+1, \overline{x})$, $0 \le i \le m$, $0 \le x_j \le K_j$, $j = 1, \ldots, L$,

(ii) $V_n^\alpha(PM, \overline{x}) \le V_n^\alpha(m+1, \overline{x})$, $0 \le x_j \le K_j$, $j = 1, \ldots, L$.

Proof. We will prove the lemma by induction on n. The lemma is valid for $n = 0$, since $V_0^\alpha(s) = 0, s \in S$. We assume that it is valid for $n-1 (\ge 0)$. We will show that it is also valid for n. First, we prove part (ii) and then part (i).

Part (ii): Let $D = V_{n-1}^\alpha(m+1, (\overline{x}-\overline{d})^+) - V_{n-1}^\alpha(0, (\overline{x}-\overline{d})^+)$.

For $0 \le x_j \le K_j$, $j = 1, \ldots, L$, we have that

$$
\begin{aligned}
V_n^\alpha(PM, \overline{x}) &= c_p + \sum_{j=1}^{L} h_j x_j + \frac{\sum_{j=1}^{L} (d_j - x_j)^+}{\sum_{j=1}^{L} d_j} C + \alpha a_I V_{n-1}^\alpha\left(0, \left(\overline{x}-\overline{d}\right)^+\right) \\
&\quad + \alpha(1-a_I) V_{n-1}^\alpha\left(PM, \left(\overline{d}-\overline{x}\right)^+\right) \\
&\le c_f + \sum_{j=1}^{L} h_j x_j + \frac{\sum_{j=1}^{L} (d_j - x_j)^+}{\sum_{j=1}^{L} d_j} C \\
&\quad + \alpha a_I V_{n-1}^\alpha\left(0, \left(\overline{x}-\overline{d}\right)^+\right) + \alpha(1-a_I) V_{n-1}^\alpha\left(m+1, \left(\overline{x}-\overline{d}\right)^+\right) \\
&= c_f + \sum_{j=1}^{L} h_j x_j + \frac{\sum_{j=1}^{L} (d_j - x_j)^+}{\sum_{j=1}^{L} d_j} C + \alpha V_{n-1}^\alpha\left(m+1, \left(\overline{x}-\overline{d}\right)^+\right) - \alpha a_I D \\
&\le c_f + \sum_{j=1}^{L} h_j x_j + \frac{\sum_{j=1}^{L} (d_j - x_j)^+}{\sum_{j=1}^{L} d_j} C + \alpha V_{n-1}^\alpha\left(m+1, \left(\overline{x}-\overline{d}\right)^+\right) - \alpha b_I D \\
&= V_n^\alpha(m+1, \overline{x}).
\end{aligned}
\tag{2.4}
$$

The first inequality follows from Condition 4 and from part (ii) of the induction hypothesis. The second inequality follows from Condition 3 and the inequality $D \ge 0$ which is a consequence of part (i) of the induction hypothesis.

Part (i): We have to show that

$$V_n^\alpha(m,\overline{x}) \le V_n^\alpha(m+1,\overline{x}), \qquad 0 \le x_j \le K_j, \quad j=1,\dots,L, \tag{2.5}$$

$$V_n^\alpha(i,\overline{x}) \le V_n^\alpha(i+1,\overline{x}), \quad 0 \le i \le m-1, \qquad 0 \le x_j \le K_j, \quad j=1,\dots,L. \tag{2.6}$$

Inequality (2.5) is easily verified using (2.1) with $i=m$ and part (ii) above

$$V_n^\alpha(m,\overline{x}) \le V_n^\alpha(PM,\overline{x}) \le V_n^\alpha(m+1,\overline{x}). \tag{2.7}$$

