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Research Article

Stochastic Stabilization of Nonholonomic Mobile Robot with Heading-Angle-Dependent Disturbance

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The problem of exponential stabilization for nonholonomic mobile robot with dependent stochastic disturbance of heading angle is considered in this paper. An integrator backstepping controller based on state-scaling method is designed such that the state of the closed-loop system, starting from a nonzero initial heading angle, is regulated to the origin with exponential rate in almost surely sense. For zero initial heading angle, a controller is designed such that the heading angle is driven away from zero while the position variables are bounded in a neighborhood of the origin. Combing the above two cases results in a switching controller such that for any initial condition the configuration of the robot can be regulated to the origin with exponential rate. The efficiency of the proposed method is demonstrated by a detailed simulation.

1. Introduction

In the past decades, there has been increasing attention devoted to the control of nonholonomic systems such as knife edge, rolling disk, tricycle-type robot, and car-like robot with trailers (see, [1, 2] and the references therein). From Brockett's necessary condition [3], it is well known that the nonholonomic systems cannot be stabilized to the origin by any static continuous state feedback, so the classical smooth control theory cannot be applied directly. This motivates researchers to seek for novel approaches such as discontinuous feedback and time-varying feedback. The discontinuous feedback uses the state-scaling technique and switching control strategy [4, 5], which usually results in an exponential convergence. The time-varying feedback provides smooth controllers, but its convergence rate usually is slow [6, 7]. All the above references considered the nonholonomic systems in the deterministic case, while the nonholonomic systems with stochastic disturbance have rarely been researched up to now.

The purpose of this paper is to consider the posture (including position and direction) adjustment of nonholonomic mobile robot with stochastic disturbance dependent of heading angle. By a state transformation, the mentioned control model can be rewritten as

$$
\begin{aligned}
dx_1 &= v dt + \varphi_1 dW_1, \\
dx_2 &= x_3 v dt + x_3 \varphi_1 dW_1, \\
dx_3 &= (u - x_2 v) dt + \varphi_2 dW_2 - x_2 \varphi_1 dW_1,
\end{aligned}
\tag{1.1}
$$

where u is the forward velocity, v is the steering velocity, φ_1 and φ_2 are two smooth functions, and W_1 and W_2 are two independent standard Wiener processes. It seems that the stabilization can be achieved by extending the backstepping procedure based on state-scaling technique [5] to the stochastic case.

Our main contribution consists of the following aspects. (i) For nonzero initial value of x_1, by imposing a reasonable assumption on function φ_1, the state x_1 can be easily exponentially regulated to zero via control v. However, in doing so, v will converge to zero as t goes to infinity. This phenomenon causes serious trouble in controlling x_2-subsystem via the virtual control x_3 because, in the limit ($\lim_{t\to\infty} v = 0$, a.s.), x_2-subsystem is uncontrollable. The variable x_3 appears in the drift term $x_3 v$ and the diffusion term $x_3\varphi_1$ of x_2-subsystem simultaneously. This leads to a new problem that the desirable control α and its square term α^2 appear in the same procedure of backstepping control (see, (4.11)), which is distinct from the traditional stochastic backstepping method as used in [8]. (ii) For a nonzero $x_1(t_0)$, the transformation $z_1 = x_2/x_1$ is used in controller design, therefore, it cannot work for systems with initial state whose $x_1(t_0) = 0$, which motivates us to drive x_1 away from zero in a small distance during a shorter time interval by designing v. (iii) For any initial condition, a switching control is given by combining the above two cases. Different from the usual switching schemes depending only on state, our switching controller depends on a stopping time as well as state. Therefore, the measurement of the stopping time is expected. It is proved that all signals in the closed-loop system converge to zeros with exponential rate. The discontinuous switching function is replaced with a continuous one to eliminate the trembling phenomenon in simulation.

This paper is organized as follows. Section 2 begins with some mathematical preliminaries. The model of a wheeled mobile robot with stochastic disturbance is presented in Section 3. Backstepping stabilizer based on state-scaling technique is investigated for the case of $x_1(t_0) \neq 0$ in Section 4. In Section 5, for the case of $x_1(t_0) = 0$, a controller is designed such that there exists a time interval in which x_1 is driven away from zero and the other signals are bounded in probability. Section 6 formulates the main stabilization results for any initial condition. A simulation is given in Section 7. Finally, Section 8 draws the conclusion.

The following notations are used throughout the paper: C^i denotes the set of all functions with continuous ith partial derivative; for any vector x in $\mathbb{R}^n$, $|x|$ means its Euclidean norm and x^T is its transpose; for any Matrix X in $R^{m\times n}$, $|X|$ denotes the Frobenius norm defined by $|X| = (\mathrm{Tr}\{XX^T\})^{1/2}$, where $\mathrm{Tr}(\cdot)$ denotes the matrix trace; $\mathcal{K}$ denotes the set of all functions: $\mathbb{R}_+ \to \mathbb{R}_+$, which are continuous, strictly increasing and vanish at zero; $\mathcal{K}_\infty$ denotes the set of all functions which are of class $\mathcal{K}$ and unbounded.

2. Mathematical Preliminaries

Consider the nonlinear stochastic system

$$dx = f(x,t)dt + g(x,t)dW, \qquad x(t_0) = x_0 \in \mathbb{R}^n, \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the state, $f(0,t) = 0$, $g(0,t) = 0$, and W is an r-dimensional independent standard Wiener process.

The following notion of boundedness on an interval in probability can be seen as a slight extension from that used in [9].

