Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2012, Article ID 848942, 14 pages
doi:10.1155/2012/848942

Research Article

Leader-Following Consensus in Networks of Agents with Nonuniform Time-Varying Delays

Zhao-Jun Tang,[1,2] Ting-Zhu Huang,[1] Jiang-Ping Hu,[3] and Jin-Liang Shao[1]

[1] *School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 611731, China*
[2] *School of Science, Chongqing Jiaotong University, Chongqing 400074, China*
[3] *School of Automation Engineering, University of Electronic Science and Technology of China, Chengdu 611731, China*

Correspondence should be addressed to Ting-Zhu Huang, tingzhuhuang@126.com

Received 30 October 2011; Revised 26 March 2012; Accepted 27 March 2012

Academic Editor: Victoria Vampa

This paper is concerned with a leader-following consensus problem for networks of agents with fixed and switching topologies as well as nonuniform time-varying communication delays. By employing Lyapunov-Razumikhin function, a necessary and sufficient condition is derived in the case of fixed topology, and a sufficient condition is obtained in the case when the interconnection topology is switched and satisfies certain condition. Simulation results are provided to illustrate the theoretical results.

1. Introduction

In recent years, consensus problems of multiagent systems have received compelling attention from various research communities. This is mainly due to their wide applications in many areas such as synchronization of coupled oscillators, flocking, rendezvous, distributed sensor fusion in sensor networks, and distributed formation control (see [1]). The study of consensus problems is of great importance to understand many complex phenomena related to animal behaviors, such as flocking of birds, schooling of fish, and swarming of bees.

Consensus problems have a long history in the field of computer science, particularly in automata theory and distributed computation [2]. Vicsek et al. [3] proposed a simple model of a system of several autonomous agents, and demonstrated by simulation that all agents eventually reach an agreement. Jadbabaie et al. [4] provided a theoretical explanation for the observed behavior of the Vicsek model. Up to now, a variety of topics related to consensus problems have been addressed such as consensus with switching topologies, consensus with time-delays, finite-time consensus, consensus over random networks, consensus with measurement noises.

In multiagent systems, time-varying delays may arise naturally, for example, because of the moving of the agents, the congestion of the communication channels, the asymmetry of interactions, and the finite transmission speed due to the physical characteristic of the medium transmitting the information [5]. It has been observed from numerical experiments that consensus protocols without considering time delays may lead to unexpected instability. Therefore, it is important and meaningful to consider the consensus problems when communication is affected by time-delays. Consensus problems with communication delays have been addressed by many researchers. Olfati-Saber and Murray [2] studied the average consensus of first-order multiagent systems with constant and uniform communication time delays under fixed topology. The upper bound on the admissible delays was derived by means of frequency domain approach, and it was shown that it is inversely proportional to the largest eigenvalue of the Laplacian matrix of the network topology. Bliman and Ferrari-Trecate [6] generalized the results of Reference [2] in considering uniform and nonuniform time-varying time-delays. Consensus in networks of agents with single-integrator dynamics and double-integrator dynamics as well as multiple time-varying delays was addressed in [5, 7–10]. Note that the maximum allowable upper bound of time-delays in these literatures is presented in terms of linear matrix inequalities (LMIs). By employing Lyapunov-Razuminkhin function, Hu and Hong [11] and Hu and Lin [12] investigated the consensus problems of second-order multiagent systems with fixed and switching topologies in the presence of uniform communication delays.

In this paper, we are interested in the leader-following consensus of multiagent systems with fixed and switching topologies as well as multiple time-varying delays. With the help of Lyapunov-Razumikhin function, we derive the maximal allowable upper bound of communication delays such that all the agents can follow the considered leader. In [13], the authors also studied the leader-following consensus problem with multiple time-varying delays, but the proposed protocol requires that the time-delays can be detectable. We assume that the time-delays are all unknown in the present paper. The objective of this paper is to generalize the results of [11] by considering nonuniform time-varying communication delays. Obviously, it is more practical to consider nonuniform time-delays than uniform time-delays. It is worthy to note that we derive an explicit formula for the bound of the allowable time-delays by means of Lyapunov-Razumikhin function whereas the bound is presented in terms of LMIs in [5, 7–10].

The following notations will be used throughout this paper. Let I be an identity matrix with appropriate dimension. i is the imaginary unit. For a given matrix A, A^T denotes its transpose; $\|A\|$ denotes its spectral norm; $\Lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote its maximum and minimum eigenvalues, respectively. $\|\cdot\|$ denotes the Euclidean norm for a given vector. Given a complex number $\mu \in \mathbb{C}$, $\mathrm{Re}(\mu)$ and $\mathrm{Im}(\mu)$ are its real part and imaginary part, respectively. A matrix A is said to be positive stable if all its eigenvalues have positive real parts.

2. Problem Formulation

Consider a multiagent system consisting of n agents, and a leader. We first describe the interconnection topology among the n agents by a simple digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, n\}$ is the set of nodes representing the n agents and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges of the graph. An edge of $\mathcal{G}$ is denoted by (i, j), representing that agent i can directly receive information from agent j. A path in a digraph is a sequence of ordered edges of the form (i_k, i_{k+1}), $k = 1, \ldots, m-1$. We say that node j is reachable from node i if there is a path from node i to node j. The set of neighbors of node i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$.

The weighted adjacency matrix of the digraph $\mathcal{G}$ is denoted by $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, where $a_{ij} > 0$ if $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$, otherwise. Moreover, we assume that $a_{ii} = 0$ for all $i \in \mathcal{V}$. The indegree and outdegree of node i are defined as $\deg_{\text{in}}(i) = \sum_{j=1}^{n} a_{ji}$ and $\deg_{\text{out}}(i) = \sum_{j=1}^{n} a_{ij}$, respectively. A digraph is said to be balanced if $\deg_{\text{in}}(i) = \deg_{\text{out}}(i)$. The Laplacian matrix $L = [l_{ij}]$ associated with digraph $\mathcal{G}$ is defined as

$$l_{ij} = \begin{cases} -a_{ij}, & i \neq j, \\ \displaystyle\sum_{k=1, k\neq i}^{n} a_{ik}, & i = j. \end{cases} \tag{2.1}$$

The definition of L clearly implies that L must have a zero eigenvalue corresponding to a eigenvector $\mathbf{1}$, where $\mathbf{1} \triangleq [1, \ldots, 1]^T \in \mathbb{R}^n$. Moreover, 0 is a simple eigenvalue of L if and only if $\mathcal{G}$ has a spanning tree [14].