For $0 \le i \le m-1$ and $0 \le x_j < K_j,\ j=1,\dots,L$, we have that

$$\begin{aligned} V_n^\alpha(i,\overline{x}) &= \min\left\{ \min_{J\subseteq\{1,\dots,L\}} \left[\sum_{j\in J} c_j(i) + \sum_{j=1}^{L} h_j x_j + \frac{\sum_{j\notin J}(d_j-x_j)^+}{\sum_{j=1}^{L} d_j} C + \alpha \sum_{r=0}^{m+1} p_{ir} V_{n-1}^\alpha(r,\overline{x}') \right], V_n^\alpha(PM,\overline{x}) \right\} \\ &\le \min\left\{ \min_{J\subseteq\{1,\dots,L\}} \left[\sum_{j\in J} c_j(i+1) + \sum_{j=1}^{L} h_j x_j + \frac{\sum_{j\notin J}(d_j-x_j)^+}{\sum_{j=1}^{L} d_j} C + \alpha \sum_{r=0}^{m+1} p_{i+1,r} V_{n-1}^\alpha(r,\overline{x}') \right], \right. \\ &\qquad \left. V_n^\alpha(PM,\overline{x}) \right\} = V_n^\alpha(i+1,\overline{x}). \end{aligned} \tag{2.8}$$

The above inequality follows from Condition 1 and the inequality

$$\sum_{r=0}^{m+1} p_{ir} V_{n-1}^\alpha(r,\overline{x}') \le \sum_{r=0}^{m+1} p_{i+1,r} V_{n-1}^\alpha(r,\overline{x}') \tag{2.9}$$

which is implied by part (i) of the induction hypothesis and Condition 6. Hence (2.6) has been proved for $x_j \in \{0,\dots,K_j-1\},\ j=1,\dots,L$. Similarly, we obtain (2.6) if $x_j = K_j$ for some values of $j \in \{1,\dots,L\}$. □

Since the state space S is finite, and the state $(0,\overline{0})$ is accessible from every other state under any stationary policy, it follows that there exist numbers $v(s)$, $s \in S$ and a constant g that satisfy the average-cost optimality equations (see Corollary 2.5 in [13], page 98). For the states $(i,\overline{x})$, $0 \le i \le m$, $0 \le x_j < K_j,\ j=1,\dots,L$, the optimality equations take the following form:

$$\begin{aligned} v(i,\overline{x}) = \min&\left\{ \min_{J\subseteq\{1,\dots,L\}} \left[\sum_{j\in J} c_j(i) + \sum_{j=1}^{L} h_j x_j + \frac{\sum_{j\notin J}(d_j-x_j)^+}{\sum_{j=1}^{L} d_j} C - g + \sum_{r=0}^{m+1} p_{ir} v(r,\overline{x}') \right], \right. \\ &\left. v(PM,\overline{x}) \right\}. \end{aligned} \tag{2.10}$$

If $x_j = K_j$ for some values of $j \in \{1,\dots,L\}$, we must change in the above equations $c_j(i)$ to $\tilde{c}_j(i)$ for these values of j. In view of part (i) of the above lemma we have the following result.

Corollary 2.2. $v(i,\overline{x}) \le v(i+1,\overline{x}),\ 0 \le i \le m,\ 0 \le x_j \le K_j,\ j = 1,\dots,L.$

The following proposition gives a characterization of the form of the optimal policy.

Proposition 2.3. *For fixed contents $x_1,\dots,x_L$ ($0 \le x_j \le K_j$, $j = 1,\dots,L$) of buffers $B_1,\dots,B_L$, there exists a critical working condition $\mathbf{i}^*(x_1,\dots,x_L)$ such that the policy that minimizes the expected long-run average cost per unit time initiates a preventive maintenance of the installation if and only if its working condition $\mathbf{i} \in \{0,\dots,m\}$ is greater than or equal to $\mathbf{i}^*(x_1,\dots,x_L)$.*

Proof. Suppose that for some fixed $\overline{x} = (x_1,\dots,x_L)$ such that $0 \le x_j < K_j$, $1 \le j \le L$, the optimal policy initiates a preventive maintenance of the installation at state $(i,\overline{x})$ where $i \in \{0,\dots,m-1\}$. This implies that

$$v(PM,\overline{x}) \le \min_{J\subseteq\{1,\dots,L\}} \left[\sum_{j\in J} c_j(i) + \sum_{j=1}^{L} h_j x_j + \frac{\sum_{j\notin J}\left(d_j - x_j\right)^{+}}{\sum_{j=1}^{L} d_j} C - g + \sum_{r=0}^{m+1} p_{ir} v\left(r,\overline{x}'\right)\right]. \tag{2.11}$$