Definition 2.1. A stochastic process $x(t)$ is said to be bounded on $t \in [t_0, T]$, where $T \leq \infty$, in probability if the random variable $|x(t)|$ satisfies

$$\lim_{R\to\infty} \sup_{t\in[t_0,T]} P\{|x(t)| > R\} = 0. \tag{2.2}$$

For this notion, a corresponding criterion can be easily obtained following the line of [10].

Lemma 2.2. *Consider system* (2.1) *defined in* $[t_0, T]$, *where* $T \leq \infty$. *Assume that there exist a function* $V \in C^2$, *class* $\mathcal{K}_\infty$ *functions* $\underline{\alpha}(|x|)$ *and* $\overline{\alpha}(|x|)$, *a positive constant* c, *and a nonnegative constant* d *such that for all* $x_0 \in \mathbb{R}^n$, *(i)* $\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|)$, *and (ii)* $\mathcal{L}V(x) = V_x(x)f(x,t) + (1/2)\operatorname{Tr}[g^T(x,t)V_{xx}g(x,t)] \leq -cV(x) + d$, *for all* $t \in [t_0, T]$, *then system* (2.1) *has a unique solution on* $[t_0, T]$, *which is bounded on* $t \in [t_0, T]$ *in probability.*

To find condition to let state scaling make sense, the following lemma proved by Mao in [11, pages 51, 120] is recited as follows.

Lemma 2.3. *For system* (2.1) *defined on* $t \in [t_0, T]$, *where* $T \leq \infty$, *assume that there exist two constants* K_1 *and* K_2 *such that*

(i) *(lipschitz condition) for all* $x, y \in \mathbb{R}^n$ *and* $t \in [t_0, T]$

$$\left|f(x,t) - f(y,t)\right|^2 \vee \left|g(x,t) - g(y,t)\right|^2 \leq K_1\left|x - y\right|^2; \tag{2.3}$$

(ii) *(Linear growth condition) for all* $(x,t) \in \mathbb{R}^n \times [t_0, T]$

$$\left|f(x,t)\right|^2 \vee \left|g(x,t)\right|^2 \leq K_2\left(1 + |x|^2\right), \tag{2.4}$$

then there exists a unique solution $x(t) := x(t_0, x_0, t)$ *to system* (2.1) *and for all* $x_0 \neq 0$ *in* $\mathbb{R}^n$,

$$P(x(t, t_0, x_0) \neq 0) = 1, \quad \forall T \geq t \geq t_0 \tag{2.5}$$

(i.e., almost all the sample path of any solution starting from a nonzero state will never reach the origin).

The concepts of moment exponential stability and almost surely exponential stability together with their criteria can be found in [12, page 166], which are presented here for self-sufficiency.

Definition 2.4. For $p > 0$, system (2.1) is said to be pth moment exponential stable if

$$\limsup_{t\to\infty}\frac{1}{t}\log E|x(t,t_0,x_0)|^p < 0 \tag{2.6}$$

for each $x_0 \in \mathbb{R}^n$. Moreover, system (2.1) is said to be almost surely exponential stable if

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t,t_0,x_0)| < 0, \quad \text{a.s.}, \ \forall x_0 \in \mathbb{R}^n. \tag{2.7}$$

Lemma 2.5. *Assume that there exist a function $V \in C^2$ and positive constants c_1, c_2, c and p such that (i) $c_1|x|^p \le V(x) \le c_2|x|^p$, and (ii) $\mathcal{L}V(x) \le -cV(x)$, for all $x_0 \in \mathbb{R}^n$ and $t \ge t_0$, then system* (2.1) *has a unique solution on $[t_0,\infty)$, which is pth moment exponential stable. Moreover, if further assume that (iii) there exists a positive constant K such that*

$$\left|f(x,t)\right| \vee \left|g(x,t)\right| \le K|x| \tag{2.8}$$

for all $(x,t) \in \mathbb{R}^n \times [t_0,\infty)$, then system (2.1) *is almost surely exponential stable.*

3. Problem Formulation

A nonholonomic mobile robot of tricycle type in the presence of stochastic disturbance can be described by

$$\begin{aligned} d\theta &= v dt + \varphi_1(\theta)dW_1, \\ dx_c &= \left(u dt + \varphi_2(\theta)dW_2\right)\cos\theta, \\ dy_c &= \left(u dt + \varphi_2(\theta)dW_2\right)\sin\theta, \end{aligned} \tag{3.1}$$

where u is the forward velocity, v is the steering velocity, (x_c, y_c) is the position of the mass center of the robot moving in the plane, θ is the heading angle from the horizontal axis, W_1 and W_2 are two independent standard Wiener processes, and φ_1 and φ_2 are two unknown scaler-valued smooth functions. A tricycle-type robot is described by Figure 1.

Performing the change of coordinate

$$x_1 = \theta, \qquad x_2 = x_c\sin\theta - y_c\cos\theta, \qquad x_3 = x_c\cos\theta + y_c\sin\theta, \tag{3.2}$$

system (3.1) can be transformed into system (1.1). The control objective is to design a state-feedback controller such that all the signals in the closed-loop system are globally exponentially regulated to the origin in probability. For this end, the following assumptions are imposed throughout this paper.