In order to study a leader-following problem, we also concern another digraph $\overline{\mathcal{G}}$, which consists of digraph $\mathcal{G}$, node 0, and edges from some nodes to node 0. We say that node 0 is globally reachable in $\overline{\mathcal{G}}$ if node 0 is reachable from any node in $\mathcal{G}$. The leader adjacency matrix associated with $\overline{\mathcal{G}}$ is defined as a diagonal matrix B with diagonal elements b_i, where $b_i > 0$ if $(i, 0)$ is an edge of $\overline{\mathcal{G}}$ and $b_i = 0$, otherwise.

In this paper, we consider the following double integrator system of n agents:

$$\begin{aligned} \dot{x}_i &= v_i, \\ \dot{v}_i &= u_i, \quad i = 1, 2, \ldots, n, \end{aligned} \tag{2.2}$$

where x_i, v_i, $u_i \in \mathbb{R}$ denote the position, velocity, and control input of agent i, respectively. The dynamics of the leader is expressed as follows:

$$\dot{x}_0 = v_0, \tag{2.3}$$

where v_0 is the desired constant velocity.

Let $\tau_{ij}(t)$ denote the communication time-delay from agent j to agent i. Similarly to [2], we assume that communication delays between agents are symmetrical, that is, $\tau_{ij}(t) = \tau_{ji}(t)$. Our control goal is to let the n agents follow the considered leader in the sense of both position and velocity, namely, $x_i \to x_0, v_i \to v_0$ $(i = 1, \ldots, n)$ as $t \to \infty$. For this end, we study the following neighbor-based protocol:

$$\begin{aligned} u_i(t) = {} & \sum_{i=1}^{n} a_{ij}(t)\left[x_j\left(t - \tau_{ij}(t)\right) - x_i\left(t - \tau_{ij}(t)\right)\right] + b_i(t)\left[x_0(t - \tau_{i0}(t)) - x_i(t - \tau_{i0}(t))\right] \\ & + k(v_0 - v_i(t)), \end{aligned} \tag{2.4}$$

where k is a control parameter.

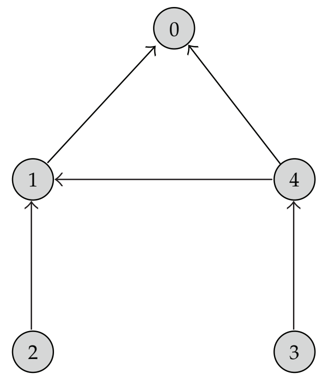


Figure 1: Interconnection topology $\overline{\mathcal{G}}_1$.

The communication topology among the group of agents may change dynamically due to link failure or creation, for instance, because of the limited detection range of agents, existence of the obstacles. In order to describe the switching topologies, we define a piecewise constant switching signal $\sigma(t)$ (σ in short) : $[0,\infty) \to \mathcal{P} = \{1,2,\ldots,N\}$, where N denotes the total number of all possible interaction topologies. The collection of all possible interaction topologies $\{\overline{\mathcal{G}}_1,\ldots,\overline{\mathcal{G}}_N\}$ is a finite set. For convenience, let

$$\mathfrak{D} = \{\tau_p(t) : p \in \mathcal{I}\}, \qquad \mathcal{I} = \{1,\ldots,m\}, \tag{2.5}$$

be the collection of independent time-delays affecting the communication links, where $\tau_p(t)$ are piecewise continuous functions. It is clear that $m \le ((n+1)n)/2$ because the delays are symmetrical. We assume that the nonuniform time-varying delays are uniformly bounded; namely, there exists a constant $\tau \ge 0$ such that $0 \le \tau_p(t) \le \tau$, $\forall p \in \mathcal{I}$. The associated edges, with the time-delay $\tau_p(t)$ and switching signal σ, define a subgraph $\overline{\mathcal{G}}_{p,\sigma}$. The corresponding Laplacian matrix associated with $\mathcal{G}_{p,\sigma}$ and the leader adjacency matrix associated with $\overline{\mathcal{G}}_{p,\sigma}$ are denoted by $L_{p,\sigma}$ and $B_{p,\sigma}$, respectively. It is clear that

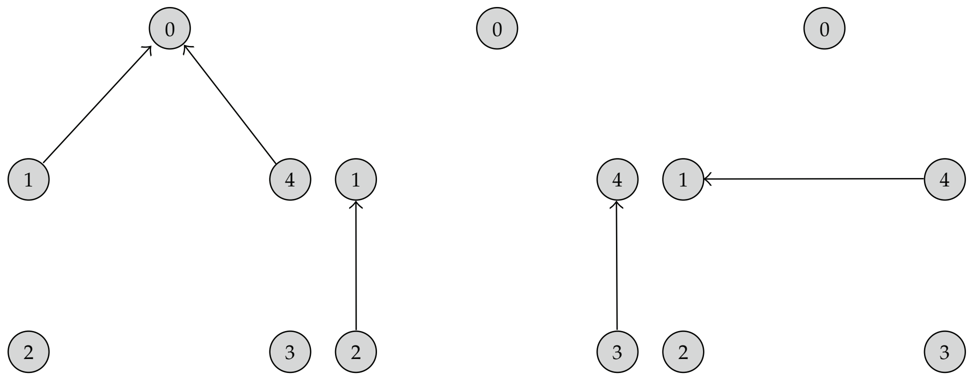
$$L_{p,\sigma}\mathbf{1} = 0, \quad p \in \mathcal{I}, \qquad \sum_{p=1}^{m} L_{p,\sigma} = L_\sigma, \qquad \sum_{p=1}^{m} B_{p,\sigma} = B_\sigma. \tag{2.6}$$

To illustrate these relationships, an example is given as follows.