To show that the optimal policy prescribes a preventive maintenance on the installation at state $(i+1,\overline{x})$ it is enough to show that

$$v(PM,\overline{x}) \le \min_{J\subseteq\{1,\dots,L\}} \left[\sum_{j\in J} c_j(i+1) + \sum_{j=1}^{L} h_j x_j + \frac{\sum_{j\notin J}\left(d_j - x_j\right)^{+}}{\sum_{j=1}^{L} d_j} C - g + \sum_{r=0}^{m+1} p_{i+1,r} v\left(r,\overline{x}'\right)\right]. \tag{2.12}$$

From Conditions 1 and 6 and Corollary 2.2, it follows that the right-hand side of (2.12) is greater than or equal to the right-hand side of (2.11). Hence (2.11) implies (2.12). The same result is obtained similarly if $x_j = K_j$ for some values of $j \in \{1,\dots,L\}$. □

Remark 2.4. In the above proposition, if for fixed contents $x_1,\dots,x_L$ of the buffers $\mathbf{i}^*(x_1,\dots,x_L) = m+1$, then the optimal policy never initiates a preventive maintenance of the installation whenever the buffer B_j, $j = 1,\dots,L$, contains x_j units or raw material j.

2.2. Numerical Results

Example 2.5. Suppose that $L = 2, m = 5, a_I = 0.6, b_I = 0.4, c_p = 10, c_f = 15, K_1 = 5, K_2 = 20, h_1 = 1, h_2 = 1, p_1 = 2, d_1 = 1, p_2 = 2, d_2 = 1, c_1(i) = 0.8(i+1), \tilde{c}_1(i) = 0.5(i+1), c_2(i) = 0.7(i+1), \tilde{c}_2(i) = 0.5(i+1), 0 \le i \le m$, and $p_{ir} = 1/(m+2-i), 0 \le i \le m, i \le r \le m+1$. It can be readily checked that these probabilities satisfy Condition 5. We computed the optimal policy if C is equal to 0.5 or 15.5 by implementing the value-iteration algorithm (see Chapter 3 of Tijms [14]). Our numerical results verify the result of Proposition 2.3. In Table 1 we present the critical numbers $i^*(x_1,x_2),\ 0 \le x_1 \le 5,\ 0 \le x_2 \le 20$. In each cell of this table the first number corresponds to $C = 0.5$ and the second number corresponds to $C = 15.5$. The minimum average cost was found to be 7.49 if $C = 0.5$ which is, as expected, considerably smaller than the minimum average cost if $C = 15.5$, which was found to be 11.63.

Table 1: The critical numbers $i^*(x_1, x_2)$ for $C = 0.5, 15.5$.

$x_2 \backslash x_1$	0	1	2	3	4	5
0	3, 6	3, 5	3, 5	4, 6	4, 6	4, 6
1	3, 5	2, 4	2, 4	1, 6	1, 4	1, 4
2	3, 5	2, 3	1, 2	0, 3	0, 3	2, 3
3	4, 6	1, 4	0, 3	0, 2	0, 2	3, 3
4	3, 6	1, 1	0, 0	0, 0	0, 0	3, 3
5	4, 6	1, 4	0, 0	0, 2	0, 1	3, 3
6	4, 6	1, 4	0, 2	0, 1	0, 1	2, 4
7	4, 6	0, 4	0, 2	0, 1	0, 1	2, 3
8	4, 6	0, 4	0, 2	0, 0	0, 0	1, 3
9	4, 6	0, 4	0, 2	0, 0	0, 0	2, 3
10	4, 6	0, 4	0, 2	0, 0	0, 0	1, 3
11	4, 6	0, 4	0, 2	0, 0	0, 0	1, 3
12	4, 6	0, 4	0, 2	0, 0	0, 0	1, 3
13	4, 6	0, 4	0, 1	0, 0	0, 0	1, 3
14	4, 6	0, 4	0, 1	0, 0	0, 0	1, 3
15	4, 6	0, 4	0, 1	0, 0	0, 0	1, 3
16	4, 6	0, 4	0, 1	0, 0	0, 0	1, 2
17	4, 6	0, 4	0, 1	0, 0	0, 0	1, 3
18	4, 6	0, 4	0, 1	0, 0	0, 0	1, 2
19	4, 6	0, 4	0, 1	0, 0	0, 0	1, 3
20	4, 6	0, 4	0, 1	0, 0	0, 0	1, 2

