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Research Article

Controllability of Second-Order Semilinear Impulsive Stochastic Neutral Functional Evolution Equations

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We consider a class of impulsive neutral second-order stochastic functional evolution equations. The Sadovskii fixed point theorem and the theory of strongly continuous cosine families of operators are used to investigate the sufficient conditions for the controllability of the system considered. An example is provided to illustrate our results.

1. Introduction

Controllability, as a fundamental concept of control theory, plays an important role both in stochastic and deterministic control problems. The study of controllability of linear and nonlinear systems represented by infinite-dimensional systems in Banach spaces has been raised by many authors recently, see Chang [1], Sakthivel [2], Ren and Sakthivel [3], Ntouyas and Regan [4], Kang et al. [5], Sakthivel and Mahmudov [6], and Shubov et al. [7]. With the help of fixed point theorem, Luo [8, 9] and Burton [10–13] have investigated the problem of controllability of the systems in Banach spaces.

Recently, stochastic partial differential equations (SPDEs) arise in the mathematical modeling of various fields in physics and engineering science cited by Sobczyk [14]. Among them, several properties of SPDEs such as existence, controllability, and stability are studied for the first-order equations. But in many situations, it is useful to investigate the second-order abstract differential equations directly rather than to convert them to first-order systems introduced by Fitzgibbon [15]. The second-order stochastic differential equations are the right model in continuous time to account for integrated processes that can be made stationary. For instance, it is useful for engineers to model mechanical vibrations or charge on a capacitor or condenser subjected to white noise excitation by second-order stochastic differential equations. A useful tool for the study of abstract second-order equations is the fixed point theory and the theory of strongly continuous cosine families.

In the past decades, the theory of impulsive differential equations or inclusions is emerging as an active area of investigation due to the application in area such as mechanics, electrical engineering, medicine biology, and ecology, see Benchohra and Henderson [16], Liu and Willms [17], Hernández et al. [18], Prato and Zabczyk [19], and Fattorini [20]. As an adequate model, impulsive differential equations are used to study the evolution of processes that are subject to sudden changes in their states.

The focus of this paper is the controllability of mild solutions for a class of impulsive neutral second-order stochastic evolution equations of the form:

$$\begin{aligned} &d\left[x'(t) - D(x_t)\right] = \left[Ax(t) + Bu(t) + f(t, x_t)\right]dt + g(t, x_t)dw(t), \quad t \in [0, T], \quad t \neq t_k \\ &\Delta x(t_k) = I_k(x(t_k)), \quad \Delta x'(t_k) = \widetilde{I}_k(x(t_k)), \quad k = 1, \dots, n, \ x(0) = \phi, \ x'(0) = y_0. \end{aligned} \tag{1.1}$$

Here, $x(\cdot)$ is a stochastic process taking values in a real separable Hilbert space H with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. $A : D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous cosine family on H. W is a given K-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$ defined on a filtered complete probability space $(\Omega, F, \{F_t\}_{t\geq 0}, P)$ and K is another separable Hilbert space with inner product $(\cdot,\cdot)_K$ and norm $\|\cdot\|_K$. The fixed time $t_k, k = 1, \dots, n$, satisfies $0 < t_1 < \cdots < t_n < T$, $x(t_k^+)$ and $x(t_k^-)$ denote the right and left limits of $x(t)$ at $t = t_k$, and $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ represents the jump in the state x at time t_k, where $I_k \in C(H, H)$ $(k = 1, 1, 2, \dots, m)$ are bounded which determine the size of the jump. Similarly $x'(t_k^+)$ and $x'(t_k^-)$ denote, respectively, the right and left limits of x' at t_k. f, B, g are appropriate mappings specified later; x_0 and y_0 are F_0-measurable random variables with finite second moment. The main contributions are as follows. The Sadovskii fixed point theorem and the theory of strongly continuous cosine families of operators are used to investigate the sufficient conditions for the controllability of the system considered. The differences of using the fixed point theorem between our proposed method and others are that Sadovskii fixed point theorem is much easier in application, and the condition is easier to be satisfied than other fixed point theorem. To our best knowledge, there are few works about the controllability for mild solutions to second-order semilinear impulsive stochastic neutral functional evolution equations, motivated by the previous problems, our current consideration is on second-order semilinear impulsive stochastic neutral functional evolution equations. We will apply the Sadovskii fixed point theorem to investigate the controllability of mild solution of this class of equations.