Example 2.1. Consider a multiagent system consisting of four agents and a leader with the interconnection topology $\overline{\mathcal{G}}_1$ shown in Figure 1. We assume that $\overline{\mathcal{G}}_1$ has $0-1$ weights, and $\tau_{10}(t) = \tau_{40}(t) = \tau_1(t)$, $\tau_{21}(t) = \tau_{34}(t) = \tau_2(t)$, $\tau_{41}(t) = \tau_3(t)$. The subgraphs $\overline{\mathcal{G}}_{11}$, $\overline{\mathcal{G}}_{21}$, $\overline{\mathcal{G}}_{31}$ are shown in Figure 2, and one can obtain the following:

$$L_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \quad L_{11} = 0, \quad L_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad L_{31} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \tag{2.7}$$

and $B_1 = B_{11} = \mathrm{diag}\{1,0,0,1\}$, $B_{21} = B_{31} = 0$. Obviously, (2.6) is true.



(a) Subgraph $\overline{\mathcal{G}}_{11}$ (b) Subgraph $\overline{\mathcal{G}}_{21}$ (c) Subgraph $\overline{\mathcal{G}}_{31}$

Figure 2: The subgraphs of $\overline{\mathcal{G}}_1$ associated to delays τ_1, τ_2, and τ_3.

Write $x = (x_1, \ldots, x_n)^T$, $v = (v_1, \ldots, v_n)^T$. With protocol (2.4), (2.2) can be written in the following matrix form:

$$\dot{x} = v,$$
$$\dot{v} = -\sum_{p=1}^{m} \left(L_{p,\sigma} + B_{p,\sigma}\right) x\left(t - \tau_p(t)\right) + \sum_{p=1}^{m} B_{p,\sigma} \mathbf{1} x_0\left(t - \tau_p(t)\right) - k\left(v - v_0 \mathbf{1}\right). \tag{2.8}$$

Let $\overline{x} = x - x_0\mathbf{1}$, $\overline{v} = v - v_0\mathbf{1}$. We can obtain an error dynamics of system (2.8) as follows:

$$\dot{\varepsilon} = \mathcal{A}_0 \varepsilon(t) + \sum_{p=1}^{m} \mathcal{A}_{p,\sigma} \varepsilon\left(t - \tau_p(t)\right), \tag{2.9}$$

where

$$\varepsilon = \begin{pmatrix} \overline{x} \\ \overline{v} \end{pmatrix}, \quad \mathcal{A}_0 = \begin{pmatrix} 0 & I \\ 0 & -kI \end{pmatrix}, \quad \mathcal{A}_{p,\sigma} = \begin{pmatrix} 0 & 0 \\ -H_{p,\sigma} & 0 \end{pmatrix}, \quad H_{p,\sigma} = L_{p,\sigma} + B_{p,\sigma}. \tag{2.10}$$

Before ending this section, we introduce Lyapunov-Razumikhin Theorem, which plays a key role in the convergence analysis of system (2.9).

Consider the following system:

$$\dot{x} = f(x_t), \quad t > 0,$$
$$x(\theta) = \varphi(\theta), \quad \theta \in [-r, 0], \tag{2.11}$$

where $x_t(\theta) = x(t + \theta)$, $\forall \theta \in [-r, 0]$ and $f(0) = 0$. Let $C([-r, 0], \mathbb{R}^n)$ be a Banach space of continuous functions defined on an interval $[-r, 0]$, taking values in $\mathbb{R}^n$ with the topology of uniform convergence, and with a norm $\|\varphi\|_c = \max_{\theta \in [-r,0]} \|\varphi(\theta)\|$.

Lemma 2.2 (Lyapunov-Razumikhin Theorem [15]). *Let ϕ_1, ϕ_2, and ϕ_3 be continuous, nonnegative, nondecreasing functions with $\phi_1(s) > 0$, $\phi_2(s) > 0$, $\phi_3(s) > 0$ for $s > 0$ and $\phi_1(0) = \phi_2(0) = 0$. For system* (2.11)*, suppose that the function $f : C([-r,0],\mathbb{R}^n) \to \mathbb{R}$ takes bounded sets of $C([-r,0],\mathbb{R}^n)$ in bounded sets of $\mathbb{R}^n$. If there is a continuous function $V(t,x)$ such that*

$$\phi_1(\|x\|) \le V(t,x) \le \phi_2(\|x\|), \quad t \in \mathbb{R},\ x \in \mathbb{R}^n. \tag{2.12}$$

In addition, there exists a continuous nondecreasing function $\phi(s)$ with $\phi(s) > s$, $s > 0$ such that the derivative of V along the solution $x(t)$ of (2.11) *satisfies*

$$\dot{V}(t,x) \le -\phi_3(\|x\|) \text{ if } V(t+\theta, x(t+\theta)) < \phi(V(t,x(t))), \quad \theta \in [-r,0], \tag{2.13}$$

then the solution $x = 0$ is uniformly asymptotically stable.

Usually, $V(t,x)$ is called a Lyapunov-Razumikhin function if it satisfies (2.12) and (2.13) in Lemma 2.2.

3. Main Results

3.1. Fixed Interconnection Topology

Consider system (2.9) with fixed interconnection topology. In this case, the subscript σ can be dropped. Rewritte (2.9) as

$$\dot{\varepsilon} = \mathcal{A}_0\varepsilon(t) + \sum_{p=1}^{m}\mathcal{A}_p\varepsilon\big(t-\tau_p(t)\big). \tag{3.1}$$

To derive a delay-dependent stability criteria, we make the following model transformation. With the observation that

$$\varepsilon\big(t-\tau_p(t)\big) = \varepsilon(t) - \int_{-\tau_p(t)}^{0}\dot{\varepsilon}(t+s)ds, \tag{3.2}$$

it follows from (3.1) that

$$\varepsilon\big(t-\tau_p(t)\big) = \varepsilon(t) - \sum_{i=0}^{m}\mathcal{A}_i\int_{-\tau_p(t)}^{0}\varepsilon(t+s-\tau_i(t))ds, \tag{3.3}$$

where $\tau_0(t) \equiv 0$. Substituting (3.3) into system (3.1) leads to

$$\dot{\varepsilon} = \sum_{p=0}^{m}\mathcal{A}_p\varepsilon(t) - \sum_{p=1}^{m}\sum_{i=0}^{m}\mathcal{A}_p\mathcal{A}_i\int_{-\tau_p(t)}^{0}\varepsilon(t+s-\tau_i(t))ds. \tag{3.4}$$