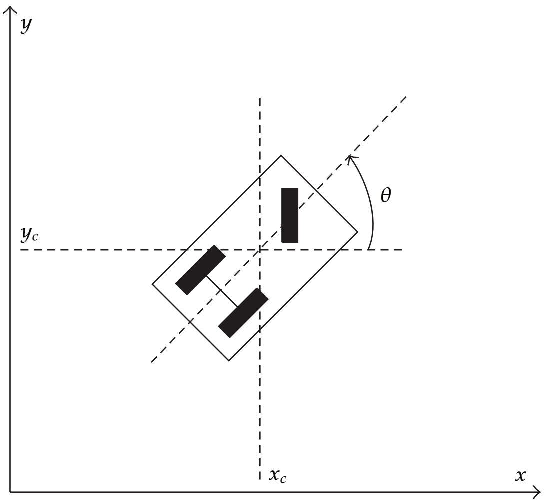


Figure 1: A nonholonomic mobile robot.

(A1) There exists a positive constant k such that

$$\left|\varphi_1(x_1) - \varphi_1(x_2)\right| \leq k|x_1 - x_2|, \quad \varphi_1(0) = 0. \tag{3.3}$$

(A2) There exist a positive constant l and a smooth nonnegative function ϕ such that

$$\varphi_2^2(x_1) \leq l x_1^2 \phi(x_1). \tag{3.4}$$

Remark 3.1. System (1.1) is similar to the class of systems in strict-feedback form driven by Wiener processes, which motivates us to investigate the backstepping controller design that had been extensively researched by [8, 13]. Assumptions (A1) and (A2) are given to diffusion terms as same as those imposed to drift terms in [5] in the deterministic case. For nonzero initial value of x_1, by imposing (A1) on function φ_1, the state x_1 can be regulated to zero with exponential rate but never reach zero (see the subsequent subsection), which is the key to introduce a state-scaling transformation to deal with other troubles (see Section 1).

4. Controller Design for the Case of $x_1(t_0) \neq 0$

4.1. Design of Controller v

It can be seen that the state x_1 of system (1.1) can be globally exponentially regulated to zero via a static feedback control law. In fact, we can introduce a Lyapunov function

$$V_0 = \frac{1}{4} x_1^4 \tag{4.1}$$

whose infinite generator along the first equation of (1.1) satisfies

$$\mathcal{L}V_0 \leq x_1^3 v + \frac{3}{2} x_1^4 k^2. \tag{4.2}$$

By choosing the control law v as

$$v = -\lambda x_1, \tag{4.3}$$

where $\lambda \geq 2k^2$ is a positive parameter (further requirements for λ will be given later), (4.2) becomes

$$\mathcal{L}V_0 \leq -\lambda V_0. \tag{4.4}$$

By substituting (4.3) into the first equation of (1.1), one has

$$dx_1 = -\lambda x_1 dt + \varphi_1 dW_1, \tag{4.5}$$

which, together with assumption (A1) and Lemma 2.3, means that there exists a unique solution to (4.5) and that any solution starting from a nonzero state will never reach the origin in almost surely sense. From assumption (A1), (4.1) and (4.4), according to Lemma 2.3, the solution exponentially converges to zero, that is, $\limsup_{t\to\infty}(1/t)\log|x_1(t)| < -\lambda$, a.s., which means that

$$|x_1(t)| < |x_1(t_0)| e^{-\lambda(t-t_0)}, \quad \text{a.s.} \tag{4.6}$$

4.2. State-Scaling Transformation

We have designed controller v such that state $x_1(t)$ can be globally exponentially regulated to zero. Consequently, v will converge to zero as t goes to ∞. This causes trouble in the control of x_2-subsystem and x_3-subsystem. To overcome this difficulty, we introduce a state-scaling transformation defined by

$$z_1 = \frac{x_2}{x_1}. \tag{4.7}$$

According to the comment in the end of Section 1, the transformation (4.7) makes sense in almost surely sense, for the initial value $x_1(t_0) \neq 0$. From (1.1), (4.3), and (4.7), we have

$$dz_1 = \left[\lambda z_1 - \lambda x_3 - \left(\frac{\varphi_1}{x_1}\right)^2 x_3 + \left(\frac{\varphi_1}{x_1}\right)^2 z_1\right] dt + \left[\left(\frac{\varphi_1}{x_1}\right) x_3 - \left(\frac{\varphi_1}{x_1}\right) z_1\right] dW_1. \tag{4.8}$$

4.3. Backstepping Controller Design of u

In this part, controller u will be constructed, based on backstepping techniques, under the assumption $x_1(t_0) \neq 0$.

Step 1. Begin with z_1-subsystem of (4.8), where x_3 is regarded as a virtual control. Introducing the transformation

$$z_2 = x_3 - \alpha \tag{4.9}$$

and choosing Lyapunov function

$$V_1 = V_0 + \frac{1}{4} z_1^4, \tag{4.10}$$

it comes from (4.8)–(4.10) that

$$\begin{aligned} \mathcal{L}V_1 = {} & z_1^3 \left[\lambda z_1 - \lambda(z_2 + \alpha) - \left(\frac{\varphi_1}{x_1} \right)^2 (z_2 + \alpha) + \left(\frac{\varphi_1}{x_1} \right)^2 z_1 \right] \\ & + 3z_1^2 \left(\frac{\varphi_1}{x_1} \right)^2 (z_2 + \alpha)^2 + 3z_1^4 \left(\frac{\varphi_1}{x_1} \right)^2 + \mathcal{L}V_0. \end{aligned} \tag{4.11}$$