The rest of this paper is arranged as follows. In Section 2, we briefly present some basic notations and preliminaries. Section 3 is devoted to the controllability of mild solutions for the system (1.1) and an example is given to illustrate our results in Section 4. Conclusion is given in Section 5.

2. Preliminaries

In this section, we briefly recall some basic definitions and results for stochastic equations in infinite dimensions and cosine families of operators. We refer to Prato and Zabczyk [19] and Fattorini [20] for more details. Throughout this paper, let *L(K,H)* be the set of all linear bounded operators from K into H, equipped with the usual operator norm $\|\cdot\|$. Let (Ω, F, P) be a complete probability space furnished with a normal filtration$\{F_t\}_{t\geq 0}$. Suppose $\{\beta_k\}_{k\geq 1}$ is a sequence of real independent one-dimensional standard Brownian motions over (Ω, F, P). Set

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k(t)e_k, \quad t \geq 0, \tag{2.1}$$

where $\{e_k\}_{k\geq 1}$ is the complete orthonormal system in K and $\lambda_k,\ k \geq 1$, a bounded sequence of nonnegative real numbers. Let $Q \in L(K, K)$ be an operator defined by $Qe_k = \lambda_k e_k,\ k = 1, 2, \ldots,$ with $\operatorname{tr} Q = \sum_{k=1}^{\infty} \lambda_k < \infty$. The K-valued stochastic process $W = (W_t)_{t\geq 0}$ is called a Q-Wiener process. Let $L_2^0 = L_2(Q^{1/2}K, H)$ be the space of all Hilbert-Schmidt operators from $Q^{1/2}K$ to H with the inner product $\langle \varphi, \phi \rangle_{L_2^0} = \operatorname{tr}[\varphi Q \phi^*]$.

The collection of all strongly measurable, square-integrable H-valued random variables, denoted by $L^2(\Omega, H)$, is a Banach space equipped with norm $\|x\|_{L^2} = (E\|x\|^2)^{1/2}$. An important subspace of $L^2(\Omega, H)$ is given by

$$L_0^2(\Omega, H) = \left\{ L^2(\Omega, H) \ni x \text{ is } F_0 - \text{measurable} \right\}. \tag{2.2}$$

Let

$$\begin{aligned}
\wp &:= D([0,T], H) \\
&= \left\{ x : [0,T] \longrightarrow H,\ x|_{(t_k, t_{k+1}]} \in C((t_k, t_{k+1}], H),\ \text{and there exists } x\left(t_k^+\right) \text{ for } k = 1, 2, \ldots, n \right\}, \\
\overline{\wp} &:= \overline{D}([0,T], H) \\
&= \left\{ x \in \wp,\ x|_{(t_k, t_{k+1}]} \in C^1((t_k, t_{k+1}], H),\ \text{and there exists } \ x'\left(t_k^+\right) \text{ for } k = 1, 2, \ldots, n \right\}.
\end{aligned} \tag{2.3}$$

It is obvious that $D([0,T], H)$ and $\overline{D}([0,T], H)$ are Banach spaces endowed with the norm

$$\|x\|_{\wp} = \left(\sup_{t\in[0,T]} E\|x(t)\|^2 \right)^{1/2} \tag{2.4}$$

and $\|x\|_{\overline{\wp}} = \|x\|_{\wp} + \|x'\|_{\wp}$, respectively.

To simplify the notations, we put $t_0 = 0$, $t_{m+1} = T$, and for $u = H_2$, we denote by $\widetilde{u}_k \in C([t_k, t_{k+1}], L^2(\Omega, H))$, $k = 0, 1, \ldots, m$, the function given by

$$\widetilde{u}_k(t) = \begin{cases} u(t), & t \in (t_k, t_{k+1}], \\ u\left(t_k^+\right), & t = t_k. \end{cases} \tag{2.5}$$

Moreover, for $B \subset H_2$ we denote $\widetilde{B}_k = \{\widetilde{u}_k : u \in B\}, k = 1, \ldots, m$. To prove our results, we need the following lemma introduced in Hernández et al. [18].

Lemma 2.1. *A set $B \subset \wp$ is relatively compact in $\wp$ if and only if the set $\widetilde{B}_k$ is relatively compact in $C([t_k, t_{k+1}], H)$, for every $k = 0, 1, \ldots, m$.*

Now, we recall some facts about cosine families of operators, see Fattorini [20] and Travis and Webb [21].