Noting that $\mathcal{A}_p\mathcal{A}_i = 0$ for $p,\ i = 1,\ldots,m$, we have

$$\dot{\varepsilon} = F\varepsilon(t) - \sum_{p=1}^{m} C_p \int_{-\tau_p(t)}^{0} \varepsilon(t+s)ds, \tag{3.5}$$

where

$$F = \sum_{p=0}^{m} \mathcal{A}_p = \begin{pmatrix} 0 & I \\ -H & -kI \end{pmatrix}, \quad C_p = \mathcal{A}_p\mathcal{A}_0 = \begin{pmatrix} 0 & 0 \\ 0 & -H_p \end{pmatrix}, \quad H = L + B. \tag{3.6}$$

The process of transforming a system represented by (3.1) to one represented by (3.5) is known as a model transformation. The stability of the system represented by (3.5) implies the stability of the original system [16].

To get the main result of this subsection, we need the following lemmas.

Lemma 3.1 (see [17]). *Given a complex-coefficient polynomial,*

$$f(s) = s^2 + (a + \mathrm{i}b)s + c + \mathrm{i}d, \tag{3.7}$$

where $a, b, c, d \in \mathbb{R}$, $f(s)$ is Hurwitz stable if and only if $a > 0$ and $abd + a^2c - d^2 > 0$.

Lemma 3.2. *Let*

$$F = \begin{pmatrix} 0 & I_n \\ -H & -kI_n \end{pmatrix}, \quad k > 0. \tag{3.8}$$

Then F is Hurwitz stable if and only if H is positive stable and

$$k > \max_{\mu_i \in \Lambda(H)} \left\{ \frac{|\mathrm{Im}(\mu_i)|}{\sqrt{\mathrm{Re}(\mu_i)}} \right\}. \tag{3.9}$$

Proof. Note that the characteristic polynomial of F is given by

$$\begin{aligned} \det(sI_{2n} - F) &= \det\left(\begin{bmatrix} sI_n & -I_n \\ H & (s+k)I_n \end{bmatrix} \right) \\ &= \det(s(s+k)I_n + H), \end{aligned} \tag{3.10}$$

where we have used Schur formula [18] to obtain the second equality. Let μ_i be the ith eigenvalue of H, and we have

$$\det(sI_{2n} - F) = \prod_{i=1}^{n} \left(s^2 + ks + \mu_i \right). \tag{3.11}$$

Denote that $f(s,\mu_i) = s^2 + ks + \mu_i$. It follows from Lemma 3.1 that $f(s,\mu_i)$ is Hurwitz stable if and only if $\mathrm{Re}(\mu_i) > 0$ and $k > |\mathrm{Im}(\mu_i)|/\sqrt{\mathrm{Re}(\mu_i)}$. Therefore, all eigenvalues of F have negative real parts if and only if $\mathrm{Re}(\mu_i) > 0$ and $k > |\mathrm{Im}(\mu_i)|/\sqrt{\mathrm{Re}(\mu_i)}$ for any $\mu_i \in \Lambda(H)$, which implies the conclusion. □

Lemma 3.3 (see [11]). *The matrix $H = L + B$ is positive stable if and only if node 0 is globally reachable in $\overline{\mathcal{G}}$.*

Now we state one of our main results.

Theorem 3.4. *Consider system* (3.1) *and take*

$$k > k^* = \max_{\mu_i \in \Lambda(H)} \left\{ \frac{|\mathrm{Im}(\mu_i)|}{\sqrt{\mathrm{Re}(\mu_i)}} \right\}, \tag{3.12}$$

where $H = L + B$, k is the control parameter in protocol (2.4). *Then, there exists a constant $\tau^* > 0$ (which will be defined in the following* (3.20)*) such that when $\tau < \tau^*$,*

$$\lim_{t\to\infty} \varepsilon(t) = 0; \tag{3.13}$$

namely, the n agents can follow the leader (in the sense of both position and velocity), if and only if node 0 is globally reachable in $\overline{\mathcal{G}}$.

Proof. (Sufficiency). Since node 0 is globally reachable in $\overline{\mathcal{G}}$, it follows from Lemma 3.3 that H is positive stable. Thus, it follows from (3.12) and Lemma 3.2 that F is Hurwitz stable. Hence, by Lyapunov theorem [19], there exists a positive definite matrix $P \in \mathbb{R}^{2n\times 2n}$ such that

$$PF + F^T P = -I_{2n}. \tag{3.14}$$

Take a Lyapunov-Razumikhin function

$$V(\varepsilon) = \varepsilon^T P \varepsilon. \tag{3.15}$$

Along the solution of system (2.9), from (3.5), we have

$$\dot{V}(\varepsilon) = \varepsilon^T \left(PF + F^T P\right)\varepsilon - \sum_{p=1}^{m} 2\varepsilon^T P C_p \int_{-\tau_p(t)}^{0} \varepsilon(t+s)ds. \tag{3.16}$$

Note that $2a^T b \le a^T \Psi a + b^T \Psi^{-1} b$ holds for any appropriate positive definite matrix Ψ. Then, we have

$$\dot{V}(\varepsilon) \le \varepsilon^T \left(PF + F^T P\right)\varepsilon + \sum_{p=1}^{m} \left[\tau_p(t)\varepsilon^T P C_p P^{-1} C_p^T P \varepsilon + \int_{-\tau_p(t)}^{0} \varepsilon^T(t+s) P \varepsilon(t+s) ds \right]. \tag{3.17}$$

Take $\phi(s) = qs$ for some constant $q > 1$. In the case of

$$V(\varepsilon(t+\theta)) < qV(\varepsilon(t)), \quad -\tau \leq \theta \leq 0, \tag{3.18}$$

we have

$$\dot{V}(\varepsilon) \leq -\varepsilon^T\varepsilon + \tau\sum_{p=1}^{m}\varepsilon^T\left(PC_pP^{-1}C_p^TP + qP\right)\varepsilon, \tag{3.19}$$

by recalling that $\tau_p(t) \leq \tau$. As a result, if

$$\tau < \tau^* = \frac{1}{\left\|\sum_{p=1}^{m}(PC_pP^{-1}C_p^TP + qP)\right\|}, \tag{3.20}$$

then $\dot{V}(\varepsilon) \leq -\eta\varepsilon^T\varepsilon$ for some constant $\eta > 0$. Therefore, the conclusion follows by Lemma 2.2.