We observe that for $x_1 \in \{0, \ldots, 5\}$ and $x_2 \in \{0, \ldots, 20\}$ the critical number that corresponds to $C = 0.5$ is smaller than or equal to the critical number that corresponds to $C = 15.5$. This is intuitively reasonable since if the cost due to production delay takes large values it seems disadvantageous to have all or most of the buffers empty when a maintenance is performed on the installation. Therefore in this case it seems preferable to initiate a preventive maintenance of the installation only if its degree of deterioration is relatively high. We also observe that when $C = 15.5$ and buffer B_1 or buffer B_2 is empty, the optimal policy in most cases never initiates a preventive maintenance of the installation. For example $i^*(0, 3) = 6 = m + 1$. It can be also seen from the Table 1 that $i^*(x_1, x_2)$ is not a monotone function with respect to $x_1 \in \{0, \ldots, 5\}$ and with respect to $x_2 \in \{0, \ldots, 20\}$ for constant x_2 and x_1, respectively. Note that, when $i < i^*(x_1, x_2), 0 \le x_1 \le 5, 0 \le x_2 \le 20$, the value iteration algorithm gives the optimal action for the operation of the installation. For example, if $C = 0.5$, $x_1 = 0$, $x_2 = 18$, the optimal action when the working condition of the installation is 3 is to transfer raw material 1 to buffer 1 (i.e., $J = \{1\}$). If $C = 15.5$, $x_1 = 1$, $x_2 = 1$, the optimal action when the working condition of the installation is 2 is to transfer raw material 1 to buffer 1 and raw material 2 to buffer 2 (i.e., $J = \{1, 2\}$).

Example 2.6. Suppose that $L = 2, m = 15, a_I = 0.3, b_I = 0.2, c_p = 10, c_f = 15, h_1 = 1, h_2 = 1, C = 80$, $p_1 = 4, d_1 = 2, p_2 = 3, d_2 = 2, c_1(i) = 1.5(i + 1), \tilde{c}_1(i) = 0.75(i + 1), c_2(i) = 2(i + 1), \tilde{c}_2(i) = i + 1, 0 \le i \le m, p_{ir} = 1/(m + 2 - i), 0 \le i \le m, i \le r \le m + 1$. In Table 2 below we present the minimum average cost g obtained by the value-iteration algorithm for $K_1 \in \{1, \ldots, 10\}$ and $K_2 \in \{5, 10\}$.

Table 2: The minimum average cost as K_1 or K_2 varies.

K_1	$K_2 = 5$	$K_2 = 10$
1	51.20	51.17
2	48.55	48.49
3	47.55	47.50
4	45.94	45.88
5	45.60	45.56
6	44.87	44.83
7	44.78	44.75
8	44.49	44.45
9	44.46	44.43
10	44.39	44.37

From the above table we see that as K_1 or K_2 increases, the minimum average cost decreases. This is intuitively reasonable because in this example it seems favourable to have buffers with large, capacities since the cost rate C due to production delay is relatively large while the probabilities a_I, b_I of successful maintenances and the storage cost rates h_1, h_2 are relatively small.

3. The Problem when the Production Unit Deteriorates Stochastically

We consider the same production-inventory system (see Figure 1) as the one introduced in the previous section with the following modifications: (i) the installation is always in operative condition while the production unit may experience a failure as time evolves and (ii) as long as the buffer B_j, $j = 1, \ldots, L$, is not empty and the production unit is in operative condition, the production unit may pull the raw material j from buffer B_j at a constant rate of $d_j(> p_j)$ units of raw material j per unit of time. When the buffer B_j is empty and the production unit is in operative condition, the production unit reduces its pull rate from d_j to p_j.