Definition 2.2. (1) The one-parameter family $\{C(t) : t \in \mathbb{R}\} \subset L(H, H)$ is said to be a strongly continuous cosine family if the following hold:

(1) $C(0) = I$;

(2) $C(t)x$ is continuous in t on $\mathbb{R}$ for any $x \in H$;

(3) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $t, s \in \mathbb{R}$.

(2) The corresponding strongly continuous sine family $\{S(t) : t \in \mathbb{R}\} \subset L(H, H)$ is defined by

$$S(t)x = \int_0^t C(s)x\,ds, \qquad t \in \mathbb{R}, x \in H. \tag{2.6}$$

(3) The (infinitesimal) generator $A : H \to H$ of $\{C(t) : t \in R\}$ is given by

$$Ax = \left.\frac{d^2}{dt^2} C(t)x\right|_{t=0}, \tag{2.7}$$

for all $x \in D(A) = \{x \in H : C(\cdot)x \in C^2(\mathbb{R}, H)\}$.

It is known that the infinitesimal generator A is a closed, densely defined operator on H, and the following properties hold, see Travis and Webb [21].

Proposition 2.3. *Suppose that A is the infinitesimal generator of a cosine family of operators $\{C(t) : t \in \mathbb{R}\}$. Then, the following hold*

(i) *There exist a pair of constants $M_A \geq 1$ and $\alpha \geq 0$ such that $\|C(t)\| \leq M_A e^{\alpha|t|}$ and hence, $\|S(t)\| \leq M_A e^{\alpha|t|}$.*

(ii) *$A\int_s^r S(u)x\,du = [C(r) - C(s)]x$, for all $0 \leq s \leq r < \infty$.*

(iii) *There exist $N \geq 1$ such that $\|S(s) - S(r)\| \leq N|\int_s^r e^{\alpha|s|}ds|$, for all $0 \leq s \leq r < \infty$.*

The uniform boundedness principle: as a direct consequence we see that both $\{C(t) : t \in [0, T]\}$ and $\{S(t) : t \in [0, T]\}$ are uniformly bounded by $M^* = M_A e^{\alpha|T|}$.

At the end of this section we recall the fixed point theorem of Sadovskii [22] which is used to estimate the controllability of the mild solution to the system (1.1).

Lemma 2.4. *Let Φ be a condensing operator on a Banach space H. If $\Phi(N) \subset N$ for a convex, closed, and bounded set N of H, then Φ has a fixed point in H.*

3. Main Results

In this section we consider the system (1.1). We first present the definition of mild solutions for the system.

Definition 3.1. An F_t-adapted stochastic process $x(t) : [0,T] \to H$ is said to be a mild solution of the system (1.1) if

(1) $x_0, y_0 \in L_0^2(\Omega, H)$;

(2) $\Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k))$, $\Delta x'(t_k) = x'(t_k^+) - x'(t_k^-) = \widetilde{I}_k(x(t_k))$, $k = 1, \ldots, n$;

(3) $x(t)$ satisfies the following integral equation:

$$\begin{aligned} x(t) = {} & C(t)\phi(0) + S(t)\left[y_0 - D(0,\phi)\right] + \int_0^t C(t-s)D(s,x_s)ds \\ & + \int_0^t S(t-s)Bu(s)ds + \int_0^t S(t-s)f(s,x_s)ds \\ & + \int_0^t S(t-s)g(s,x_s)dW(s) + \sum_{0<t_k<t} C(t-t_k)I_k(x(t_k)) \\ & + \sum_{0<t_k<t} S(t-t_k)\widetilde{I}_k(x(t_k)). \end{aligned} \tag{3.1}$$

In this paper, we will work under the following assumptions.