(Necessity). System (3.1) is asymptotically stable for any time delays $\tau_p(t) < \tau^*$, $p \in \mathcal{I}$. In particular, let $\tau_p(t) \equiv 0$, $p \in \mathcal{I}$. By (3.1), the system $\dot{\varepsilon} = F\varepsilon(t)$ is asymptotically stable, and hence all eigenvalues of F have negative real parts. Therefore, it follows from Lemma 3.2 that H is positive stable, and the conclusion follows by Lemma 3.3. □

Remark 3.5. From the proof of Theorem 3.4, we can see that many zoom techniques have to be applied during the derivation of τ^*, and hence our estimate τ^* may be very conservative.

3.2. Time-Varying Topology

In this subsection, we consider the case of switching topologies. Similar to the case of fixed topology, we can obtain that

$$\varepsilon(t-\tau_p(t)) = \varepsilon(t) - \int_{-\tau_p(t)}^{0}\left(\mathcal{A}_0\varepsilon(t+s) + \sum_{i=1}^{m}\mathcal{A}_{i,\sigma}\varepsilon(t+s-\tau_i(t))\right)ds. \tag{3.21}$$

Then, from (2.9), we have

$$\dot{\varepsilon} = \left(\mathcal{A}_0 + \sum_{p=1}^{m}\mathcal{A}_{p,\sigma}\right)\varepsilon(t) - \sum_{p=1}^{m}\mathcal{A}_{p,\sigma}\mathcal{A}_0\int_{-\tau_p(t)}^{0}\varepsilon(t+s)ds, \tag{3.22}$$

by noting that $\mathcal{A}_{p,\sigma}\mathcal{A}_{i,\sigma} = 0$ for $p, i = 1, \ldots, m$. Denote the following:

$$F_\sigma = \mathcal{A}_0 + \sum_{p=1}^{m}\mathcal{A}_{p,\sigma} = \begin{pmatrix} 0 & I_n \\ -H_\sigma & -kI_n \end{pmatrix},$$
$$C_{p,\sigma} = \mathcal{A}_{p,\sigma}\mathcal{A}_0 = \begin{pmatrix} 0 & 0 \\ 0 & -H_{p,\sigma} \end{pmatrix}, \tag{3.23}$$

where $H_{p,\sigma} = L_{p,\sigma} + B_{p,\sigma}$. Then (3.22) can be rewritten as

$$\dot{\varepsilon} = F_\sigma \varepsilon(t) + \sum_{p=1}^{m} C_{p,\sigma} \int_{-\tau_p(t)}^{0} \varepsilon(t+s)ds. \tag{3.24}$$

To obtain the main result of this subsection, we introduce the following assumption.

Assumption 3.6. The weights of digraph $\overline{\mathcal{G}}$ satisfy the following conditions:

(1) $\sum_{j=1}^{n} a_{kj} \geq \sum_{j=1}^{n} a_{jk}$, $k \in \mathcal{V}, k \notin \mathcal{O}$,

(2) $2b_k + \sum_{j=1}^{n} a_{kj} > \sum_{j=1}^{n} a_{jk}$, $\forall k \in \mathcal{O}$,

where $\mathcal{O} \triangleq \{i \mid b_i > 0,\ i \in \mathcal{V}\}$ namely, $\mathcal{O}$ denotes the index set of neighbors of vertex 0.

Lemma 3.7 (see [20]). *Assume that the weights of $\overline{\mathcal{G}}$ satisfy (Assumption 3.6), and node 0 is globally reachable in $\overline{\mathcal{G}}$. Then $H + H^T$ is positive definite, where $H = L + B$.*

Remark 3.8. In the study of leader-following consensus for second-order multiagent systems with switching topologies [11, 21–24], it was assumed that $\mathcal{G}$ is balanced and vertex 0 is globally reachable in $\overline{\mathcal{G}}$ so that a common Lyapunov function can be established. The theoretical base is the conclusion that $H + H^T$ is positive definite if $\mathcal{G}$ is balanced, and vertex 0 is globally reachable in $\overline{\mathcal{G}}$. Noticing that $\sum_{j=1}^{n+1} a_{kj}$ and $\sum_{j=1}^{n+1} a_{jk}$ denote the out-degree and in-degree of node k in $\overline{\mathcal{G}}$, respectively, (Assumption 3.6) contains that the out-degree of node k is greater or equal to its in-degree for each $k \in \mathcal{V}$ as a special case. Then (Assumption 3.6) is much weaker than the balanced constraint on $\mathcal{G}$, and the corresponding results in the above literatures can be improved accordingly.

For convenience, denote that $\mu = \max_{t\geq 0}\{\lambda_{\max}(H_{\sigma(t)} H_{\sigma(t)}^T)\}$, $\lambda = \min_{t\geq 0}\{\lambda_{\min}(H_{\sigma(t)} + H_{\sigma(t)}^T)\}$, which are well defined by noting that the set $\mathcal{P}$ is finite. The main result of this subsection is as follows.