We assume that the production unit is monitored at discrete equidistant time epochs $\tau = 0, 1, \ldots$ (say every day), and is classified into one of $n+2$ working conditions $0, \ldots, n+1$. We suppose that working condition i is better than working condition $i + 1$. Working condition 0 means that the production unit is new (or functioning as good as new), while working condition $n + 1$ means that the production unit does not function, and it cannot pull the materials from the buffers. The intermediate working conditions $1, \ldots, n$ are operative. If the working condition at time epoch τ is i then the working condition at time epoch $\tau + 1$ will be r with probability q_{ir}. The probability that the deterioration process of the production unit reaches eventually the failure state $n + 1$ from any initial working condition i is assumed to be nonzero. If at a time epoch τ the production unit is found to be at the failure state $n + 1$, a corrective maintenance is compulsory. If it is found to be at any working condition $i \leq n$, a preventive maintenance is optional. The production unit does not operate when it is under preventive maintenance, and it does not pull any raw material from its buffer. When a preventive or a corrective maintenance is performed and the buffer B_j, $j = 1, \ldots, L$, contains x_j units of raw material j, the installation transfers $\min(p_j, K_j - x_j)$ units of raw material j to buffer B_j during one unit of time. The preventive and corrective maintenance times (expressed in time units) are geometrically distributed with probability of success a_P and b_P, respectively. Both maintenances bring the production unit to its perfect condition 0. The state space of the system is the set $\widetilde{S} = \{0, \ldots, n+1, PM\} \times \{0, \ldots, K_1\} \times \cdots \times \{0, \ldots, K_L\}$, where PM represents the situation that the production unit is under a preventive repair. The possible actions are the same as the ones considered in the problem studied in Section 2 with the following modification: action $J \subseteq \{1, \ldots, L\}$ is the action of pulling raw materials only from buffers B_j, $j \in J$. If at a time epoch τ the production unit is found to be at working condition $i \in \{0, \ldots, n\}$, then we may choose either the action of initiating a preventive maintenance or action $J \subseteq \{1, \ldots, L\}$.

The cost structure includes operating costs of the production unit, storage costs, maintenance costs, costs due to production delay, and penalty costs. The storage costs h_j, $j \in \{1, \ldots, L\}$, and the maintenance costs c_p, c_f are defined exactly in the same way as in the problem studied in the previous section. If the working condition of the production unit is $i \in \{0, \ldots, n\}$ and the buffer B_j, $j \in \{1, \ldots, L\}$, is nonempty (or empty), the cost of pulling d_j (or p_j) units of raw material j from buffer B_j during one unit of time is equal to $c_j(i)$ (or $\tilde{c}_j(i)$). We assume that the cost rate due to production delay as long as a maintenance of the production unit lasts is equal to $C > 0$. Therefore, if at a time epoch τ the action $J \subseteq \{1, \ldots, L\}$ is selected and the content of buffer B_j, $j = 1, \ldots, L$, is $x_j \in \{0, \ldots, K_j\}$, then the cost due to production delay until time epoch $\tau + 1$ is equal to $C(\sum_{j \notin J} d_j + \sum_{j \in J} (d_j - p_j - x_j)^+) / \sum_{j=1}^{L} d_j$, where $(d_j - p_j - x_j)^+ = \max(d_j - p_j - x_j, 0)$ is the unavailable quantity of raw material $j \in J$ during one unit of time. A penalty cost per unit time which is equal to $P_j > 0$, $j = 1, \ldots, L$, is also imposed for each unit or raw material j that is not stored in buffer B_j during a corrective or a preventive maintenance of the production unit when the buffer B_j is full. This cost is due to the necessary labor for transferring and storing the raw material in another place until the completion of the maintenance. We assume that Conditions 1–5 on the cost structure that we introduced for the problem studied in the previous section are valid if we replace b_I with b_P, a_I with a_P, and m with n. We consider a discrete-time Markov decision process in which we aim to find a stationary policy which minimizes the expected long-run average cost per unit of time. Note that for $L = 1$, the problem was studied in Pavitsos and Kyriakidis [2].

Since the state space $\tilde{S}$ is finite and the state $(0, K_1, \ldots, K_L)$ is accessible from every other state under any stationary policy, it follows that there exist numbers $w(s)$, $s \in S$ and a constant g that satisfy the average-cost optimality equations. For the states $(i, \overline{x})$, $0 \le i \le n, 0 < x_j \le K_j$, $j = 1, \ldots, L$, the optimality equations have the following form:

$$w(i, \overline{x}) = \min\left\{ \min_{J \subseteq \{1, \ldots, L\}} A(J),\ w(PM, \overline{x}) \right\}, \tag{3.1}$$

where

$$\begin{aligned} A(J) = {} & \sum_{j \in J} c_j(i) + \sum_{j=1}^{L} h_j x_j + \frac{\sum_{j \notin J} d_j + \sum_{j \in J} (d_j - p_j - x_j)^+}{\sum_{j=1}^{L} d_j} C \\ & + \sum_{j \notin J} P_j (p_j + x_j - K_j)^+ - g + \sum_{r=0}^{m+1} q_{ir} w(r, \overline{x}'), \end{aligned} \tag{3.2}$$

$$\begin{aligned} w(PM, \overline{x}) = {} & \sum_{j=1}^{L} h_j x_j + C + \sum_{j=1}^{L} P_j (p_j + x_j - K_j)^+ \\ & - g + a_P w(0, \min(x_1 + p_1, K_1), \ldots, \min(x_L + p_L, K_L)) \\ & + (1 - a_P) w(PM, \min(x_1 + p_1, K_1), \ldots, \min(x_L + p_L, K_L)). \end{aligned} \tag{3.3}$$