(A1) The cosine family of operators $\{C(t) : t \in [0,T]\}$ on H and the corresponding sine family $\{S(t) : t \in [0,T]\}$ are compact for $t > 0$, and there exists a positive constant M such that

$$\|C(t)\| \leq M, \qquad \|S(t)\| \leq M. \tag{3.2}$$

(A2) D, f, g are continuous functions, and there exist some positive constants M_D, M_f, M_g, such that D, f, g satisfy the following Lipschitz condition:

$$\begin{aligned} &\|D(t,\varphi) - D(t,\phi)\| \leq M_D\|\varphi - \phi\|, \\ &\|f(t,\varphi) - f(t,\phi)\| \leq M_f\|\varphi - \phi\|, \\ &\|g(t,\varphi) - g(t,\phi)\| \leq M_g\|\varphi - \phi\|, \end{aligned} \tag{3.3}$$

for all $\varphi, \phi \in H$, $k = 1, \ldots, n$ and $t \in [0, T]$, and there exist positive constants $\overline{M}_D$, $\overline{M}_f$, $\overline{M}_g$ that satisfy the following linear growth condition:

$$\begin{aligned} \|D(t,\varphi)\|^2 &\le \overline{M}_D\left(\|\varphi\|^2 + 1\right), \\ \|f(t,\varphi)\|^2 &\le \overline{M}_f\left(\|\varphi\|^2 + 1\right), \\ \|g(t,\varphi)\|^2 &\le \overline{M}_g\left(\|\varphi\|^2 + 1\right) \end{aligned} \tag{3.4}$$

for all $\varphi, \phi \in H$, $k = 1, \ldots, n$ and $t \in [0, T]$.

(A3) $I_k, \widetilde{I}_k : H \to H$ are continuous and there exist positive constants M_k, N_k such that

$$\|I_k(x) - I_k(y)\| \le M_k\|x - y\|^2, \qquad \left\|\widetilde{I}_k(x) - \widetilde{I}_k(y)\right\| \le N_k\|x - y\|^2 \tag{3.5}$$

for each $x, y \in H$, $k = 1, \ldots, n$.

(A4) B is a continuous operator from Ω to H and the linear operator $W : L_0^2(\Omega, H) \to X$ defined by

$$Wu = \int_0^T S(T-s)Bu(s)ds \tag{3.6}$$

has a bounded invertible operator W^{-1} which takes values in $L_0^2(\Omega, H)/\ker W$ such that $\|B\| \le M_1$, $\|W^{-1}\| \le M_2$, for some positive constants M_1, M_2.

We formulate and prove conditions for the approximate controllability of semilinear control differential systems

Theorem 3.2. *Assume that (A1)–(A4) are satisfied and* $x_0, y_0 \in L_0^2(\Omega, H)$, *then the system* (1.1) *is controllable on* $[0, T]$ *provided that*

$$\begin{aligned} 8M^2\Bigg[&T\overline{M}_D^2 + T\overline{M}_f^2 + \operatorname{tr}(Q)\overline{M}_g^2 + 2M^2\sum_{k=1}^{n}M_k + 2M^2\sum_{k=1}^{n}N_k \\ &+ 8M_2\left(T\overline{M}_D^2 + T\overline{M}_f^2 + \operatorname{tr}(Q)\overline{M}_g^2 + 2M^2\sum_{k=1}^{n}M_k + 2M^2\sum_{k=1}^{n}N_k\right)\Bigg] < 1. \end{aligned} \tag{3.7}$$

Proof. Define the control process with final value $\xi = x(T)$

$$
\begin{aligned}
u_x^T(t) = W^{-1}\Bigg\{ \xi - S(T)\left[y_0 - D(0,\phi)\right] - C(T)\phi(0) - \int_0^T C(T-s)D(s,x_s)ds \\
- \int_0^T S(T-s)f(s,x_s)ds - \int_0^T S(T-s)g(s,x_s)dW(s) \\
- \sum_{0<t_k<t} C(T-t_k)I_k(x(t_k)) - \sum_{0<t_k<t} S(T-t_k)\widetilde{I}_k(x(t_k)) \Bigg\}(t).
\end{aligned}
\tag{3.8}
$$

Let $B_N = \{x \in H_2 : \|x\|_{\wp}^2 \le N\}$, for every positive integer N. It is clear that B_N is a bounded closed convex set in H_2 for each N. Define an operator $\pi : H_2 \to H_2$ by

$$
\begin{aligned}
(\pi x)(t) = C(t)\phi(0) + S(t)\left[y_0 - D(0,\phi)\right] + \int_0^t C(t-s)D(s,x_s)ds + \int_0^t S(t-s)Bu(s)ds \\
+ \int_0^t S(t-s)f(s,x_s)ds + \int_0^t S(t-s)g(s,x_s)dW(s) \\
+ \sum_{0<t_k<t} C(t-t_k)I_k(x(t_k)) + \sum_{0<t_k<t} S(t-t_k)\widetilde{I}_k(x(t_k)).
\end{aligned}
\tag{3.9}
$$