Theorem 3.9. *Suppose that the weights of $\overline{\mathcal{G}}_\sigma$ satisfy (Assumption 3.6) and node 0 is globally reachable in $\overline{\mathcal{G}}_\sigma$ for any $t \geq 0$. Consider system* (2.9) *and take*

$$k > k^* = \frac{\mu}{2\lambda} + 1. \tag{3.25}$$

If $\tau < \tau_1^*$ (which will be defined in the following (3.32)*), then*

$$\lim_{t\to\infty} \varepsilon(t) = 0. \tag{3.26}$$

Proof. Take a Lyapunov-Razumikhin function $V(\varepsilon) = \varepsilon^T \Phi \varepsilon$ with positive definite matrix

$$\Phi = \begin{pmatrix} kI_n & I_n \\ I_n & I_n \end{pmatrix}, \qquad k > 1. \tag{3.27}$$

Similar to the analysis in the proof of Theorem 3.4, we have

$$\dot{V}(\varepsilon) \leq \varepsilon^T \left(\Phi F_\sigma + F_\sigma^T \Phi\right)\varepsilon + \sum_{p=1}^{m} \left[\tau_p(t)\varepsilon^T \Phi C_{p,\sigma}\Phi^{-1}C_{p,\sigma}^T \Phi\varepsilon + \int_{-\tau_p(t)}^{0} \varepsilon^T(t+s)\Phi\varepsilon(t+s)ds\right]. \tag{3.28}$$

Take $\phi(s) = qs$ for some constant $q > 1$. In the case of

$$V(\varepsilon(t+\theta)) < qV(\varepsilon(t)), \quad -\tau \leq \theta \leq 0, \tag{3.29}$$

we have

$$\dot{V}(\varepsilon) \leq -\varepsilon^T Q_\sigma \varepsilon + \tau \sum_{p=1}^{m} \varepsilon^T \left(\Phi C_{p,\sigma}\Phi^{-1}C_{p,\sigma}^T \Phi + q\Phi\right)\varepsilon, \tag{3.30}$$

where

$$Q_\sigma = -\left(\Phi F_\sigma + F_\sigma^T \Phi\right) = \begin{pmatrix} H_\sigma + H_\sigma^T & H_\sigma^T \\ H_\sigma & 2(k-1)I \end{pmatrix}. \tag{3.31}$$

According to Schur complement [19], Q_σ is positive definite for any $t \geq 0$ if k satisfies (3.25). Hence, if

$$\tau < \tau_1^* = \frac{\min_{t\geq 0}\{\lambda_{\min}(Q_\sigma)\}}{\max_{t\geq 0}\left\|\sum_{p=1}^{m}\left(\Phi C_{p,\sigma}\Phi^{-1}C_{p,\sigma}^T \Phi + q\Phi\right)\right\|}, \tag{3.32}$$

which is well defined by noting that the set $\mathcal{P}$ is finite, then $\dot{V}(\varepsilon) \leq -\eta_1 \varepsilon^T \varepsilon$ for some η_1. Therefore, the conclusion follows by Lemma 2.2. □

Remark 3.10. For the first-order multiagent systems, it was shown that the consensus can be achieved provided that the network topology jointly contains a spanning tree [14, 25]. However, if the group of agents is governed by second-order dynamics, the consensus depends not only on the topology condition but also on the coupling strength between neighboring agents, and it was shown that consensus may fail to be achieved even if the network topology contains a spanning tree [26]. It should be pointed out that (Assumption 3.6) is not necessary to ensure the consensus, and it is of great interest to consider the more general condition on the network topology.

4. Simulations

In this section, two examples are provided to illustrate the theoretical results. For simplicity, we assume that each interconnection topology has 0-1 weights in the following two examples.

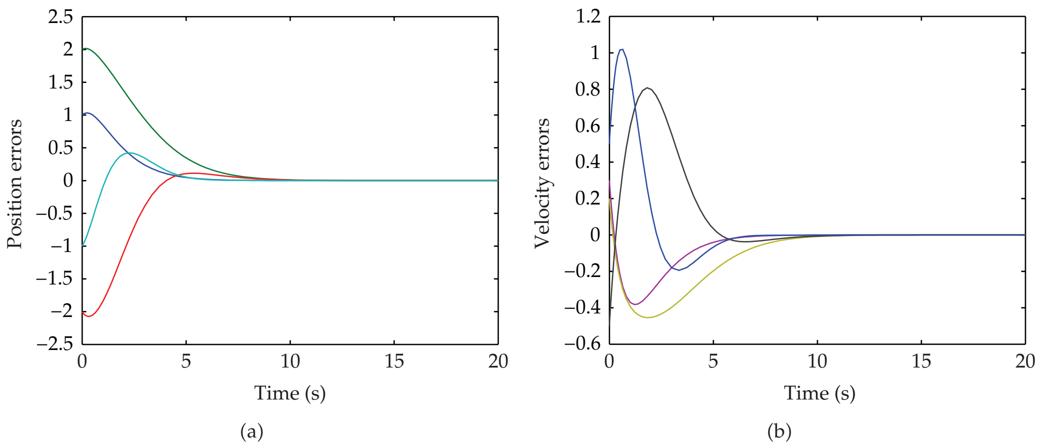


Figure 3: Position errors and velocity errors of Example 4.1.

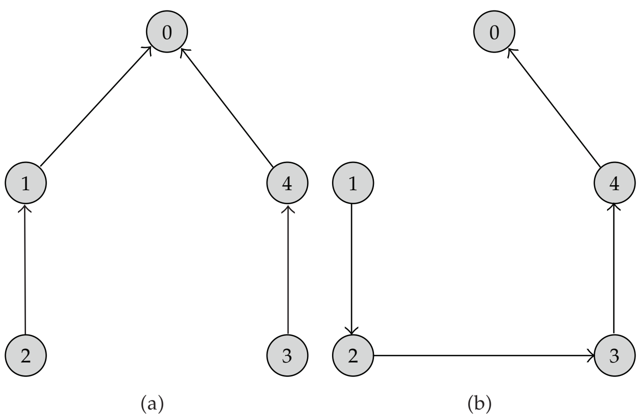


Figure 4: Interconnection topologies $\overline{\mathcal{G}_2}$ and $\overline{\mathcal{G}_3}$.

