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Research Article

Strong Uniform Attractors for Nonautonomous Suspension Bridge-Type Equations

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We discuss long-term dynamical behavior of the solutions for the nonautonomous suspension bridge-type equation in the strong Hilbert space $D(A) \times H^2(\Omega) \cap H_0^1(\Omega)$, where the nonlinearity $g(u,t)$ is translation compact and the time-dependent external forces $h(x,t)$ only satisfy condition (C^*) instead of translation compact. The existence of strong solutions and strong uniform attractors is investigated using a new process scheme. Since the solutions of the nonautonomous suspension bridge-type equation have no higher regularity and the process associated with the solutions is not continuous in the strong Hilbert space, the results are new and appear to be optimal.

1. Introduction

Consider the following equations:

$$\begin{gathered} u_{tt} + u_{xxxx} + \delta u_t + ku^+ = l + \epsilon h(x,t), \quad \text{in } (0,L) \times \mathbb{R}, \\ u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0, \quad t \in \mathbb{R}. \end{gathered} \tag{1.1}$$

Suspension bridge equations (1.1) have been posed as a new problem in the field of nonlinear analysis [1] by Lazer and McKenna in 1990. This model has been derived as follows. In the suspension bridge system, suspension bridge can be considered as an elastic and unloaded beam with hinged ends. $u(x,t)$ denotes the deflection in the downward direction; δu_t represents the viscous damping. The restoring force can be modeled owing to the cable with one-sided Hooke's law so that it strongly resists expansion but does not resist compression. The simplest function to model the restoring force of the stays in the suspension bridge can be denoted by a constant k times u, the expansion, if u is positive, but zero, if u is negative, corresponding to compression; that is, ku^+, where

$$u^+ = \begin{cases} u, & \text{if } u > 0, \\ 0, & \text{if } u \leqslant 0. \end{cases} \tag{1.2}$$

Besides, the right-hand side of (1.1) also contains two terms: the large positive term l corresponding to gravity, and a small oscillatory forcing term $\epsilon h(x,t)$ possibly aerodynamic in origin, where ϵ is small.

There are many results for (1.1) (cf. [1–9]), for instance, the existence, multiplicity, and properties of the traveling wave solutions, and so forth.

In the study of equations of mathematical physics, attractor is a proper mathematical concept about the depiction of the behavior of the solutions of these equations when time is large or tends to infinity, which describes all the possible limits of solutions. In the past two decades, many authors have proved the existence of attractor and discussed its properties for various mathematical physics models (e.g., see [10–12] and the reference therein). About the long-time behavior of suspension bridge-type equations, for the autonomous case, in [13, 14] the authors have discussed long-time behavior of the solutions of the problem on $\mathbb{R}^2$ and obtained the existence of global attractors in the space $H_0^2(\Omega) \times L^2(\Omega)$ and $D(A) \times H_0^2(\Omega)$.

It is well known that, for a model to describe the real world which is affected by many kinds of factors, the corresponding nonautonomous model is more natural and precise than the autonomous one, moreover, it always presents as a nonlinear equation, not just a linear one. Therefore, in this paper, we will discuss the following nonautonomous suspension bridge-type equation: let Ω be an open bounded subset of $\mathbb{R}^2$ with smooth boundary, $\mathbb{R}_\tau = [\tau, +\infty]$, and we add the nonlinear forcing term $g(u,t)$ (which is dependent on deflection u and time t) to (1.1) and neglect gravity, then we can obtain the following initial-boundary value problem:

$$\begin{gathered} u_{tt} + \Delta^2 u + \alpha u_t + ku^+ + g(u,t) = h(x,t), \quad \text{in } \Omega \times \mathbb{R}_\tau, \\ u(x,t) = \Delta u(x,t) = 0, \quad \text{on } \partial\Omega \times \mathbb{R}_\tau, \\ u(x,\tau) = u_1(x), \quad u_t(x,\tau) = u_2(x), \quad x \in \Omega, \end{gathered} \tag{1.3}$$

where $u(x,t)$ is an unknown function, which could represent the deflection of the road bed in the vertical plane; $h(x,t)$ and $g(u,t)$ are time dependant external forces; ku^+ represents the restoring force, k denotes the spring constant; αu_t represents the viscous damping, α is a given positive constant.

To our knowledge, this is the first time to consider the nonautonomous dynamics of (1.3) with the time dependant external forces $h(x,t)$ and $g(u,t)$ in the strong topological space $D(A) \times H^2(\Omega) \cap H_0^1(\Omega)$. At the same time, in mathematics, we only assume that the force term $h(x,t)$ satisfies the so-called condition (C^*) (introduced in [15]), which is weaker than translation compact assumption (see [10] or Section 2 below).

This paper is organized as follows. At first, in Section 2, we give (recall) some preliminaries, including the notation we will use, the assumption on nonlinearity $g(\cdot,t)$, and some general abstract results about nonautonomous dynamical system. Then, in Section 3 we prove our main result about the existence of strong attractor for the nonautonomous dynamical system generated by the solution of (1.3).

2. Notation and Preliminaries

With the usual notation, we introduce the spaces $H = L^2(\Omega)$, $V = H^2(\Omega) \cap H_0^1(\Omega)$, $D(A) = \{u \in H^2(\Omega) \cap H_0^1(\Omega) \mid Au \in L^2(\Omega)\}$, where $A = \Delta^2$. We equip these spaces with inner product and norm $\langle \cdot, \cdot \rangle$, $\|\cdot\|$, $\langle \cdot, \cdot \rangle_1$, $\|\cdot\|_1$ and $\langle \cdot, \cdot \rangle_2$, $\|\cdot\|_2$, respectively,

$$\begin{gathered}
\langle u, v \rangle = \int_\Omega u(x)v(x)dx, \quad \|u\|^2 = \int_\Omega |u(x)|^2 dx, \quad \forall u, v \in H, \\
\langle u, v \rangle_1 = \int_\Omega \Delta u(x)\Delta v(x)dx, \quad \|u\|_1^2 = \int_\Omega |\Delta u(x)|^2 dx, \quad \forall u, v \in V, \\
\langle u, v \rangle_2 = \int_\Omega \Delta^2 u(x)\Delta^2 v(x)dx, \quad \|u\|_2^2 = \int_\Omega \left|\Delta^2 u(x)\right|^2 dx, \quad \forall u, v \in D(A).
\end{gathered} \tag{2.1}$$

Obviously, we have

$$D(A) \subset V \subset H = H^* \subset V^*, \tag{2.2}$$

where H^*, V^* is dual space of H, V, respectively, the injections are continuous, and each space is dense in the following one.

In the following, the assumption on the nonlinearity g is given. Let g be a C^1 function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ and satisfy

$$\liminf_{|u| \to \infty} \frac{G(u,s)}{u^2} \geqslant 0, \tag{2.3}$$

where $G(u,s) = \int_0^u g(w,s)dw$, and there exists $C_0 > 0$, such that

$$\liminf_{|u| \to \infty} \frac{\langle u, g(u,s)\rangle - C_0 G(u,s)}{u^2} \geqslant 0. \tag{2.4}$$

Suppose that γ is an arbitrary positive constant, and

$$\left|g_u(u,s)\right| \leqslant C_1(1 + |u|^\gamma), \qquad \left|g_s(u,s)\right| \leqslant C_1\left(1 + |u|^{\gamma+1}\right), \tag{2.5}$$

$$G_s(u,s) \leqslant \delta^2 G(u,s) + C_2, \quad \forall (u,s) \in \mathbb{R} \times \mathbb{R}, \tag{2.6}$$

where δ is a sufficiently small constant.