Now let us show that π has a fixed point in H_2 which is a solution of (1.1) by Lemma 2.4. This will be done in the next lemmas. □

Lemma 3.3. *There exists a positive integer N such that $\pi(B_N) \subset B_N$.*

Proof. This proof can be done by contradiction. In fact, if it is not true, then for each positive number N and $t^N \in [0,T]$, there exists a function $x^N \in B_N$, but $\pi(x^N)(t^N) \notin B_N$. That is, $E\|\pi(x^N)(t^N)\|^2 > N$. By applying assumptions (A1)–(A4) one can obtain the following estimates:

$$
\begin{aligned}
E\left\|\sum_{0<t_k<t^N} S\left(t^N - t_k\right)\widetilde{I}_k\left(x^N(t_k)\right)\right\|^2 &\le NM^2 \sum_{0<t_k<T} E\left\|\widetilde{I}_k\left(x^N(t_k)\right) - \widetilde{I}_k(0) + \widetilde{I}_k(0)\right\| \\
&\le 2NM^2\left(\sum_{k=1}^{N} N_k E\left\|x^N(t_k)\right\|^2 + \sum_{k=1}^{N}\left\|\widetilde{I}_k(0)\right\|^2\right),
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
E\left\|\sum_{0<t_k<t^N} C\left(t^N - t_k\right)I_k\left(x^N(t_k)\right)\right\|^2 &\le NM^2 \sum_{0<t_k<T} E\left\|I_k\left(x^N(t_k)\right) - I_k(0) + I_k(0)\right\| \\
&\le 2NM^2\left(\sum_{k=1}^{N} M_k E\left\|x^N(t_k)\right\|^2 + \sum_{k=1}^{N}\|I_k(0)\|^2\right),
\end{aligned}
\tag{3.11}
$$

$$E\left\|\int_0^{t^N} S\left(t^N-s\right)g(s,x_s)dW(s)\right\|^2 \le \operatorname{tr}(Q)M^2\int_0^{t^N} E\|g(s,x_s)\|^2 ds \\ \le \operatorname{tr}(Q)M^2\overline{M}_g^2\int_0^{t^N} E\left(\|\varphi\|^2+1\right)ds, \tag{3.12}$$

$$E\left\|\int_0^{t^N} C\left(t^N-s\right)D(x_s)ds\right\|^2 \le TM^2\overline{M}_D^2\int_0^{t^N} E\left(\|\varphi\|^2+1\right)ds, \tag{3.13}$$

$$E\left\|\int_0^{t^N} S\left(t^N-s\right)f(s,x_s)ds\right\|^2 \le TM^2\overline{M}_f^2\int_0^{t^N} E\left(\|\varphi\|^2+1\right)ds, \tag{3.14}$$

$$\begin{aligned} &E\left\|\int_0^{t^N} S\left(t^N-s\right)Bu(s)ds\right\|^2 \\ &\le 8M_2M^2\Bigg(\|\xi\|^2+\|\varphi(0)\|^2+y_0^2 \\ &\quad+(T+1)\overline{M}_D^2\int_0^{t^N} E\left(\|\varphi\|^2+1\right)ds \\ &\quad+T\overline{M}_f^2\int_0^{t^N} E\left(\|\varphi\|^2+1\right)ds \\ &\quad+\overline{M}_g^2\int_0^{t^N} E\left(\|\varphi\|^2+1\right)ds \\ &\quad+2N\sum_{k=1}^{N}N_kE\left\|x^N(t_k)\right\|^2 \\ &\quad+2N\sum_{k=1}^{N}M_kE\left\|x^N(t_k)\right\|^2\Bigg) := M^2U \end{aligned} \tag{3.15}$$