Here, the terms α and α^2 appear in the same time, which is different from the traditional backstepping procedure. Considering assumption (A1) and the characters of terms of (4.11), the virtual control is chosen as

$$\alpha = c_1 z_1, \tag{4.12}$$

where $c_1 > 0$ is a design parameter. By the aid of (4.4), (4.12), and $(\varphi_1 / x_1)^2 \le k^2$ (that comes from assumption (A1)), (4.11) can be rewritten as

$$\begin{aligned} \mathcal{L}V_1 \le {} & \lambda z_1^4 - \lambda z_1^3 z_2 - c_1 \lambda z_1^4 + k^2 \left| z_1^3 \right| |z_2| c_1 k^2 z_1^4 \\ & + k^2 z_1^4 + 6k^2 z_1^2 z_2^2 + 6c_1^2 k^2 z_1^4 + 3k^2 z_1^4 - \lambda \frac{x_1^4}{4}. \end{aligned} \tag{4.13}$$

Submitting the inequalities

$$\begin{gathered} -\lambda z_1^3 z_2 \le \frac{3d}{4} \lambda z_1^4 + \frac{1}{4d^3} \lambda z_2^4, \qquad k^2 z_1^3 |z_2| \le \frac{3}{4} k^2 z_1^4 + \frac{1}{4} k^2 z_2^4, \\ 6k^2 z_1^2 z_2^2 \le 3k^2 z_1^4 + 3k^2 z_2^4, \end{gathered} \tag{4.14}$$

where $d > 0$ is a design parameter, into (4.13) gives

$$\mathcal{L}V_1 \le \left(\left(1 + \frac{3d}{4} \right) \lambda - c_1 \lambda + \left(\frac{31}{4} + c_1 + 6c_1^2 \right) k^2 \right) z_1^4 + \left(\frac{\lambda}{4d^3} + \frac{k^2}{4} + 3k^2 \right) {z_2}^4 - \frac{\lambda}{4} x_1^4. \tag{4.15}$$

By selecting parameters λ and c_1 such that $c_1 \geq (4+3d)/4e, \lambda \geq (31+4c_1+24c_1^2)k^2/2c_1(1-e)$, where e is a design parameter satisfying $0 < e < 1$, it comes from (4.15) that

$$\mathcal{L}V_1 \leq -\frac{c_1\lambda(1-e)}{2} z_1^4 + \left(\frac{\lambda}{4d^3} + \frac{13k^2}{4}\right) z_2{}^4 - \frac{\lambda}{4} x_1^4. \tag{4.16}$$

Step 2. In view of (1.1) and (4.9), we have

$$\begin{aligned} dz_2 = &\left[u - x_2 v - c_1\lambda z_1 + c_1\lambda x_3 + c_1\left(\frac{\varphi_1}{x_1}\right)^2 x_3 - c_1\left(\frac{\varphi_1}{x_1}\right)^2 z_1\right] dt \\ &+ \varphi_2 dW_2 + \left[-x_2\varphi_1 - c_1\left(\frac{\varphi_1}{x_1}\right) x_3 + c_1\left(\frac{\varphi_1}{x_1}\right) z_1\right] dW_1. \end{aligned} \tag{4.17}$$

Consider the candidate Lyapunov function

$$V_2 = V_1 + \frac{1}{4} z_2^4 \tag{4.18}$$

whose infinite generator along (4.17) satisfies

$$\begin{aligned} \mathcal{L}V_2 \leq\ & z_2^3\left[u - x_2 v - c_1\lambda z_1 + c_1\lambda x_3 + c_1\left(\frac{\varphi_1}{x_1}\right)^2 x_3 - c_1\left(\frac{\varphi_1}{x_1}\right)^2 z_1\right] + \frac{3}{2} z_2^2 \varphi_2{}^2 \\ &+ \frac{9}{2} z_2^2 (x_2\varphi_1)^2 + \frac{9}{2} z_2^2 (c_1)^2 \left(\frac{\varphi_1}{x_1}\right)^2 x_3^2 + \frac{9}{2} z_2^2 (c_1)^2 \left(\frac{\varphi_1}{x_1}\right)^2 z_1^2 + \mathcal{L}V_1. \end{aligned} \tag{4.19}$$

By using Young's equality and (A1), (A2), (4.7), and (4.9), it is easy to obtain that

$$\begin{aligned} \frac{3}{2} z_2^2 \varphi_2^2 &\leq \frac{3}{4} l^2 z_2^4 \phi^2(x_1) + \frac{3}{4} l^2 x_1^4, \\ \frac{9}{2} z_2^2 (x_2\varphi_1)^2 &\leq \frac{9}{4} k^2 z_2^4 x_1^8 + \frac{9}{4} k^2 z_1^4, \\ \frac{9}{2} z_2^2 (c_1)^2 \left(\frac{\varphi_1}{x_1}\right)^2 x_3^2 &\leq 9c_1^2 k^2 z_2^4 + \frac{9}{2} c_1^4 k^2 z_2^4 + \frac{9}{2} c_1^4 k^2 z_1^4, \\ \frac{9}{2} z_2^2 (c_1)^2 \left(\frac{\varphi_1}{x_1}\right)^2 z_1^2 &\leq \frac{9}{4} c_1^2 k^2 z_2^4 + \frac{9}{4} c_1^2 k^2 z_1^4, \end{aligned} \tag{4.20}$$