As a consequence of (2.3)-(2.4), if we denote $\mathcal{G}(u,s) = \int_\Omega G(u,s)dx$, then there exist two positive constants K_1, K_2 such that

$$\mathcal{G}(\varphi,s) + m\|\varphi\|^2 + K_1 \geqslant 0, \tag{2.7}$$

$$\langle \varphi, g(\varphi,s)\rangle - C_0\mathcal{G}(\varphi,s) + m\|\varphi\|^2 + K_2 \geqslant 0, \quad \forall(\varphi,s) \in \mathbb{R}\times\mathbb{R}, \tag{2.8}$$

where $m, C_0 > 0$, and m is sufficiently small.

By virtue of (2.5), we can get

$$|g(u,s)| \leqslant C_3\left(1+|u|^{\gamma+1}\right), \qquad |G(u,s)| \leqslant C_3\left(1+|u|^{\gamma+2}\right). \tag{2.9}$$

When $A = \Delta^2$, problem (1.3) is equivalent to the following equations in H:

$$\begin{gathered} u_{tt} + \alpha u_t + Au + ku^+ + g(u,t) = h(x,t), \\ u(\tau) = u_1, \qquad u_t(\tau) = u_2. \end{gathered} \tag{2.10}$$

From the Poincaré inequality, there exists a proper constant $\lambda_1 > 0$, such that

$$\lambda_1\|u\|^2 \leqslant \|u\|_1^2, \quad \forall u \in V. \tag{2.11}$$

We introduce the Hilbert space

$$\mathcal{E}_0 = V \times H, \qquad \mathcal{E}_1 = D(A)\times V, \tag{2.12}$$

and endow this space with norm:

$$\begin{aligned} \|z\|_{\mathcal{E}_0} &= \|(u,u_t)\|_{\mathcal{E}_0} = \left(\frac{1}{2}\left(\|u\|_1^2 + \|u_t\|^2\right)\right)^{1/2}, \\ \|z\|_{\mathcal{E}_1} &= \|(u,u_t)\|_{\mathcal{E}_1} = \left(\frac{1}{2}\left(\|u\|_2^2 + \|u_t\|_1^2\right)\right)^{1/2}. \end{aligned} \tag{2.13}$$

To prove the existence of uniform attractors corresponding to (2.10), we also need the following abstract results (e.g., see [10]).

Let E be a Banach space, and let a two-parameter family of mappings $\{U(t,\tau)\} = \{U(t,\tau) \mid t \geqslant \tau, \tau \in \mathbb{R}\}$ on E:

$$U(t,\tau): E \longrightarrow E, \quad t \geqslant \tau,\ \tau \in \mathbb{R}. \tag{2.14}$$

Definition 2.1 (see [10]). Let Σ be a parameter set. $\{U_\sigma(t,\tau) \mid t \geqslant \tau, \tau \in \mathbb{R}\}$, $\sigma \in \Sigma$ is said to be a family of processes in Banach space E, if for each $\sigma \in \Sigma$, $\{U_\sigma(t,\tau)\}$ is a process; that is, the two-parameter family of mappings $\{U_\sigma(t,\tau)\}$ from E to E satisfy

$$\begin{gathered} U_\sigma(t,s) \circ U_\sigma(s,\tau) = U_\sigma(t,\tau), \quad \forall t \geqslant s \geqslant \tau,\ \tau \in \mathbb{R}, \\ U_\sigma(\tau,\tau) = I \text{ is the identity operator}, \quad \tau \in \mathbb{R}, \end{gathered} \tag{2.15}$$

where Σ is called the symbol space and $\sigma \in \Sigma$ is the symbol.

Note that the following translation identity is valid for a general family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, if a problem is the unique solvability and for the translation semigroup $\{T(l) \mid l \geqslant 0\}$ satisfying $T(l)\Sigma = \Sigma$:

$$U_\sigma(t+l,\tau+l) = U_{T(l)\sigma}(t,\tau), \quad \forall \sigma \in \Sigma, t \geqslant \tau,\ \tau \in \mathbb{R},\ l \geqslant 0. \tag{2.16}$$

A set $B_0 \subset E$ is said to be a uniformly (w.r.t $\sigma \in \Sigma$) absorbing set for the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ if for any $\tau \in \mathbb{R}$ and $B \in \mathcal{B}(E)$, there exists $t_0 = t_0(\tau, B) \geqslant \tau$ such that $\cup_{\sigma\in\Sigma} U_\sigma(t,\tau) \subseteq B_0$ for all $t \geqslant t_0$. A set $Y \subset E$ is said to be uniformly (w.r.t. $\sigma \in \Sigma$) attracting for the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ if for any fixed $\tau \in \mathbb{R}$ and every $B \in \mathcal{B}(E)$,

$$\lim_{t\to\infty}\left(\sup_{\sigma\in\Sigma} \operatorname{dist}_E(U_\sigma(t,\tau)B, Y)\right) = 0. \tag{2.17}$$

Definition 2.2 (see [10]). A closed set $A_\Sigma \subset E$ is said to be the uniform (w.r.t. $\sigma \in \Sigma$) attractor of the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ if it is uniformly (w.r.t. $\sigma \in \Sigma$) attracting (attracting property) and contained in any closed uniformly (w.r.t. $\sigma \in \Sigma$) attracting set A' of the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$: $A_\Sigma \subseteq A'$ (minimality property).

Now we recalled the results in [16].

Definition 2.3 (see [16, 17]). A family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ is said to be satisfying uniform (w.r.t. $\sigma \in \Sigma$). Condition (C) if for any fixed $\tau \in \mathbb{R}$, $B \in \mathcal{B}(E)$ and $\epsilon > 0$, there exist a $t_0 = t_0(\tau, B, \epsilon) \geqslant \tau$ and a finite dimensional subspace E_m of E such that

(i) $P_m(\cup_{\sigma\in\Sigma}\cup_{t\geqslant t_0} U_\sigma(t,\tau)B)$ is bounded;

(ii) $\|(I-P_m)(\cup_{\sigma\in\Sigma}\cup_{t\geqslant t_0} U_\sigma(t,\tau)x)\|_E \leqslant \epsilon,\ \forall x \in B$,

where $\dim E_m = m$ and $P_m : E \to E_m$ is abounded projector.

Theorem 2.4 (see [16]). *Let Σ be a complete metric space, and let $\{T(t)\}$ be a continuous invariant $T(t)\Sigma = \Sigma$ semigroup on Σ satisfying the translation identity. A family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ possesses compact uniform (w.r.t. $\sigma \in \Sigma$) attractor A_Σ in E satisfying*

$$A_\Sigma = \omega_{0,\Sigma}(B_0) = \omega_{\tau,\Sigma}(B_0), \quad \forall t \in \mathbb{R}, \tag{2.18}$$

if it

(i) *has a bounded uniformly (w.r.t.* $\sigma \in \Sigma$*) absorbing set* B_0*;*

(ii) *satisfies uniform (w.r.t.* $\sigma \in \Sigma$*) condition (C),*

where $\omega_{\tau,\Sigma}(B_0) = \cap_{t\geqslant\tau}[\cup_{\sigma\in\Sigma}\cup_{s\geqslant t}U_\sigma(s,t)B_0]$. *Moreover, if E is a uniformly convex Banach space, then the converse is true.*

Let X be a Banach space. Consider the space $L^2_{\text{loc}}(\mathbb{R};X)$ of functions $\phi(s)$, $s \in \mathbb{R}$ with values in X that are 2-power integrable in the Bochner sense. $L^2_c(\mathbb{R};X)$ is a set of all translation compact functions in $L^2_{\text{loc}}(\mathbb{R};X)$, $L^2_b(\mathbb{R};X)$ is a set of all translation bound functions in $L^2_{\text{loc}}(\mathbb{R};X)$.