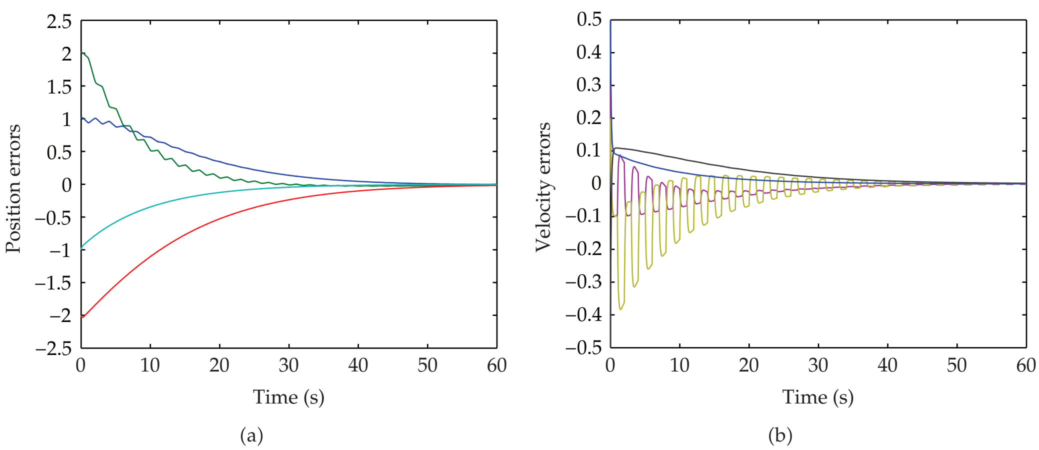


Figure 5: Position errors and velocity errors of Example 4.2.

Example 4.1. Consider a multiagent system consisting of a leader and four agents with fixed topology $\overline{\mathcal{G}}_1$ given in Figure 1. It is clear that node 0 is globally reachable in $\overline{\mathcal{G}}_1$. By simple calculation, we have $\tau^* = 0.0426$ and $k > 0$. Let $\tau_1(t) = 0.02|\sin t|$, $\tau_2(t) = 0.03|\cos t|$, $\tau_3(t) = 0.03 + 0.01\sin t$. The simulation results are obtained with $k = 2$. Figure 3 shows that the four agents can follow the considered leader.

Example 4.2. Consider a multiagent system consisting of a leader and four agents. The interconnection topology of the multiagent system switches every $1s$ in the sequence $\langle \overline{\mathcal{G}}_2, \overline{\mathcal{G}}_3 \rangle$ described as Figure 4. It is clear that the weights of $\overline{\mathcal{G}}_2$ and $\overline{\mathcal{G}}_3$ satisfy (Assumption 3.6) and node 0 is globally reachable in $\overline{\mathcal{G}}_2$ and $\overline{\mathcal{G}}_3$. Let $\tau_{10}(t) = \tau_{40}(t) = 0.1|\sin t|$, $\tau_{21}(t) = \tau_{34}(t) = 0.2|\cos t|$, $\tau_{23}(t) = 0.2 + 0.1\sin t$. The simulation results are obtained with $k = 10$. It can be seen from Figure 5 that the four agents can follow the considered leader.

5. Conclusion

In this paper, we study a leader-following consensus problem of second-order multiagent systems with fixed and switching topologies as well as nonuniform time-varying communication delays. With the help of Lyapunov-Razumikhin function, an explicit formula for the upper bound of admissible delays is obtained for both fixed and switching topologies. Future research issues will include the cases when the communication delays are asymmetric, and the velocity of the considered leader is time-varying.

Acknowledgment

The authors would like to thank the anonymous reviewers for their insightful comments and suggestions. This work was supported by NSFC (60973015, 61170311, 61104104), NSFC Tianyuan foundation (11126104), Chinese Universities Specialized Research Fund for the Doctoral Program (20110185110020), Sichuan Province Sci.&Tech. Research Project (12ZC1802).

References

[1] R. Olfati-Saber, J. A. Fax, and R. M. Murray, “Consensus and cooperation in networked multi-agent systems,” *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, 2007.

[2] R. Olfati-Saber and R. M. Murray, “Consensus problems in networks of agents with switching topology and time-delays,” *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520–1533, 2004.

[3] T. Vicsek, A. Czirk, E. Ben-Jacob, I. Cohen, and O. Shochet, “Novel type of phase transition in a system of self-driven particles,” *Physical Review Letters*, vol. 75, no. 6, pp. 1226–1229, 1995.

[4] A. Jadbabaie, J. Lin, and A. S. Morse, “Coordination of groups of mobile autonomous agents using nearest neighbor rules,” *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, 2003.

[5] Y. G. Sun, L. Wang, and G. Xie, “Average consensus in networks of dynamic agents with switching topologies and multiple time-varying delays,” *Systems & Control Letters*, vol. 57, no. 2, pp. 175–183, 2008.

[6] P.-A. Bliman and G. Ferrari-Trecate, “Average consensus problems in networks of agents with delayed communications,” *Automatica*, vol. 44, no. 8, pp. 1985–1995, 2008.

[7] Y. G. Sun and L. Wang, “Consensus problems in networks of agents with double-integrator dynamics and time-varying delays,” *International Journal of Control*, vol. 82, no. 10, pp. 1937–1945, 2009.

[8] Y. G. Sun and L. Wang, "Consensus of multi-agent systems in directed networks with uniform time-varying delays," *IEEE Transactions on Automatic Control*, vol. 54, no. 7, pp. 1607–1613, 2009.

[9] P. Lin and Y. Jia, "Average consensus in networks of multi-agents with both switching topology and coupling time-delay," *Physica A*, vol. 387, no. 1, pp. 303–313, 2008.

[10] P. Lin and Y. Jia, "Consensus of a class of second-order multi-agent systems with time-delay and jointly-connected topologies," *IEEE Transactions on Automatic Control*, vol. 55, no. 3, pp. 778–784, 2010.

[11] J. Hu and Y. Hong, "Leader-following coordination of multi-agent systems with coupling time delays," *Physica A*, vol. 374, no. 2, pp. 853–863, 2007.

[12] J. Hu and Y. S. Lin, "Consensus control for multi-agent systems with double-integrator dynamics and time delays," *IET Control Theory & Applications*, vol. 4, no. 1, pp. 109–118, 2010.

[13] W. Zhu and D. Cheng, "Leader-following consensus of second-order agents with multiple time-varying delays," *Automatica*, vol. 46, no. 12, pp. 1994–1999, 2010.

[14] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 655–661, 2005.

[15] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional-Differential Equations*, vol. 99, Springer, New York, NY, USA, 1993.