which are submitted into (4.19) to give

$$\begin{aligned}\mathcal{L}V_2 \leq\ & z_2^3\Bigg[u - x_2 v - c_1\lambda z_1 + c_1\lambda x_3 + c_1\left(\frac{\varphi_1}{x_1}\right)^2 x_3 - c_1\left(\frac{\varphi_1}{x_1}\right)^2 z_1 + \frac{3}{4}l^2\phi^2(x_1)z_2 \\ & + \frac{9}{4}k^2x_1^8z_2 + 9c_1^2k^2z_2 + \frac{9}{2}c_1^4k^2z_2 + \frac{9}{4}c_1^2k^2z_2 + \frac{1}{4d^3}\lambda z_2 + \frac{1}{4}k^2z_2 + 3k^2z_2\Bigg] \\ & + \frac{3}{4}l^2x_1^4 + \left(\frac{9}{4}k^2 + \frac{9}{2}c_1^4k^2 + \frac{9}{4}c_1^2k^2 - \frac{1}{2}c_1\lambda(1-e)\right)z_1^4 - \frac{\lambda}{4}x_1^4.\end{aligned} \tag{4.21}$$

By giving a further requirement to the parameter $\lambda \geq \max\{(9k^2+18c_1^4k^2+9c_1^2k^2)/c_1(1-e), 6l^2\}$ and choosing the control

$$u = u_1 + u_2, \tag{4.22}$$

we have

$$\mathcal{L}V_2 \leq -\frac{\lambda}{8}x_1^4 - \frac{1}{4}c_1\lambda(1-e)z_1^4 - c_2z_2^4 \leq -cV_2, \tag{4.23}$$

where $c = \min\{\lambda/2, c_1\lambda(1-e), 4c_2\}$ and

$$\begin{aligned}u_1 &= -c_2z_2 + x_2v - c_1\lambda x_3 - \frac{3}{4}l^2\phi^2(x_1)z_2 \\ &\quad - \frac{9}{4}k^2x_1^8z_2 - 9c_1^2k^2z_2 - \frac{9}{4}c_1^4k^2z_2 - \frac{1}{d^3}4\lambda z_2 - 3k^2z_2, \\ u_2 &= c_1\lambda z_1 - c_1\left(\frac{\varphi_1}{x_1}\right)^2 x_3 + c_1\left(\frac{\varphi_1}{x_1}\right)^2 z_1 - \frac{9}{4}c_1^2k^2z_2 - \frac{1}{4}k^2z_2.\end{aligned} \tag{4.24}$$

Summing up all the requirements to λ leads to

$$\lambda \geq \max\left\{2k^2, \frac{31k^2 + 4c_1k^2 + 24c_1^2k^2}{2c_1(1-e)}, \frac{9k^2 + 18c_1^4k^2 + 9c_1^2k^2}{c_1(1-e)}, 6l^2\right\}. \tag{4.25}$$

Remark 4.1. It is noteworthy that the terms in control u is separated into two groups. The terms caused by the state scaling are put in u_2, in other words, if the transformation $z_1 = x_2/x_1$ is replaced with nonscaling one $z_1 = x_2$, the terms in u_1 will still remain in u. This will be used in the subsequent section.

4.4. Stability Analysis

It is position to give stability conclusion for the case of $x_1(t_0) \neq 0$.

Theorem 4.2. *Under assumptions (A1) and (A2), for every* $x_1(t_0) \neq 0$ *and any* $x_2(t_0)$, $x_3(t_0)$, *with an appropriate choice of the design parameters* λ *and* c_1, *the closed-loop system consists of* (3.1), (3.2), (4.3), *and* (4.22) *has a unique solution which is 4th moment exponential stable.*

Proof. The existence and uniqueness of solution comes from (4.18) and (4.23), according to Lemma 2.5. It can also be further concluded that

$$\limsup_{t\to\infty} \frac{1}{t} \log E|z_i|^4 < 0 \quad (i = 1, 2) \tag{4.26}$$

for each $x_0 \in \mathbb{R}^n$. From (4.6), (4.7), and (4.26), we have

$$\limsup_{t\to\infty} \frac{1}{t} \log E|x_2|^4 \le \limsup_{t\to\infty} \frac{1}{t} \left(\log|x_1(t)| - 4\lambda(t - t_0)\right) + \limsup_{t\to\infty} \frac{1}{t} \log E|z_1|^4 < 0. \tag{4.27}$$

From (4.6), (4.9), (4.12), and (4.26), we have

$$\limsup_{t\to\infty} \frac{1}{t} \log E|x_3|^4 = \limsup_{t\to\infty} \frac{1}{t} \log E|z_2 + c_1 z_1|^4 < 0. \tag{4.28}$$

Combining (4.6), (4.27), and (4.28) gives

$$\limsup_{t\to\infty} \frac{1}{t} \log E|(x_1, x_2, x_3)|^4 < 0, \tag{4.29}$$

which completes the proof. □

5. Controller Design to Drive x_1 Away from Zero

Considering the transformation $z_1 = x_2/x_1$, the control u given by (4.22) will escape to infinite for an initial state with element $x_1(t_0) = 0$. The first thing before the controller (4.22) does work is to drive the state $x_1(t)$ away from zero in a small distance denoted by r. For a given $r > 0$, define a stopping time $\tau_r = \inf\{t : t \ge t_0, |x_1(t)| \ge r\}$. To let the state x_1 leave zero, the control v can be chosen as

$$v = -\lambda \tag{5.1}$$

during $[t_0, \tau_r]$, where λ is the same design parameter used in (4.25) (some explanation will be given latter). In this case, the x_1-subsystem becomes