In [15], the authors have introduced a new class of functions which are translation bounded but not translation compact. In the third section, let the forcing term $h(x,t)$ satisfy condition (C^*), we can prove the existence of compact uniform (w.r.t. $\sigma \in \mathscr{H}(\sigma_0)$, $\sigma_0(s) = (g_0(u,s),h(x,s))$) attractor for nonautonomous suspension bridge equation in $\mathcal{E}_1$.

Definition 2.5 (see [15]). Let X be a Banach space. A function $f \in L^2_b(\mathbb{R};X)$ is said to satisfy condition (C^*) if for any $\epsilon > 0$, there exists a finite dimensional subspace X_1 of X such that

$$\sup_{t\in\mathbb{R}}\int_t^{t+1}\|(I-P_m)f(s)\|_X^2\,ds < \epsilon, \tag{2.19}$$

where $P_m : X \to X_1$ is the canonical projector.

Denote by $L^2_{c^*}(\mathbb{R};X)$ the set of all functions satisfying condition (C^*). From [15], we can see that $L^2_c(\mathbb{R};X) \subset L^2_{c^*}(\mathbb{R};X) \subset L^2_b(\mathbb{R};X)$.

Remark 2.6. In fact, the function satisfying condition (C^*) implies the dissipative property in some sense, and the condition (C^*) is very natural in view of the compact condition, uniform condition (C).

Lemma 2.7 (see [15]). *If* $f \in L^2_{c^*}(\mathbb{R};X)$*, then for any* $\epsilon > 0$ *and* $\tau \in \mathbb{R}$ *we have*

$$\sup_{t\geqslant\tau}\int_\tau^t e^{-\delta(t-s)}\|(I-P_m)f(s)\|_X^2\,ds \leqslant \epsilon, \tag{2.20}$$

where $P_m : X \to X_1$ *is the canonical projector and* δ *is a positive constant.*

In order to define the family of processes of (2.10), we also need the following results.

Proposition 2.8 (see [10]). *If X is reflexive separable, then*

(i) *for all* $h_1 \in \mathscr{H}(h_0)$, $\|h_1\|_{L^2_b(\mathbb{R};X)} \leqslant \|h_0\|_{L^2_b(\mathbb{R};X)}$*;*

(ii) *the translation group* $\{T(t)\}$ *is weakly continuous on* $\mathscr{H}(h_0)$*;*

(iii) $T(t)\mathscr{H}(h_0) = \mathscr{H}(h_0)$ *for all* $t \in \mathbb{R}^+$.

Proposition 2.9 (see [10]). *Let* $g_0(s) \in L^2_c(\mathbb{R};X)$*, then*

(i) *for all* $g_1 \in \mathscr{H}(g_0)$, $g_1 \in L^2_c(\mathbb{R};X)$*, and the set* $\mathscr{H}(g_0)$ *is bound in* $L^2_b(\mathbb{R};X)$*;*

(ii) *the translation group* $\{T(t)\}$ *is continuous on* $\mathscr{H}(g_0)$ *with the topology of* $L^2_{\text{loc}}(\mathbb{R},X)$*;*

(iii) $T(t)\mathscr{H}(g_0) = \mathscr{H}(g_0)$ *for all* $t \in \mathbb{R}^+$.

3. Uniform Attractors in $\mathcal{E}_1$

To describe the asymptotic behavior of the solutions of our system, we set $h_0 \in L^2_{c^*}(\mathbb{R}_\tau; V) \subset L^2_b(\mathbb{R}_\tau; V)$ and $\mathscr{H}(h_0) = [h_0(x, s+h) \mid h \in \mathbb{R}]_{L^{2,w}_{\text{loc}}(\mathbb{R}_\tau;V)}$, where $[\,]$ denotes the closure of a set in topological space $L^{2,w}_{\text{loc}}(\mathbb{R}_\tau; V)$. If $h \in \mathscr{H}(h_0)$, then $h \in L^2_b(\mathbb{R}_\tau; V)$, that is to be

$$\sup_{t \geqslant \tau} \int_t^{t+1} \|h(x,s)\|_1 \, ds < \infty, \tag{3.1}$$

where $\|\cdot\|_1$ denotes the norm in V.

3.1. *Existence and Uniqueness of Strong Solutions*

At first, we give the concept of strong solutions for the initial-boundary value problem (2.10).

Definition 3.1. Set $I = [\tau, T]$, for $T > \tau \geqslant 0$. We suppose that $k > 0$, $h \in L^2_b(\mathbb{R}_\tau; V)$, $g \in C^1(\mathbb{R} \times \mathbb{R}; \mathbb{R})$ satisfying (2.3)–(2.6) and $g(0,0) = 0$. The function $z = (u, u_t) \in L^\infty(I; \mathcal{E}_1)$ is said to be a strong solution to problem (2.10) in the time interval I, with initial data $z(\tau) = z_\tau = (u_1, u_2) \in \mathcal{E}_1$, provided

$$\langle u_{tt}, \overline{v} \rangle + \alpha \langle u_t, \overline{v} \rangle + \int_\Omega \Delta u \Delta \overline{v} dx + \int_\Omega g(u,t) \overline{v} dx + k \langle u^+, \overline{v} \rangle = \int_\Omega h(x,t) \overline{v} dx, \tag{3.2}$$

for all $\overline{v} \in V$ and a.e. $t \in I$.

Then, by using the methods in [18] (Galerkin approximation method), we can get the following result about the existence and uniqueness of strong solutions.

Theorem 3.2 (existence and uniqueness of strong solutions). *Define $I = [\tau, T]$, for all $T > \tau$. Let $k > 0$, $h \in L^2_b(\mathbb{R}_\tau; V)$, $g \in C^1(\mathbb{R} \times \mathbb{R}; \mathbb{R})$ satisfying* (2.3)–(2.6)*. Then for any given $z_\tau \in \mathcal{E}_1$, there is a unique solution $z = (u, u_t)$ for problem* (2.10) *in $\mathcal{E}_1$. Furthermore, for $i = 1, 2$, let $\{z^i_\tau, h_i\}$ ($z^i_\tau \in \mathcal{E}_1$ and $h_i \in L^2_b(\mathbb{R}_\tau; V)$) be two initial conditions, and denote by z_i corresponding solutions to problem* (2.10)*. Then the estimates hold as follows: for all $\tau \leqslant t \leqslant T + \tau$,*

$$\|z_1(t) - z_2(t)\|^2_{\mathcal{E}_1} \leqslant Q\left(\left\|z^i_\tau\right\|_{\mathcal{E}_1}, T\right)\left(\left\|z^1_\tau - z^2_\tau\right\|^2_{\mathcal{E}_1} + \|h_1 - h_2\|^2_{L^2_b(\mathbb{R}_\tau;V)}\right). \tag{3.3}$$

Thus, (2.10) will be written as an evolutionary system introduced $z(t) = (u(t), u_t(t))$ and $z_\tau = z(\tau) = (u_1, u_2)$ for brevity, as $\|z\|^2_{\mathcal{E}_1} = (1/2)(\|u\|^2_2 + \|u_t\|^2_1)$, the system (2.10) can be written in the operator form

$$\partial_t z = A_{\sigma(t)}(z), \qquad z|_{t=\tau} = z_\tau, \tag{3.4}$$

where $\sigma(s) = (g(u,s), h(x,s))$ is the symbol of (3.4). If $z_\tau \in \mathcal{E}_1$, then problem (3.4) has a unique solution $z(t) \in L^\infty(\mathbb{R}_\tau, \mathcal{E}_1)$. This implies that the process $\{U_\sigma(t,\tau)\}$ given by the formula $U_\sigma(t,\tau) z_\tau = z(t)$ is defined in $\mathcal{E}_1$.