[16] K. Gu, V. Kharitonov, and J. Chen, *Stability of Time-Delay Systems*, Birkhauser, 2003.

[17] P. C. Parks and V. Hahn, *Stability Theory*, Prentice Hall, Upper Saddle River, NJ, USA, 1992.

[18] Y. Cao, W. Ren, and Y. Li, "Distributed discrete-time coordinated tracking with a time-varying reference state and limited communication," *Automatica*, vol. 45, no. 5, pp. 1299–1305, 2009.

[19] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, NY, USA, 1994.

[20] Z.-J. Tang, T.-Z. Huang, J.-L. Shao, and J.-P. Hu, "Leader-following consensus for multi-agent systems via sampled-data control," *IET Control Theory & Applications*, vol. 5, no. 14, pp. 1658–1665, 2011.

[21] K. Peng and Y. Yang, "Leader-following consensus problem with a varying-velocity leader and time-varying delays," *Physica A*, vol. 388, no. 2-3, pp. 193–208, 2009.

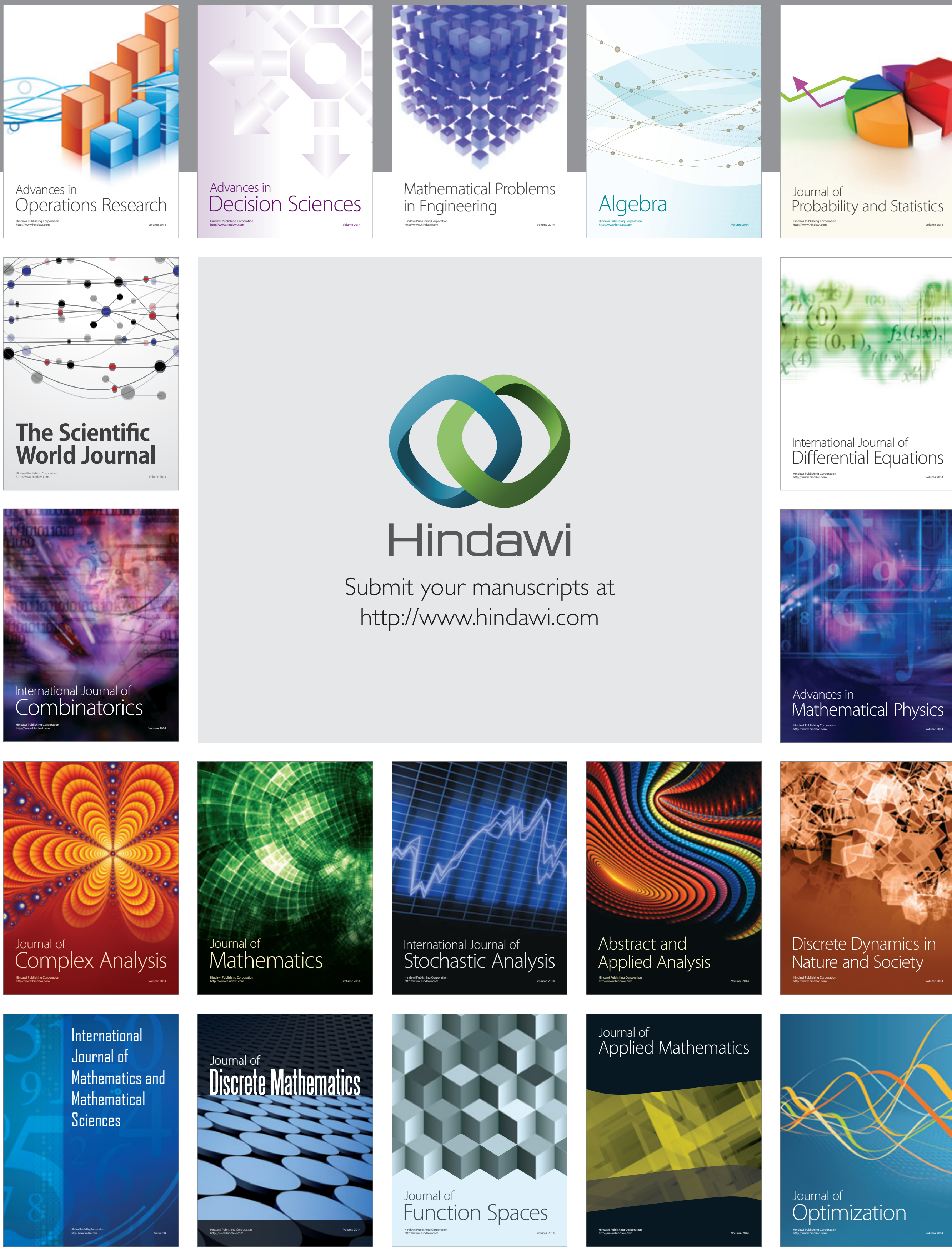
[22] J. Hu and G. Feng, "Distributed tracking control of leader-follower multi-agent systems under noisy measurement," *Automatica*, vol. 46, no. 8, pp. 1382–1387, 2010.

[23] Y. Sun, D. Zhao, and J. Ruan, "Consensus in noisy environments with switching topology and time-varying delays," *Physica A*, vol. 389, no. 19, pp. 4149–4161, 2010.

[24] D. Cheng, J. Wang, and X. Hu, "An extension of LaSalle's invariance principle and its application to multi-agent consensus," *IEEE Transactions on Automatic Control*, vol. 53, no. 7, pp. 1765–1770, 2008.

[25] L. Moreau, "Stability of multiagent systems with time-dependent communication links," *IEEE Transactions on Automatic Control*, vol. 50, no. 2, pp. 169–182, 2005.

[26] W. Ren and E. Atkins, "Distributed multi-vehicle coordinated control via local information exchange," *International Journal of Robust and Nonlinear Control*, vol. 17, no. 10-11, pp. 1002–1033, 2007.


Advances in Operations Research
Advances in Decision Sciences
Mathematical Problems in Engineering
Algebra
Journal of Probability and Statistics
The Scientific World Journal
International Journal of Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
International Journal of Stochastic Analysis
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
Journal of Applied Mathematics
Journal of Optimization