$$dx_1(t) = -\lambda dt + \varphi_1 dW_1. \tag{5.2}$$

By defining $\overline{\tau}_r = \inf\{t : t \ge t_0, x_1(t) \le -r\}$, the expectation of τ_r satisfies $E(\tau_r - t_0) \le E(\overline{\tau}_r - t_0) = r/\lambda$, therefore, $P(\tau_r - t_0 \ge T) \le r/\lambda T$, which implies that

$$P(\tau_r = \infty) = 0, \quad \forall r > 0. \tag{5.3}$$

The existence and uniqueness of solution of x_1-subsystem in $[t_0, \tau_{\mathrm{r}}]$ comes from assumption (A1) and Lemma 2.3. Since during the interval $[t_0, \tau_r]$, the controller (4.22) cannot be used. A new scheme for u is expected to bound the states x_2 and x_3 in a neighborhood of the origin in this interval when x_1 is being driven away from the origin. Substituting $v = -\lambda$ into the last two equations of (1.1) gives

$$\begin{aligned} dx_2 &= -\lambda x_3 dt + x_3 \varphi_1 dW_1, \\ dx_3 &= (u + \lambda x_2) dt + \varphi_2 dW_2 - x_2 \varphi_1 dW_1. \end{aligned} \tag{5.4}$$

Since (5.4) is a standard strict-feedback form, viewing x_1 as an external bounded input, backstepping controller can be designed to make the states x_2 and x_3 to be bounded in probability in $[t_0, \tau_r]$.

Introduce the transformation

$$z_1 = x_2, \qquad z_2 = x_3 - \alpha, \tag{5.5}$$

which implies that

$$\begin{aligned} dz_1 &= -\lambda x_3 dt + \varphi_1 x_3 dW_1, \\ dz_2 &= (u - x_2 v + c_1 \lambda x_3) dt + \varphi_2 dW_2 + \left(-x_2 \varphi_1 - c_1 \varphi_1 x_3\right) dW_1, \end{aligned} \tag{5.6}$$

where $\alpha = c_1 z_1$ is used as in (4.12) with a design parameter $c_1 > 0$. A careful observation indicates that all the terms in (5.6) have the corresponding terms in (4.8) and (4.17), that is, if φ_1 / x_1 in the latter is replaced with φ_1, then we can obtain the terms in the former.

In $[t_0, \tau_r]$, we have $|x_1| \le r$. To design controller u to guarantee the boundedness of x_2 and x_3, a candidate Lyapunov function is given as follows:

$$V = \frac{1}{4} z_1^4 + \frac{1}{4} z_2^4. \tag{5.7}$$

Just for simplicity, we will design the controller as consistent as possible with state-scaling case in the proceeding section. By selecting $r \le 1$, according to assumption (A1), we can see that in the nonscaling case, we have $\varphi_1^2 \le k^2 r^2 \le k^2$, which is corresponding to $(\varphi_1 / x_1)^2 \le k^2$ used in state-scaling case (4.13). Comparing (5.5)–(5.7) with the corresponding equalities in the proceeding section, it can be found that, in the nonscaling case, some terms used in (4.22) (that are included in u_2) disappear and the others (that are contained in u_1) have the same forms with the same or milder requirements to the parameters c_1 and λ. Therefore, by choosing

$$u = u_1, \tag{5.8}$$

we have

$$\mathcal{L} V \le \frac{3}{4} l^2 x_1^4 - \frac{1}{4} c_1 \lambda (1 - e) z_1^4 - c_2 {z_2}^4 \le -\overline{c} V + d_c, \tag{5.9}$$

where $d_c = (3/4) l^2 r^4$ and $\overline{c} = \min\{c_1 \lambda (1 - e), 4 c_2\}$, which implies that

$$EV(z(t)) \le e^{-\overline{c}(t - t_0)} EV(z(t_0)) - \frac{d_c}{\overline{c}} e^{-\overline{c}(t - t_0)} + \frac{d_c}{\overline{c}}, \quad t \in [t_0, \tau_r]. \tag{5.10}$$

The stability analysis before τ_r can be included in the following result.

Theorem 5.1. *Under assumptions (A1) and (A2), for every* $x_1(t_0) = 0$ *and any* $x_2(t_0), x_3(t_0)$*, for any* $0 < r \leq 1$*, with an appropriate choice of the design parameters* λ *and* c_1*, the closed-loop system consists of* (3.1), (3.2), (5.1)*, and* (5.8) *has a unique solution, and all the signals are bounded in probability in the interval* $[t_0, \tau_r]$.

Proof. According to Lemma 2.2, the existence and uniqueness of z_1 and z_2 in $[t_0, \tau_r]$ come from (5.7) and (5.9). Noting the existence and uniqueness of x_1, (4.7), (4.9), and (4.12), the existence and uniqueness of x_2 and x_3 can be concluded on $[t_0, \tau_r]$. Following the same line, the boundedness of x_i ($i = 1, 2, 3$) on $t \in [t_0, \tau_r]$ can be obtained, which complete the proof. □

6. Design of Switching Controller

Since $|x_1(\tau_r)| = r > 0$, at the stochastic moment $t = \tau_r$, we switch the control laws v and u from (5.1) and (5.8) to (4.3) and (4.22), respectively. According to Theorem 4.2, the solution of the closed-loop system converges to the origin with exponential rate on $[\tau_r, \infty)$ for any $r > 0$. A switching control scheme on $[t_0, \infty)$ can be given as

$$\begin{gathered} v = -\lambda(1 + (x_1 - 1)s), \qquad u = u_1 + u_2 s, \\ z_1 = \frac{x_2}{1 + (x_1 - 1)s}, \qquad z_2 = x_3 - \alpha, \quad \alpha = c_1 z_1, \end{gathered} \tag{6.1}$$

where the switching signal is defined by

$$s(t) = \begin{cases} 0, & t \in [0, \tau_r), \\ 1, & t \in [\tau_r, \infty). \end{cases} \tag{6.2}$$

By summing up the above arguments, the main result in this paper can be presented now.