Now we define the symbol space. A fixed symbol $\sigma_0(s) = (g_0(u,s), h_0(x,s))$ can be given, where $h_0(x,s)$ is in $L^2_{c^*}(\mathbb{R}_\tau; V)$, the function $g_0(u,s) \in L^2_c(\mathbb{R}_\tau; \mathcal{M})$ satisfying (2.3)–(2.6), and $\mathcal{M}$ is a Banach space,

$$\mathcal{M} = \left\{ g \in C^1(\mathbb{R}\times\mathbb{R}, \mathbb{R}) \frac{\|g(u)\|_1 + \|g_s(u)\|_1}{\|u\|_1^{\gamma+1}+1} + \frac{\|g_u(u)\|_1}{\|u\|_1^{\gamma}+1} < \infty \right\}, \tag{3.5}$$

endowed with the following norm:

$$\|g\|_{\mathcal{M}} = \sup_{u\in\mathbb{R}} \left\{ \frac{\|g(u)\|_1 + \|g_s(u)\|_1}{\|u\|_1^{\gamma+1}+1} + \frac{\|g_u(u)\|_1}{\|u\|_1^{\gamma}+1} \right\}. \tag{3.6}$$

Obviously, the function $\sigma_0(s) = (g_0(u,s), h_0(x,s))$ is in $L^2_c(\mathbb{R}_\tau; \mathcal{M}) \times L^2_{c^*}(\mathbb{R}_\tau; V)$. we define $\mathcal{H}(\sigma_0) = \mathcal{H}(g_0) \times \mathcal{H}(h_0) = [g_0(u, s+l) \mid l \in \mathbb{R}]_{L^{2,w}_{\text{loc}}(\mathbb{R}_\tau;\mathcal{M})} \times [h_0(x, s+l) \mid l \in \mathbb{R}]_{L^{2,w}_{\text{loc}}(\mathbb{R}_\tau;V)}$, where $[\,]$ denotes the closure of a set in topological space $L^{2,w}_{\text{loc}}(\mathbb{R}_\tau;\mathcal{M})$ (or $L^{2,w}_{\text{loc}}(\mathbb{R}_\tau;V)$). So, if $(g,h) \in \mathcal{H}(\sigma_0)$, then $g(u,t)$ and $h(x,t)$ all satisfy condition (C^*).

Applying Propositions 2.8 and 2.9 and Theorem 3.2, we can easily know that the family of processes $\{U_\sigma(t,\tau)\} : \mathcal{E}_1 \to \mathcal{E}_1$, $\sigma \in \mathcal{H}(\sigma_0)$, $t \geqslant \tau$ are defined. Furthermore, the translation semigroup $\{T(l) \mid l \in \mathbb{R}^+\}$ satisfies that for all $l \in \mathbb{R}^+$, $T(l)\mathcal{H}(\sigma_0) = \mathcal{H}(\sigma_0)$, and the following translation identity:

$$U_\sigma(t+l, \tau+l) = U_{T(l)\sigma}(t,\tau), \quad \forall \sigma \in \mathcal{H}(\sigma_0),\ \text{for } t \geqslant \tau \geqslant 0,\ l \geqslant 0 \tag{3.7}$$

holds.

Then for any $\sigma \in \mathcal{H}(\sigma_0)$, the problem (3.4) with σ instead of σ_0 possesses a corresponding to process $\{U_\sigma(t,\tau)\}$ acting on $\mathcal{E}_1$.

Consequently, for each $\sigma \in \mathcal{H}(\sigma_0)$, $\sigma_0(s) = (g_0(u,s), h_0(x,s))$ (here $h_0(x,s) \in L^2_{c^*}(\mathbb{R}_\tau; V)$, $g_0(u,s) \in L^2_c(\mathbb{R}_\tau;\mathcal{M})$ satisfying (2.3)–(2.6)), we can define a process

$$\begin{gathered} U_\sigma(t,\tau) : \mathcal{E}_1 \longrightarrow \mathcal{E}_1, \\ z_\tau = (u_1, u_2) \longrightarrow (u(t), u_t(t)) = U_\sigma(t,\tau) z_\tau, \end{gathered} \tag{3.8}$$

and $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$ is a family of processes on $\mathcal{E}_1$.

3.2. A Priori Estimates

3.2.1. A Priori Estimates in $\mathcal{E}_0$

Theorem 3.3. *Assume that $z(t)$ is a solution of* (2.10) *with initial data $z_0 \in B$. If the nonlinearity $g(u,t)$ satisfies* (2.3)–(2.6)*, $h_0 \in L^2_{c^*}(\mathbb{R}_\tau; H)$, $h \in \mathcal{H}(h_0)$, $k > 0$, then there is a positive constant μ_0 such that for any bounded (in $\mathcal{E}_0$) subset B, there exists $t_0 = t_0(\|B\|_{\mathcal{E}_0})$ such that*

$$\|z(t)\|^2_{\mathcal{E}_0} = \frac{1}{2}\left(\|u\|_1^2 + \|u_t\|^2\right) \leqslant \mu_0^2, \quad t \geqslant t_0 = t_0(\|B\|_{\mathcal{E}_0}). \tag{3.9}$$

Proof. Now we will prove that $z = (u, u_t)$ are bounded in $\mathcal{E}_0 = V \times H$.

We assume that ϱ is positive and satisfies

$$0 < \varrho(\alpha - \varrho) < \lambda_1. \tag{3.10}$$

Multiplying (2.10) by $v(t) = u_t(t) + \varrho u(t)$ and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|v\|^2 + \|u\|_1^2\right) + \varrho\|u\|_1^2 + (\alpha - \varrho)\|v\|^2 - \varrho(\alpha - \varrho)\langle u, v\rangle + k\langle u^+, v\rangle + \langle g(u,t), v\rangle = \langle h(t), v\rangle. \tag{3.11}$$

We can easily see that

$$\varrho(\alpha - \varrho)\langle u, v\rangle \leqslant (\alpha - \varrho)\frac{\|v\|^2}{4} + (\alpha - \varrho)\varrho^2\|u\|^2, \tag{3.12}$$

$$\langle h(t), v\rangle \leqslant (\alpha - \varrho)\frac{\|v\|^2}{4} + \frac{\|h(t)\|^2}{\alpha - \varrho}. \tag{3.13}$$

Then, substituting (3.12)-(3.13) into (3.11), we can obtain that

$$\frac{d}{dt}\left(\|v\|^2 + \|u\|_1^2\right) + 2\varrho\|u\|_1^2 + (\alpha - \varrho)\|v\|^2 - 2\varrho^2(\alpha - \varrho)\|u\|^2 + 2k\langle u^+, v\rangle + 2\langle g, v\rangle \leqslant 2\frac{\|h(t)\|^2}{\alpha - \varrho}. \tag{3.14}$$