If $x_j = 0$ for some values of $j \in \{1, \ldots, L\}$, we must change in (3.2) $c_j(i)$ to $\tilde{c}_j(i)$ for these values. Note that $\overline{x}'$ in (3.2) is a vector with L components in which the jth component is equal to $(x_j + p_j - d_j)^+$ if $j \in J$, while, if $j \notin J$, it is equal to $(x_j - d_j)^+$. It is possible to prove that $w(i, \overline{x}) \leq w(i+1, \overline{x}), 0 \leq i \leq n, 0 \leq x_j \leq K_j$, $j = 1, \ldots, L$, using the dynamic programming equation for the corresponding finite-horizon problem. The method is exactly the same as that used for the proof of Corollary 2.2 in the previous section and, therefore, we omit the details. An immediate consequence of the above monotonicity result is that the result of Proposition 2.3 is valid for the problem of the optimal preventive maintenance of the production unit.

3.1. Numerical Examples

Example 3.1. Suppose that $L = 2$, $K_1 = 7$, $K_2 = 10$, $n = 10$, $a_P = 0.6$, $b_P = 0.4$, $c_p = 0.4$, $c_f = 0.8$, $h_1 = 1$, $h_2 = 2$, $C = 0.5$, $P_1 = 1$, $P_2 = 1$, $p_1 = 1$, $d_1 = 2$, $p_2 = 1$, $d_2 = 2$, $c_1(i) = 2(i+1)$, $\tilde{c}_1(i) = 1.5(i+1)$, $c_2(i) = 3(i+1)$, $\tilde{c}_2(i) = 2.5(i+1)$, $0 \leq i \leq n$, and $p_{ir} = 1/(n+2-i)$, $0 \leq i \leq n$, $i \leq r \leq n+1$. We computed the optimal policy by implementing the value-iteration algorithm. The minimum average cost was found to be 15.67. In Table 3, we present the critical numbers $i^*(x_1, x_2)$, $0 \leq x_1 \leq 7$, $0 \leq x_2 \leq 10$.

We can see from Table 3 that $i^*(x_1, x_2)$ is not a monotone function with respect to $x_1 \in \{0, \ldots, 7\}$ or with respect to $x_2 \in \{0, \ldots, 10\}$, respectively. Note that, when $i < i^*(x_1, x_2)$, $0 \leq x_1 \leq 7$, $0 \leq x_2 \leq 10$, the value-iteration algorithm gives the optimal action for the operation of the production unit. For example, if $x_1 = 0, x_2 = 9$, the optimal action when the working condition of the installation is $i \in \{0, 1, 2, 3, 4\}$ is to pull raw material 1 from buffer B_1 and raw material 2 from buffer B_2 (i.e., $J = \{1, 2\}$), while, when the working condition is $i \in \{5, 6, 7\}$ the optimal condition is to pull only raw material 2 from buffer B_2 (i.e., $J = \{2\}$).

Example 3.2. Suppose that $L = 2$, $K_1 = 5$, $K_2 = 15$, $n = 10$, $a = 0.6$, $b = 0.4$, $c_p = 0.5$, $c_f = 0.8$, $C = 0.5$, $P_1 = 1$, $P_2 = 1, p_1 = 1$, $d_1 = 2$, $p_2 = 1$, $d_2 = 2$, $c_1(i) = 2(i+1)$, $\tilde{c}_1(i) = 1.5(i+1)$, $c_2(i) = 3(i+1)$, $\tilde{c}_2(i) = 2.5(i+1)$, $0 \leq i \leq n$, and $p_{ir} = 1/(n+2-i)$, $0 \leq i \leq n$, $i \leq r \leq n+1$. In Figure 2 we present the graph of the minimum average cost $g(h_1)$ as a function of $h_1 \in \{1, \ldots, 10\}$, if $h_2 = 1$, and the graph of the minimum average cost $g(h_2)$ as a function of $h_2 \in \{1, \ldots, 10\}$, if $h_1 = 1$. We observe that $g(h_i), i = 1, 2$, increases as h_i increases. The increase of the minimum average cost is more intense when h_2 increases. This can be explained by the fact that the capacity of buffer B_2 is considerably greater than the capacity of buffer B_1.