Theorem 6.1. *Under assumptions (A1) and (A2), for* $x_1(t_0) = 0$ *and any* $x_2(t_0)$, $x_3(t_0)$*, with an appropriate choice of the design parameters* λ, c_1 *and* r*, the closed-loop system consists of* (3.1), (3.2)*, and* (6.1) *has a unique solution on* $[t_0, \infty)$*, which is 4th moment exponential stable.*

Proof. For any $0 < r \leq 1$, the existence of a.s. finite stopping time τ_r can be concluded from (5.3). The existence and uniqueness of solution of the closed-loop system can be proved by Theorem 5.1 on $[t_0, \tau_r]$ and by Theorem 4.2 on $[\tau_r, \infty)$, respectively. In the interval $[t_0, \tau_r]$, there holds $|x_1| \leq r$, and from (5.10), we have $EV(z_1^4(t) + z_2^4(t)) \leq \overline{d}$, for all $t \in [t_0, \tau_r]$, where $\overline{d} = E((1/4)(z_1^4(t_0) + z_2^4(t_0))) + (d_c/\overline{c})$, which implies that there exists a constant $\overline{\overline{d}}$ such that

$$EV\left(x_1^4(t) + x_2^4(t) + x_3^4(t)\right) \leq \overline{\overline{d}}, \quad \forall t \in [t_0, \tau_r]. \tag{6.3}$$

In the interval $[\tau_r, \infty)$, similar to (4.29), we have

$$\limsup_{t \to \infty} \frac{1}{t} \log E|(x_1, x_2, x_3)|^4 < 0. \tag{6.4}$$

Combining (6.3) and (6.4) completes the proof. □

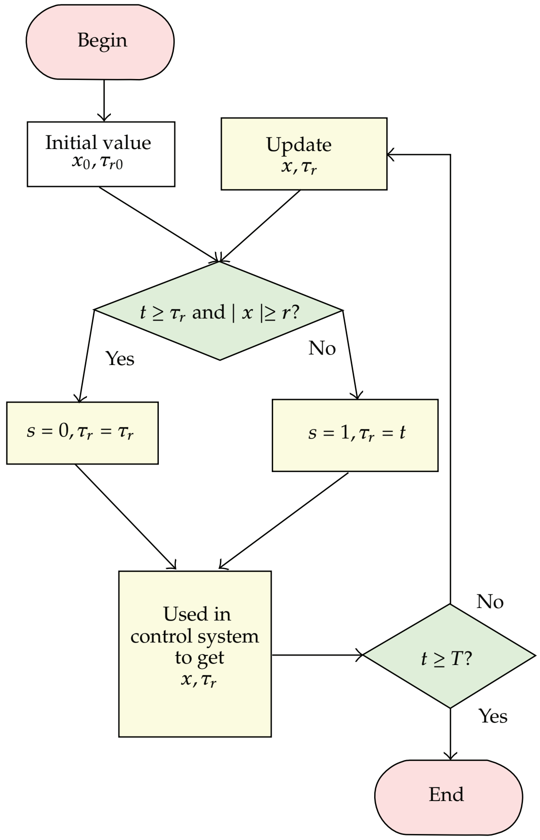


Figure 2: The Logic operation of switching.

A new question about the performing of switching signal comes forth. One scheme is presented as follows in a discrete-time form. Suppose that the running time interval is $[0, T]$ and every step equals to $\Delta \ll T$. Initial step: begin with $t = 0$, $\tau_1 = T$, and $\tau_2 = 0$. Recursive steps: perform the following procedures in turn unless otherwise stated. (a) Write down the value of $x = x(t)$ and let $t = t + \Delta$. (b) If $t > T$, then turn to (g), otherwise, perform the following calculation. (c) If $t \leq \tau_1$, then we have $t_m = t$, otherwise, we have $t_m = \tau_1$; we have $s = 0$, $\tau_1 = \tau_1$ and restore $\tau_2 = t$, otherwise, we have $s = 1$, $\tau_2 = \tau_2$ and $\tau_1 = t_m$. (e) Submitting s into control (6.1) and resolve the response $x(t)$ of closed-loop system. (f) Turn to (a). (g) Output the observed value $\tau_r = \tau_2$. (h) End the procedure. The procedure is described in Figure 2.

It should be pointed out that the switching strategy will lead to trembling phenomenon. In practice, to eliminate the trembling, the switching signal given by (6.2) can be replaced by a continuous one which depends on the measurement of τ_r. The above logic method will be used in the forthcoming simulation.