In view of (2.6) and (2.8), we can know

$$\begin{aligned}
\langle g, v\rangle &= \langle g, u_t + \varrho u\rangle \\
&= \frac{d}{dt}\int_\Omega G(u(x,t),t)dx + \varrho\langle g(u,t), u\rangle - \int_\Omega G_s(u(x,t),t)dx \\
&= \frac{d}{dt}\int_\Omega G(u(x,t),t)dx + \varrho\int_\Omega g(u(x,t),t)u(x,t)dx \\
&\quad - \varrho C_0\int_\Omega G(u(x,t),t)dx + \varrho C_0\int_\Omega G(u(x,t),t)dx - \int_\Omega G_s(u(x,t),t)dx \\
&\geqslant \frac{d}{dt}\mathcal{G}(u(x,t),t) + \varrho C_0\mathcal{G}(u(x,t),t) - \varrho\left(m\|u\|^2 + K_2\right) - \delta^2\mathcal{G}(u(x,t),t) - C_2|\Omega|, \\
k\langle u^+, v\rangle &= \frac{1}{2}\frac{d}{dt}k\|u^+\|^2 + \varrho k\|u^+\|^2.
\end{aligned} \tag{3.15}$$

Consequently,

$$\begin{aligned}
&\frac{d}{dt}\Big(\|v\|^2+\|u\|_1^2+k\|u^+\|^2+2\mathcal{G}(u(x,t),t)\Big)\\
&\quad+(\alpha-\varrho)\|v\|^2+2\frac{\varrho}{\lambda_1}(\lambda_1-\varrho(\alpha-\varrho)-m)\|u\|_1^2+2\varrho k\|u^+\|^2\\
&\quad+\Big(\varrho C_0-\delta^2\Big)2\mathcal{G}(u(x,t),t)\\
&\leqslant 2\frac{\|h(t)\|^2}{\alpha-\varrho}+2(\varrho K_2+C_2|\Omega|).
\end{aligned} \tag{3.16}$$

We introduce the functional as follows:

$$y(t)=\|v\|^2+\|u\|_1^2+k\|u^+\|^2+2\mathcal{G}(u(x,t),t)+2K_1,\quad \text{for } t\geqslant\tau. \tag{3.17}$$

Setting $\beta=\min\{\alpha-\varrho,2\varrho\lambda_1^{-1}(\lambda_1-\varrho(\alpha-\varrho)-m),2\varrho,\varrho C_0-\delta^2\}$, we choose proper positive constants m and δ, such that

$$m<\lambda_1-\varrho(\alpha-\varrho),\qquad \delta^2<\varrho C_0 \tag{3.18}$$

hold, then $\beta>0$.

We define $m_h(t)=\|h(t)\|^2$, then

$$\frac{d}{dt}y(t)+\beta y(t)\leqslant C_4+C_5m_h(t), \tag{3.19}$$

where $C_4=2(\varrho K_2+C_2|\Omega|)+2\beta K_1$, $C_5=2(\alpha-\varrho)^{-1}$.

Analogous to the proof of Lemma 2.1.3 in [10], we can estimate the integral and obtain

$$\begin{aligned}
y(t)&\leqslant y(\tau)e^{-\beta(t-\tau)}+C_4\beta^{-1}\Big(1-e^{-\beta t}\Big)+C_5\int_0^t m_h(s)e^{-\beta(t-s)}ds\\
&\leqslant y(\tau)e^{-\beta(t-\tau)}+C_4\beta^{-1}\Big(1-e^{-\beta t}\Big)+C_5\int_{t-1}^t m_h(s)e^{-\beta(t-s)}ds\\
&\quad+C_5\int_{t-2}^{t-1} m_h(s)e^{-\beta(t-s)}ds+\cdots
\end{aligned}$$

$$
\begin{aligned}
&\leqslant y(\tau)e^{-\beta(t-\tau)} + C_4\beta^{-1}\left(1-e^{-\beta t}\right) + C_5\int_{t-1}^{t} m_h(s)ds \\
&\quad + C_5e^{-\beta}\int_{t-2}^{t-1} m_h(s)ds + C_5e^{-2\beta}\int_{t-3}^{t-2} m_h(s)ds + \cdots \\
&\leqslant y(\tau)e^{-\beta(t-\tau)} + C_4\beta^{-1}\left(1-e^{-\beta t}\right) + C_5 m_h\left(1+e^{-\beta}+e^{-2\beta}+\cdots\right) \\
&\leqslant y(\tau)e^{-\beta(t-\tau)} + C_4\beta^{-1}\left(1-e^{-\beta t}\right) + C_5 m_h\left(1+\beta^{-1}\right) \\
&\leqslant y(\tau)e^{-\beta(t-\tau)} + C_4\beta^{-1} + C_5 m_h\left(1+\beta^{-1}\right), \quad \text{for } t \geqslant \tau,
\end{aligned}
\tag{3.20}
$$

where $m_h = \sup_{t\geqslant\tau}\int_t^{t+1} m_h(s)ds$.

By virtue of (2.7), we can get

$$
2\mathcal{G}(u,t) \geqslant -2m\|u\|^2 - 2K_1 \geqslant -2m\lambda_1^{-1}\|u\|_1^2 - 2K_1. \tag{3.21}
$$

Choosing $m \leqslant \lambda_1/4$, we obtain from (3.17)

$$
\begin{aligned}
y(t) &= \|u\|_1^2 + \left\|u_t + \varrho u\right\|^2 + k\|u^+\|^2 + 2\mathcal{G}(u,t) + 2K_1 \\
&\geqslant \frac{1}{2}\|u\|_1^2 + \left\|u_t + \varrho u\right\|^2 + k\|u^+\|^2 \\
&\geqslant \|z(t)\|_{\mathcal{E}_0}^2.
\end{aligned}
\tag{3.22}
$$

In consideration of (2.9) and $0 < \gamma < \infty$, we can see

$$
2\mathcal{G}(u_\tau(x),\tau) \leqslant 2C_3\int_\Omega\left(|u_\tau(x)|^{\gamma+2}+1\right)dx \leqslant C_6\left(\|u_\tau\|_1^{\gamma+2}+1\right), \tag{3.23}
$$

$$
\begin{aligned}
y(\tau) &= \|u(\tau)\|_1^2 + \left\|u_t(\tau) + \varrho u(\tau)\right\|^2 + k\left\|(u(\tau))^+\right\|^2 + 2\mathcal{G}(u(\tau),\tau) + 2K_1 \\
&\leqslant C_7\left(\|z(\tau)\|_{\mathcal{E}_0}^{\gamma+2} + 1\right).
\end{aligned}
\tag{3.24}
$$

Combining (3.20), (3.22), and (3.24), we can deduce that

$$
\begin{aligned}
\|z(t)\|_{\mathcal{E}_0}^2 &\leqslant y(\tau)e^{-\beta(t-\tau)} + C_4\beta^{-1} + C_5 m_h\left(1+\beta^{-1}\right) \\
&\leqslant C_7\left(\|z(\tau)\|_{\mathcal{E}_0}^{\gamma+2}+1\right)e^{-\beta(t-\tau)} + C_4\beta^{-1} + C_5 m_h\left(1+\beta^{-1}\right) \\
&\leqslant C_7\|z(\tau)\|_{\mathcal{E}_0}^{\gamma+2}e^{-\beta(t-\tau)} + C_8, \quad t \geqslant \tau.
\end{aligned}
\tag{3.25}
$$

Assume that $\|z(\tau)\|_{\mathcal{E}_0}^2 \leqslant R$, as $t \geqslant t_0 = t_0(\|B\|_{\mathcal{E}_0})$, we have

$$\|z(t)\|_{\mathcal{E}_0} \leqslant \mu_0. \tag{3.26}$$

We complete the proof. □

3.2.2. *A Priori Estimates in* $\mathcal{E}_1$

Lemma 3.4. *Assuming that $z(t)$ is a strong solution of (2.10) with initial data $z_0 \in B$. If the nonlinearity $g(u,t)$ satisfies (2.3)–(2.6), $h_0 \in L^2_{c^*}(\mathbb{R}_\tau; V)$, $h \in \mathscr{H}(h_0)$, $k > 0$, then there is a positive constant μ_2 such that for any bounded (in $\mathcal{E}_1$) subset B, there exists $t_1 = t_1(\|B\|_{\mathcal{E}_1})$ such that*

$$\|z(t)\|_{\mathcal{E}_1}^2 = \frac{1}{2}\left(\|u\|_2^2 + \|u_t\|_1^2\right) \leqslant \mu_2^2, \quad t \geqslant t_1 = t_1(\|B\|_{\mathcal{E}_1}). \tag{3.27}$$

Proof. Now we will prove that $z = (u, u_t)$ are bounded in $\mathcal{E}_1 = D(A) \times V$.