which gives

$$\begin{aligned} N \le E\left\|\left(\pi x^N\right)\left(t^N\right)\right\|^2 &\le 8E\left\|C\left(t^N\right)[\varphi(0)]\right\|^2+8E\left\|S\left(t^N\right)[y_0-D(0,\varphi)]\right\|^2 \\ &+8E\left\|\int_0^{t^N} C\left(t^N-s\right)D(s,\varphi)ds\right\|^2+8E\left\|\int_0^{t^N} S\left(t^N-s\right)f(s,\varphi)ds\right\|^2 \\ &+8E\left\|\int_0^{t^N} S\left(t^N-s\right)g(s,\varphi)dW(s)\right\|^2+8E\left\|\sum_{0<t_k<t^N} C\left(t^N-t_k\right)I_k\left(x^N(t_k)\right)\right\|^2 \\ &+8E\left\|\sum_{0<t_k<t^N} S\left(t^N-t_k\right)\tilde{I}_k\left(x^N(t_k)\right)\right\|^2+8E\left\|\int_0^{t^N} S\left(t^N-s\right)Bu(s)ds\right\|^2 \end{aligned}$$

$$\leq L + 8M^2\Bigg[T\overline{M}_D^2 N + T\overline{M}_f^2 N + \operatorname{tr}(Q)\overline{M}_g^2 N + 2NM^2\sum_{k=1}^{n} M_k + 2NM^2\sum_{k=1}^{n} N_k + 8M_2\Bigg(T\overline{M}_D^2 N + T\overline{M}_f^2 N + \operatorname{tr}(Q)\overline{M}_g^2 N + 2NM^2\sum_{k=1}^{n} M_k + 2NM^2\sum_{k=1}^{n} N_k\Bigg)\Bigg], \tag{3.16}$$

where

$$L = 8M^2\Bigg[E\|x_0\|^2 + E\|y_0\|^2 + T\overline{M}_D^2 + T\overline{M}_f^2 + \operatorname{tr}(Q)\overline{M}_g^2 + 2M^2\sum_{k=1}^{n} M_k + 2M^2\sum_{k=1}^{n} N_k + 8M_2\Bigg(T\overline{M}_D^2 + T\overline{M}_f^2 + \operatorname{tr}(Q)\overline{M}_g^2 + 2M^2\sum_{k=1}^{n} M_k + 2M^2\sum_{k=1}^{n} N_k\Bigg)\Bigg] \tag{3.17}$$

Dividing both sides of (3.16) by N and taking limit as $N \to \infty$, we obtain that

$$8M^2\Bigg[T\overline{M}_D^2 + T\overline{M}_f^2 + \operatorname{tr}(Q)\overline{M}_g^2 + 2M^2\sum_{k=1}^{n} M_k + 2M^2\sum_{k=1}^{n} N_k + 8M_2\Bigg(T\overline{M}_D^2 + T\overline{M}_f^2 + \operatorname{tr}(Q)\overline{M}_g^2 + 2M^2\sum_{k=1}^{n} M_k + 2M^2\sum_{k=1}^{n} N_k\Bigg)\Bigg] \geq 1 \tag{3.18}$$

which is a contradiction by (3.7). Thus, $\pi(B_N) \subset B_N$, for some positive number N.

In what follows, we aim to show that the operator π has a fixed point on B_N, which implies that (1.1) is controllable. To this end, we decompose π as follows:

$$\pi = \pi_1 + \pi_2, \tag{3.19}$$

where π_1, π_2 are defined on B_N, respectively, by

$$(\pi_1 x)(t) = S(t)\left[y_0 - D(0,\varphi)\right] + \int_0^t C(t-s)D(0,\varphi)\,ds + \int_0^t S(t-s)f(s,x_s)\,ds + \sum_{0<t_k<t} C(t-t_k)I_k(x(t_k)) + \sum_{0<t_k<t} S(t-t_k)\widetilde{I}_k(x(t_k)), \tag{3.20}$$

$$(\pi_2 x)(t) = C(t)\phi(0) + \int_0^t S(t-s)g(s,x_s)\,dW(s) + \int_0^t S(t-s)Bu(s)\,ds. \tag{3.21}$$