4. Conclusions and Future Research

We presented two discrete-time Markov decision models for the optimal condition-based preventive maintenance of a production system which consists of two machines and L intermediate buffers. The first machine transfers L different raw materials to the buffers, and the second machine draws the raw materials from the buffers. The second machine is considered to be a production unit that assembles the raw materials in order to produce the final product. In the first model, it is assumed that only the first machine deteriorates stochastically over time while the production unit is always in operative condition. It is possible to monitor the first machine at discrete equidistant time epochs and to classify it into one working condition that describes its level of deterioration. If the first machine is

Table 3: The critical numbers $i^*(x_1, x_2)$, $0 \le x_1 \le 7$, $0 \le x_2 \le 10$.

$x_2 \backslash x_1$	0	1	2	3	4	5	6	7
0	2	6	7	7	7	6	4	2
1	8	9	9	9	9	9	8	8
2	9	10	10	10	10	10	9	9
3	10	10	10	10	10	10	10	1
4	10	10	10	10	10	10	10	10
5	10	10	10	10	10	10	10	10
6	10	10	10	10	10	10	10	10
7	10	10	10	10	10	10	10	10
8	10	10	10	10	10	10	10	10
9	8	8	9	9	9	9	9	9
10	3	4	4	5	5	5	4	3

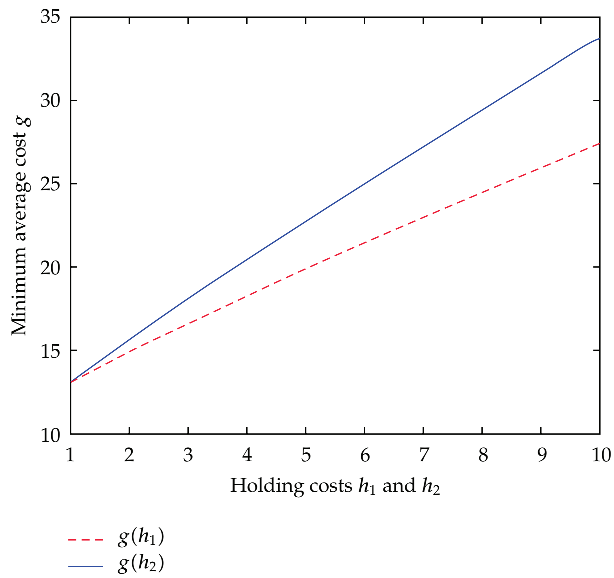

Figure 2: The minimum average cost $g(h_1)$ and $g(h_2)$.

in failed condition a corrective maintenance must be commenced; otherwise a preventive maintenance may be performed or the action of transferring raw materials to any subset of the set of L buffers may be selected. The maintenances bring the first machine to its perfect condition. In the second model it is assumed, that only the production unit deteriorates over time, and the first machine is always in operative machine. It is possible to determine the level of deterioration of the production unit after inspecting it at discrete equidistant time epochs. If the production unit is in failed condition a corrective maintenance must be started; otherwise a preventive maintenance may be initiated or the action of pulling the raw materials from any subset of the set of L buffers may be selected. Both maintenances bring the production unit to its perfect condition.

In both models we considered the problem of determining the policy that minimizes the expected long-run average cost per unit time. If the maintenance times are geometrically distributed we proved that, in both models, the optimal policy is of control-limit type, that is, for fixed contents of the buffers it prescribes a preventive maintenance of the first machine or the production unit if and only if its degree of deterioration exceeds some critical level. The proof was achieved through the corresponding finite-horizon problem.

A topic for future research could be a more complicated problem in which the first machine transfers the raw materials to the buffers and the production unit draws them from the buffers in a random manner. Another topic for future research could be the study of the maintenance problems that we would have if the maintenance times are not geometrically distributed but follow some general distributions with suitable conditions.

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