We assume that ϱ is positive and satisfies

$$0 < \varrho(\alpha - \varrho) < \lambda_1. \tag{3.28}$$

Multiplying (2.10) by $Av(t) = Au_t(t) + \varrho Au(t)$ and integrating over Ω, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|v\|_1^2 + \|u\|_2^2\right) + \varrho\|u\|_2^2 + (\alpha - \varrho)\|v\|_1^2 - \varrho(\alpha - \varrho)\langle u, v\rangle_1 + k\langle u^+, Av\rangle + \langle g(u,t), Av\rangle \\ = \langle h(t), Av\rangle,\end{aligned} \tag{3.29}$$

where $A = \Delta^2$.

We can deduce that

$$\begin{aligned}\varrho(\alpha - \varrho)\langle u, v\rangle_1 &\leqslant (\alpha - \varrho)\frac{\|v\|_1^2}{4} + (\alpha - \varrho)\varrho^2\|u\|_1^2, \\ \langle h(t), Av\rangle &\leqslant (\alpha - \varrho)\frac{\|v\|_1^2}{4} + \frac{\|h(t)\|_1^2}{\alpha - \varrho}.\end{aligned} \tag{3.30}$$

Then, substituting (3.30) into (3.29), we have

$$\begin{aligned}\frac{d}{dt}\left(\|v\|_1^2 + \|u\|_2^2\right) + 2\varrho\|u\|_2^2 + (\alpha - \varrho)\|v\|_1^2 - 2\varrho^2(\alpha - \varrho)\|u\|_1^2 + 2k\langle u^+, Av\rangle + 2\langle g, Av\rangle \\ \leqslant 2\frac{\|h(t)\|_1^2}{\alpha - \varrho}.\end{aligned} \tag{3.31}$$

In view of (2.5) and Theorem 3.3, we can see that

$$
\begin{aligned}
\langle g, Av\rangle &= \langle g, Au_t + \varrho Au\rangle \\
&= \frac{d}{dt}\langle g(u,t), Au\rangle - \langle g_u(u,t)u_t, Au\rangle - \langle g_s(u,t), Au\rangle + \varrho\langle g(u,t), Au\rangle \\
&\geqslant \frac{d}{dt}\langle g(u,t), Au\rangle + \varrho\langle g(u,t), Au\rangle - \int_\Omega \left|g_u(u,t)\right| \cdot |u_t| \cdot |Au| dx \\
&\quad - \int_\Omega \left|g_s(u,t)\right| \cdot |Au| dx \\
&\geqslant \frac{d}{dt}\langle g(u,t), Au\rangle + \varrho\langle g(u,t), Au\rangle - \int_\Omega C_1(1 + u|^\gamma) \cdot |u_t| \cdot |Au| dx \\
&\quad - \int_\Omega C_1\left(1 + |u|^{y+1}\right) \cdot |Au| dx \\
&\geqslant \frac{d}{dt}\langle g(u,t), Au\rangle + \varrho\langle g(u,t), Au\rangle - C\|u\|_2 \\
&\geqslant \frac{d}{dt}\langle g(u,t), Au\rangle + \varrho\langle g(u,t), Au\rangle - \frac{\varrho}{8}\|u\|_2^2 - C.
\end{aligned}
\tag{3.32}
$$

Exploiting $\|(u^+)_t\| \leqslant \|u_t\|$ and Theorem 3.3, we can obtain

$$
\begin{aligned}
k\langle u^+, Av\rangle &= \langle ku^+, Au_t + \varrho Au\rangle \\
&= \frac{d}{dt}k\langle u^+, Au\rangle - k\langle (u^+)_t, Au\rangle + \varrho k\langle u^+, Au\rangle \\
&\geqslant \frac{d}{dt}k\langle u^+, Au\rangle - k\|(u^+)_t\| \cdot \|Au\| + \varrho k\langle u^+, Au\rangle \\
&\geqslant \frac{d}{dt}k\langle u^+, Au\rangle - k\|u_t\| \cdot \|Au\| + \varrho k\langle u^+, Au\rangle \\
&\geqslant \frac{d}{dt}k\langle u^+, Au\rangle - k\mu_0\|Au\| + \varrho k\langle u^+, Au\rangle \\
&\geqslant \frac{d}{dt}k\langle u^+, Au\rangle - \frac{\varrho}{8}\|u\|_2^2 - C + \varrho k\langle u^+, Au\rangle.
\end{aligned}
\tag{3.33}
$$

Consequently,

$$
\begin{aligned}
&\frac{d}{dt}\left(\|v\|_1^2 + \|u\|_2^2 + 2k\langle u^+, Au\rangle + 2\langle g(u,t), Au\rangle\right) + (\alpha - \varrho)\|v\|_1^2 + 2\frac{\varrho}{\lambda_1}\left(\frac{3}{4}\lambda_1 - \varrho(\alpha - \varrho)\right)\|u\|_2^2 \\
&\quad + 2\varrho k\langle u^+, Au\rangle + 2\varrho\langle g(u,t), Au\rangle \leqslant 2\frac{\|h(t)\|_1^2}{\alpha - \varrho} + C.
\end{aligned}
\tag{3.34}
$$

Choose ϱ small enough such that

$$\frac{3}{2} - \frac{2\varrho(\alpha - \varrho)}{\lambda_1} \geqslant 1. \tag{3.35}$$

This leads to

$$\begin{aligned}
&\frac{d}{dt}\left(\|v\|_1^2 + \left\|Au + ku^+ + g(u,t)\right\|^2\right) \\
&\quad + (\alpha - \varrho)\|v\|_1^2 + \varrho\left\|Au + ku^+ + g(u,t)\right\|^2 \\
&\quad \leqslant 2\frac{\|h(t)\|_1^2}{\alpha - \varrho} + C + 2\int_\Omega g(u,t)\left(g_u(u,t)u_t + g_s(u,t)\right)dx + 2k\int_\Omega u^+ \cdot (u^+)_t dx \\
&\quad + 2k\left\langle g_u(u,t)u_t + g_s(u,t), u^+\right\rangle + 2k\left\langle g(u,t), u^+(u^+)_t\right\rangle \\
&\quad + \varrho\left\|g(u)\right\|^2 + \varrho k^2\|u^+\|^2 + 2\varrho k\left\langle g(u,t), u^+\right\rangle.
\end{aligned} \tag{3.36}$$

By (2.5), (2.9), the Hölder inequality, and Theorem 3.3, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\|v\|_1^2 + \left\|Au + ku^+ + g(u,t)\right\|^2\right) + (\alpha - \varrho)\|v\|_1^2 + \varrho\left\|Au + ku^+ + g(u,t)\right\|^2 \\
&\quad \leqslant 2\frac{\|h(t)\|_1^2}{\alpha - \varrho} + C.
\end{aligned} \tag{3.37}$$

We introduce the functional as follows:

$$\mho(t) = \|v\|_1^2 + \left\|Au + ku^+ + g(u,t)\right\|^2, \quad \text{for } t \geqslant \tau. \tag{3.38}$$

Setting $\beta = \min\{\alpha - \varrho, \varrho\}$, we define $m_h^*(t) = \|h(t)\|_1^2$, then

$$\frac{d}{dt}\mho(t) + \beta\mho(t) \leqslant C + C_5 m_h^*(t), \tag{3.39}$$

where $C_5 = 2(\alpha - \varrho)^{-1}$.