□

Lemma 3.4. *The operator π_1 as above is contractive.*

Proof. Let $x, y \in B_N$. It follows from assumptions (A1)–(A4) and Hölder's inequality that

$$\begin{aligned}
&E\left\|(\pi_1 x)(t)-(\pi_1 y)(t)\right\|^2 \\
&\quad\le 5E\left\|S(t)\left[D(0,\varphi)-D(0,\phi)\right]\right\|^2 \\
&\qquad+5E\left\|\int_0^t C(t-s)\left[D(0,\varphi)-D(0,\phi)\right]ds\right\|^2+5E\left\|\int_0^t S(t-s)\left[f(s,\varphi)-f(s,\phi)\right]ds\right\|^2 \\
&\qquad+5E\left\|\sum_{0<t_k<t} C(t-t_k)\left[I_k(x(t_k))-I_k(y(t_k))\right]\right\|^2 \\
&\qquad+5E\left\|\sum_{0<t_k<t} S(t-t_k)\left[\widetilde{I}_k(x(t_k))-\widetilde{I}_k(y(t_k))\right]\right\|^2 \\
&\quad\le 5M^2M_D^2\sup_{s\in[0,T]}E\left\|x(s)-y(s)\right\|^2+5TM^2M_D^2\sup_{s\in[0,T]}E\left\|x(s)-y(s)\right\|^2 \\
&\qquad+5TM^2M_f^2\sup_{s\in[0,T]}E\left\|x(s)-y(s)\right\|^2+5nM^2\sum_{0<t_k<t}M_kE\left\|x(t_k)-y(t_k)\right\|^2 \\
&\qquad+5nM^2\sum_{0<t_k<t}N_kE\left\|x(t_k)-y(t_k)\right\|^2
\end{aligned} \tag{3.22}$$

which deduces

$$\begin{aligned}
&\sup_{s\in[0,T]}E\left\|(\pi_1 x)(s)-(\pi_1 y)(s)\right\|^2 \\
&\qquad\le 5M^2\left[M_D^2+TM_D^2+TM_f^2+n\sum_{i=0}^{n}M_k+n\sum_{i=0}^{n}N_k\right]\sup_{s\in[0,T]}E\left\|x(s)-y(s)\right\|^2
\end{aligned} \tag{3.23}$$

and the lemma follows. □

Lemma 3.5. *The operator π_2 is compact.*

Proof. Let $N > 0$ be such that $\pi_2(B_N) \subset B_N$.

We first need to prove that the set of functions $\pi_2(B_N)$ is equicontinuous on $[0, T]$. Let $0 < \varepsilon < t < T$ and $\delta > 0$ such that $\|S(s)x - S(s')x\|^2 < \varepsilon$ and $\|C(s)x - C(s')x\|^2 < \varepsilon$, for every $s, s' \in [0, T]$ with $|s - s'| \le \delta$. For $x \in B_N$ and $0 < |h| < \delta$ with $t + h \in [0, T]$ we have

$$\begin{aligned}
&E\|(\pi_2 x)(t+h)-(\pi_2 x)(t)\|^2 \\
&\quad\le 3E\left\|\left[C(t+h)-C(t)\right]\phi(0)\right\|^2 \\
&\qquad+3E\left\|\int_0^t\left[S(t+h-s)-S(t-s)\right]g(s,x_s)dW(s)-\int_t^{t+h}S(t+h-s)g(s,x_s)dW(s)\right\|^2 \\
&\qquad+3E\left\|\int_0^t\left[S(t+h-s)-S(t-s)\right]Bu(s)ds-\int_t^{t+h}S(t+h-s)Bu(s)ds\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq 3\varepsilon E\left\|\phi(0)\right\|^2 + 6\operatorname{tr}(Q)M^2\int_t^{t+h} E\left\|g(s,x'(s),x_s)\right\|^2 ds + 6M^2\int_t^{t+h} E\|Bu(s)\|^2 ds \\
&\quad + 6M^2\int_0^t E\|Bu(s)\|^2 ds + 6\operatorname{tr}(Q)\int_0^t E\left\|[S(t+h-s)-S(t-s)]g(s,x_s)\right\|^2 ds \\
&\leq 4\varepsilon E\|x_0\|^2 + 4\varepsilon E\left\|g(x)\right\|^2 + 4\varepsilon\operatorname{tr}(Q)\int_0^t E\left\|g(s,x_s)\right\|^2 ds \\
&\quad + 4\operatorname{tr}(Q)M^2\int_t^{t+h} E\left\|g(s,s^x(s))\right\|^2 ds.
\end{aligned}
\tag{3.24}
$$

Noting that $E\|g(s,s^x(s))\|^2 \leq h_N(s) \in L^1([0,T])$, we see that $\pi_2(B_N)$ is equicontinuous on $[0, T]$.