Analogous to the proof of Theorem 3.3, we can estimate the integral and obtain

$$\begin{aligned}
\mho(t) &\leqslant \mho(\tau)e^{-\beta(t-\tau)} + C\beta^{-1}\left(1 - e^{-\beta t}\right) + C_5\int_0^t m_h^*(s)e^{-\beta(t-s)}ds \\
&\leqslant \mho(\tau)e^{-\beta(t-\tau)} + C\beta^{-1} + C_5 m_h^*\left(1 + \beta^{-1}\right), \quad \text{for } t \geqslant \tau,
\end{aligned} \tag{3.40}$$

where $m_h^* = \sup_{t\geqslant\tau}\int_t^{t+1} m_h^*(s)ds$.

Assuming that $\|\mathcal{V}(\tau)\|_{\mathcal{E}_1} \leqslant R$, as $t \geqslant t_1 = t_1(\|B\|_{\mathcal{E}_1})$, we have

$$\|\mathcal{V}(t)\|_{\mathcal{E}_1} \leqslant \mu_1^2. \tag{3.41}$$

Applying (2.9), the Hölder inequality, the Cauchy inequality, and Theorem 3.3, we can deduce from (3.41) that

$$\|Au\|^2 + \|u_t\|_1^2 \leqslant \mu_2^2, \tag{3.42}$$

where μ_2 depends on ϱ, α, k, $\|h\|_1^2$, μ_0, and μ_1.

We complete the proof. □

And then, combining Theorem 3.2 with Lemma 3.4, we can get the result as follows.

Theorem 3.5 (bounded uniformly absorbing set in $\mathcal{E}_1$). *Presuming that $g_0 \in L_c^2(\mathbb{R}_\tau; M)$ and $h_0 \in L_{c^*}^2(\mathbb{R}_\tau; V)$. Let $g \in \mathcal{H}(g_0)$ satisfy* (2.3)–(2.6), *$h \in \mathcal{H}(h_0)$, and $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0) = \mathcal{H}(g_0) \times \mathcal{H}(h_0)$ be the family of processes corresponding to* (2.10) *in $\mathcal{E}_1$, then $\{U_\sigma(t,\tau)\}$ has a uniformly (w.r.t. $\sigma \in \mathcal{H}(\sigma_0)$) absorbing set $B_1 = B_{\mathcal{E}_1}(0,\mu_2)$ in $\mathcal{E}_1$. That is, for any bounded subset $B \subset \mathcal{E}_1$, there exists $t_1 = t_1(\|B\|_{\mathcal{E}_1})$ such that*

$$\bigcup_{\sigma\in\mathcal{H}(\sigma_0)} U_\sigma(t,\tau)B \subset B_1, \quad \forall t \geqslant t_1. \tag{3.43}$$

3.3. The Existence of Uniform Attractor

We will show the existence of uniform attractor to problem (2.10) in $\mathcal{E}_1$.

Theorem 3.6 (uniform attractor). *Let $\{U_\sigma(t,\tau)\}$ be the family of processes corresponding to problem* (2.10). *If $g_0 \in L_c^2(\mathbb{R}_\tau; \mathcal{M})$ satisfyies* (2.3)–(2.6), *$h_0 \in L_{c^*}^2(\mathbb{R}_\tau; V)$, and $\sigma_0 = (g_0, h_0)$, then $\{U_\sigma(t,\tau)\}$ possesses a compact uniform (w.r.t. $\sigma \in \mathcal{H}(\sigma_0)$) attractor $\mathcal{A}_{\mathcal{H}(\sigma_0)}$ in $\mathcal{E}_1$, which attracts any bounded set in $\mathcal{E}_1$ with $\|\cdot\|_{\mathcal{E}_1}$, satisfying*

$$\mathcal{A}_{\mathcal{H}(\sigma_0)} = \omega_{0,\mathcal{H}(\sigma_0)}(B_1) = \omega_{\tau,\mathcal{H}(\sigma_0)}(B_1), \tag{3.44}$$

where B_1 is the uniformly (w.r.t. $\sigma \in \mathcal{H}(\sigma_0)$) absorbing set in $\mathcal{E}_1$.

Proof. From Theorems 2.4 and 3.5, we merely need to prove that the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \mathcal{H}(\sigma_0)$ satisfies uniform (w.r.t. $\sigma \in \mathcal{H}(\sigma_0)$) condition (C) in $\mathcal{E}_1$. We assume that $\tilde{\lambda}_i$, $i = 1, 2, \ldots$ are eigenvalue of operator A in $D(A)$, satisfying

$$0 < \tilde{\lambda}_1 < \tilde{\lambda}_2 \leqslant \cdots \leqslant \tilde{\lambda}_j \leqslant \cdots, \quad \tilde{\lambda}_j \longrightarrow \infty, \quad \text{as } j \longrightarrow \infty, \tag{3.45}$$

$\tilde{\omega}_i$ denotes eigenvector corresponding to eigenvalue $\tilde{\lambda}_i$, $i = 1, 2, 3, \ldots$, which forms an orthogonal basis in $D(A)$; at the same time they are also a group of canonical basis in V or H, and satisfy

$$A\tilde{\omega}_i = \tilde{\lambda}_i\tilde{\omega}_i, \quad \forall i \in \mathbb{N}. \tag{3.46}$$

Let $V_m = \operatorname{span}\{\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_m\}$, $P_m : V \to V_m$ is an orthogonal projector. For any $(u, u_t) \in \mathcal{E}_1$, we write

$$(u, u_t) = (u_1, u_{1t}) + (u_2, u_{2t}), \tag{3.47}$$

where $(u_1, u_{1t}) = (P_m u, P_m u_t)$.

Choose $0 < \varrho < 1$, and $0 < \varrho(\alpha - \varrho) < (1/2)\lambda_1$. Taking the scalar product with $Av_2(t) = Au_{2t}(t) + \varrho Au_2(t)$ for (2.10) in H, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|v_2\|_1^2 + \|u_2\|_2^2\right) + \varrho\|u_2\|_2^2 - \varrho(\alpha - \varrho)\langle u, v_2\rangle_1 + (\alpha - \varrho)\|v_2\|_1^2 + k\langle u^+, Av_2\rangle + \langle g(u,t), Av_2\rangle \\
&\quad = \langle h(t), Av_2\rangle,
\end{aligned} \tag{3.48}$$

where

$$\langle h(t), Av_2\rangle \leqslant \frac{(\alpha - \varrho)\|v_2\|_1^2}{8} + 2(\alpha - \varrho)^{-1}\|(I - P_m)h(t)\|_1^2, \tag{3.49}$$

$$-\langle g(u,t), Av_2\rangle \leqslant \frac{(\alpha - \varrho)\|v_2\|_1^2}{8} + 2(\alpha - \varrho)^{-1}\left\|(I - P_m)g(u,t)\right\|_1^2. \tag{3.50}$$