We next need to prove that π_2 maps B_N into a precompact set in B_N. That is, for every fixed $t \in [0,T]$, the set $V(t) = \{(\pi_2 x)(t) : x \in B_N\}$ is precompact in B_N. It is obvious that $V(0) = \{(\pi_2 x)(0)\}$ is precompact. Let $0 < t \leq T$ be fixed and $0 < \varepsilon < t$. For $x \in B_N$, define

$$
\begin{aligned}
(\pi_2^\varepsilon x)(t) &= C(t)\phi(0) + \int_0^{t-\varepsilon} S(t-s)g(s,x_s)dW(s) + \int_0^{t-\varepsilon} S(t-s)Bu(s)ds \\
&= C(t)\phi(0) + S(\varepsilon)\int_0^{t-\varepsilon} S(t-\varepsilon-s)g(s,x_s)dW(s) + S(\varepsilon)\int_0^{t-\varepsilon} S(t-\varepsilon-s)Bu(s)ds.
\end{aligned}
\tag{3.25}
$$

Since $C(t), S(t), t > 0$, are compact, it follows that $V_\varepsilon(t) = \{(\pi_2^\varepsilon x)(t) : x \in B_N\}$ is precompact in H for every $0 < \varepsilon < t$. Moreover, for each $x \in B_N$, we have

$$
\begin{aligned}
E\left\|(\pi_2 x)(t) - (\pi_2^\varepsilon x)(t)\right\|^2 &\leq 2\operatorname{tr}(Q)M^2\int_{t-\varepsilon}^t E\left\|g(s,x_s)\right\|^2 ds + 2M^2\int_{t-\varepsilon}^t E\|Bu(s)\|^2 ds \\
&\leq \varepsilon 2M^2\left[\operatorname{tr}(Q)E\left(\|\varphi\|^2+1\right)+U\right] \longrightarrow 0 \quad as \quad \varepsilon \longrightarrow 0^+
\end{aligned}
\tag{3.26}
$$

which means that there are precompact sets arbitrary close to the set $V(t)$. Thus, $V(t)$ is precompact in B_N.

Finally, from the assumptions on g, it is obvious that π_2 is continuous. Thus, Arzelá-Ascoli theorem yields that π_2 is compact. Therefore, π is a condensing map on B_N. □

4. Applications

In this section, we now give an example to illustrate the theory obtained. Considering the following impulsive neutral second-order stochastic differential equation:

$$
\begin{aligned}
&d\left[\frac{\partial x(t,z)}{\partial t} + a(t)x(t,z)\right] = \frac{\partial^2}{\partial z^2}x(t,z)dt + \sigma(t,x(t,z))dW(t), \quad t \in [0,1] \\
&x(t,0) = x(t,\pi) = 0, \quad t \in [0,1], \quad \frac{\partial x(0,z)}{\partial t} = x_1(z), \quad z \in [0,\pi] \\
&\Delta x(t_k)(z) = I_k(x(t_k))(z), \quad \Delta x'(t_k)(z) = \widetilde{I}_k(x(t_k))(z), \quad t = t_k,
\end{aligned}
\tag{4.1}
$$

to rewrite (4.1) into the abstract form of (1.1), let $H = L^2[0,\pi]$, $A : H \to H$ be an operator by $Ax = x''$ with domain

$$D(A) = \{x \in H : x,\ x' \text{ are absolutely continuous},\ x'' \in H,\ x(0) = x(\pi) = 0\}. \tag{4.2}$$

It is well known that A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in R\}$ in H and is given by

$$C(t)x = \sum_{n=1}^{\infty} \cos(nt)\langle x, e_n\rangle e_n, \quad x \in H, \tag{4.3}$$

where $e_n(\xi) = \sqrt{2/\pi}\sin(n\xi)$ and $i = 1, 2, \ldots$ is the orthogonal set of eigenvalues of A. The associated sine family $\{S(t) : t > 0\}$ is compact and is given by

$$S(t)x = \sum_{n=1}^{\infty} \frac{1}{n}\sin(nt)\langle x, e_n\rangle e_n, \quad x \in H. \tag{4.4}$$

Thus, we can impose some suitable conditions on the above functions to verify the condition in Theorem 3.2.

5. Conclusions

In this paper, we have studied the controllability of second-order impulsive evolution equations. Through the Sadovskii fixed point theorem and the theory of strongly continuous cosine families of operators, we have investigated the sufficient conditions for the controllability of the system considered. At last, an example is provided to show the usefulness and effectiveness of proposed controllability results.

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