Clearly, we can get that

$$\varrho(\alpha - \varrho)\langle u, v_2\rangle_1 \leqslant \frac{(\alpha - \varrho)\|v_2\|_1^2}{4} + (\alpha - \varrho)\varrho^2\|u_2\|_1^2, \tag{3.51}$$

$$\begin{aligned}
k\langle u^+, Av_2\rangle &= \langle ku^+, Au_t + \varrho Au\rangle \\
&= \frac{d}{dt}k\langle u^+, Au\rangle - k\langle (u^+)_t, Au\rangle + \varrho k\langle u^+, Au\rangle \\
&\geqslant \frac{1}{2}\frac{d}{dt}k\left\|(u_2)^+\right\|_1^2 + \varrho k\left\|(u_2)^+\right\|_1^2 - \frac{\varrho}{2}\|u\|_2^2 - C.
\end{aligned} \tag{3.52}$$

Combining (3.49)–(3.52), we obtain from (3.48)

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|v_2\|_1^2 + \|u_2\|_2^2 + k\left\|(u_2)^+\right\|_1^2\right) \\
&\quad + \frac{1}{2}\varrho\|u_2\|_2^2 + \frac{1}{2}(\alpha - \varrho)\|v_2\|_1^2 + \varrho k\left\|(u_2)^+\right\|_1^2 - (\alpha - \varrho)\varrho^2\|u_2\|_1^2
\end{aligned}$$

$$\begin{aligned}
&\leqslant \frac{1}{2}\frac{d}{dt}\left(\|v_2\|_1^2 + \|u_2\|_2^2 + k\left\|(u_2)^+\right\|_1^2\right) + \varrho\lambda_1^{-1}\left(\frac{1}{2}\lambda_1 - (\alpha-\varrho)\varrho\right)\|u_2\|_2^2 \\
&\quad + \frac{1}{2}(\alpha-\varrho)\|v_2\|_1^2 + \varrho k\left\|(u_2)^+\right\|_1^2 \\
&\leqslant 2(\alpha-\varrho)^{-1}\left\|(I-P_m)g(u,t)\right\|_1^2 + 2(\alpha-\varrho)^{-1}\|(I-P_m)h(t)\|_1^2 \\
&\leqslant 2C(\alpha-\varrho)^{-1}\left\|(I-P_m)g(u,t)\right\|_{\mathcal{M}}^2\left(1+\|u_2\|_1^{2\gamma+2}\right) + 2(\alpha-\varrho)^{-1}\|(I-P_m)h(t)\|_1^2.
\end{aligned} \tag{3.53}$$

We define the functional

$$\mathcal{L}(t) = \frac{1}{2}\left(\|v_2\|_1^2 + \|u_2\|_2^2 + k\left\|(u_2)^+\right\|_1^2\right), \tag{3.54}$$

and set $\omega = \min\{2\varrho\lambda_1^{-1}((1/2)\lambda_1 - (\alpha-\varrho)\varrho), \alpha-\varrho, 2\varrho\}$, then

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) + \omega\mathcal{L}(t) &\leqslant 2C(\alpha-\varrho)^{-1}\left\|(I-P_m)g(u,t)\right\|_{\mathcal{M}}^2\left(1+\left(\sqrt{2}\mu_1\right)^{2\gamma+2}\right) \\
&\quad + 2(\alpha-\varrho)^{-1}\|(I-P_m)h(t)\|_1^2, \quad \text{for } t \geqslant t_1.
\end{aligned} \tag{3.55}$$

By Gronwall Lemma, we obtain

$$\begin{aligned}
\mathcal{L}(t) &\leqslant \mathcal{L}(t_1)e^{-\omega(t-t_1)} + \frac{2}{\alpha-\varrho}\int_{t_1}^{t} e^{-\omega(t-s)}\|(I-P_m)h(s)\|_1^2 ds \\
&\quad + \frac{2C}{\alpha-\varrho}\int_{t_1}^{t} e^{-\omega(t-s)}\left\|(I-P_m)g(u,s)\right\|_{\mathcal{M}}^2 ds, \quad \text{for } t \geqslant t_1.
\end{aligned} \tag{3.56}$$

Obviously, there exists a constant $\widetilde{C}$, such that

$$\|z_2(t)\|_{\mathcal{E}_1}^2 \leqslant \mathcal{L}(t) \leqslant \widetilde{C}\|z_2(t)\|_{\mathcal{E}_1}^2, \tag{3.57}$$

so

$$\begin{aligned}
\|z_2(t)\|_{\mathcal{E}_1}^2 &\leqslant \widetilde{C}\|z_2(t_1)\|_{\mathcal{E}_1}^2 e^{-\omega(t-t_1)} \\
&\quad + \frac{2}{\alpha-\varrho}\int_{t_1}^{t} e^{-\omega(t-s)}\|(I-P_m)h(s)\|_1^2 ds \\
&\quad + \frac{2C}{\alpha-\varrho}\int_{t_1}^{t} e^{-\omega(t-s)}\left\|(I-P_m)g(u,s)\right\|_{\mathcal{M}}^2 ds.
\end{aligned} \tag{3.58}$$

Since $g \in L^2_c(\mathbb{R}_\tau, \mathcal{M}) \subset L^2_{c^*}(\mathbb{R}_\tau, \mathcal{M})$, $h \in L^2_{c^*}(\mathbb{R}_\tau, H)$, from Lemma 2.7, we can know for any $\epsilon_1 > 0$, there exists a constant m large enough such that

$$\frac{2}{\alpha - \varrho} \int_{t_1}^{t} e^{-\omega(t-s)} \|(I - P_m)h(s)\|_1^2 ds \leqslant \frac{\epsilon_1}{3}, \quad \forall h \in \mathscr{H}(h_0),$$
$$\frac{2C}{\alpha - \varrho} \int_{t_1}^{t} e^{-\omega(t-s)} \|(I - P_m)g(u,s)\|_{\mathcal{M}}^2 ds \leqslant \frac{\epsilon_1}{3}, \quad \forall g \in \mathscr{H}(g_0), \tag{3.59}$$

where $t \geqslant \tau$.

Let $t_2 = 1/\omega \ln(3\widetilde{C}\mu_1^2/\epsilon_1) + t_1$, then

$$\widetilde{C}\|z_2(t_1)\|_{\mathcal{E}_1}^2 e^{-\omega(t-t_1)} \leqslant \frac{\epsilon_1}{3}, \quad \forall t \geqslant t_2. \tag{3.60}$$

So for every $\sigma \in \mathscr{H}(\sigma_0)$, we can get

$$\|z_2(t)\|_{\mathcal{E}_1}^2 \leqslant \epsilon_1, \quad \forall t \geqslant t_2, \tag{3.61}$$

where $\|z_2(t)\|_{\mathcal{E}_1}^2 = (1/2)(\|u_2\|_2^2 + \|u_{2t}\|_1^2)$.

Therefore, the family of processes $U_\sigma(t,\tau)$, $\sigma \in \mathscr{H}(\sigma_0)$ satisfies uniformly (w.r.t. $\sigma \in \mathscr{H}(\sigma_0)$) condition (C) in $\mathcal{E}_1$. Applying Theorem 2.4, we can obtain the existence of uniform (w.r.t. $\sigma \in \mathscr{H}(\sigma_0)$) attractor of the family of processes $U_\sigma(t,\tau)$, $\sigma \in \mathscr{H}(\sigma_0)$ in $\mathcal{E}_1$, which satisfies (3.44).

We complete the proof. □

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