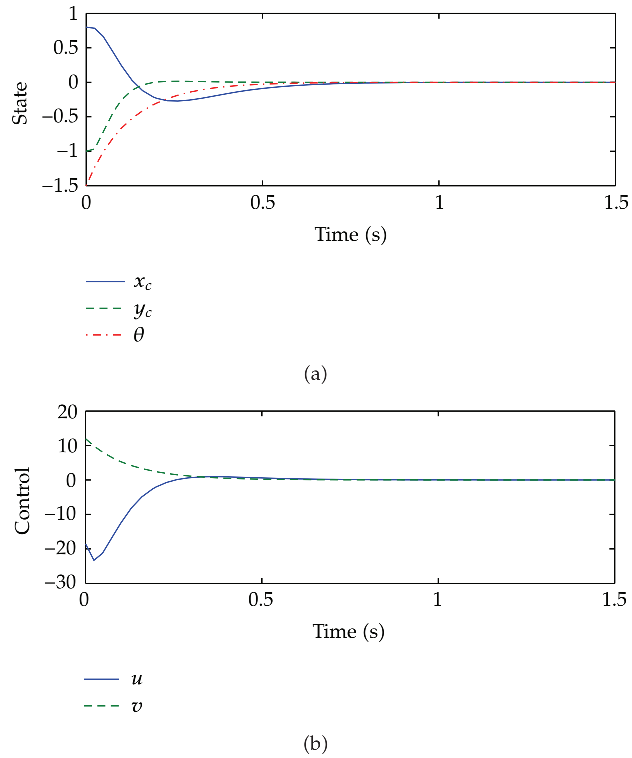


Figure 3: The responses of closed-loop system with nonzero initial heading angle.

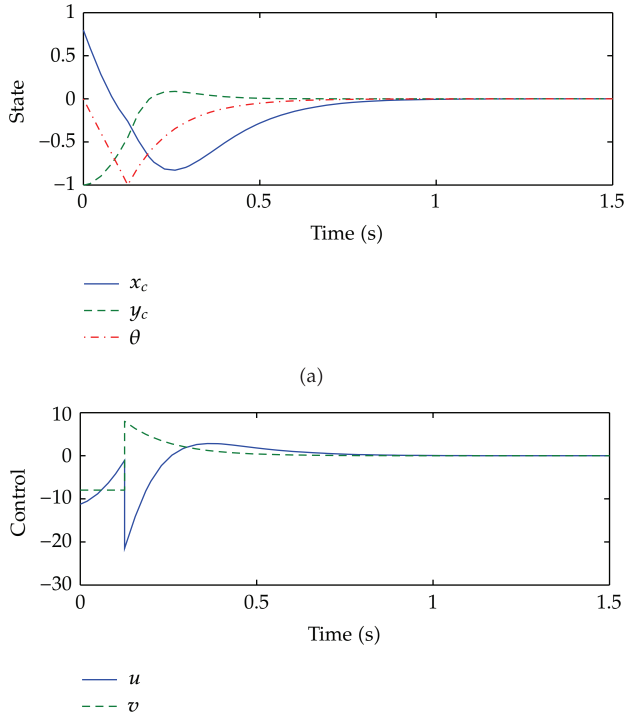
7. Simulation

Consider system (3.1) with $\varphi_1 = k\theta$ and $\varphi_2 = l\theta^2$. By letting $\phi(\theta) = \theta$, the assumptions (A1) and (A2) can be easily verified. As pointed out by [14, page 63], system (3.1) is an idealization of the following system:

$$\begin{aligned} \dot{\theta} &= v + \varphi_1(\theta)N_1, \\ \dot{x}_c &= \left(u + \varphi_2(\theta)N_2\right)\cos\theta, \\ \dot{y}_c &= \left(u + \varphi_2(\theta)N_2\right)\sin\theta \end{aligned} \tag{7.1}$$

with white noises N_1 and N_2, which is formally obtained by replacing "$dW_i(t)/dt$" by $N_i(t)$. To give approximate simulation using ordinary differential equation algorithm, system (3.1) is replaced by (7.1), where the power of each N_i equals to 1.

The following two cases are to be analyzed: (1) $\theta(0) = -1.5$, $x_c(0) = 0.8$, $y_c(0) = -1$, $k = 0.1$ and $l = 1$; (2) $\theta(0) = 0$, $x_c(0) = 0.8$, $y_c(0) = -1$, $k = 0.1$ and $l = 1$. For the first case, the state-feedback control law is given by (6.1) (not (4.3) and (4.22)) with the design parameters $d = 0.8$, $e = 0.7$, $c_1 = (1/e)(1 + (3/4)d)$, $c_2 = 3$, $r = 1$, and λ satisfying the equality of (4.25). Figure 3 demonstrates that the state of the closed-loop system can be regulated to the origin with exponential rate (in almost surely sense) without switching. For the second case, the same control (6.1) with the same design parameters as in the first case is given. From Figure 4,



(a)

(b)

Figure 4: Responses of closed-loop system with zero initial heading angle by using switching (6.2).

we can see that switching happens at the moment $\tau_r \approx 0.1258$. The state of the closed-loop system can be driven to the origin with exponential rate after moment τ_r (in almost surely sense). To eliminate the trembling phenomenon, the switching signal s given by (6.2) can be replaced by a continuous one. Figure 5 describes the responses of the closed-loop system of Case 2 with the following $s(t)$:

$$s(t) = \begin{cases} 0, & t \in [0, \tau_r), \\ \dfrac{2}{\pi} \arctan(600(t - \tau_r)), & t \in [\tau_r, \infty). \end{cases} \tag{7.2}$$

Comparison of Figure 4 with Figure 5 indicates that control magnitude in the latter is milder than that in the former.

8. Conclusions

A global exponential stabilization controller has been designed for nonholonomic mobile robot with stochastic disturbance by using the integrator backstepping procedure based on the state-scaling technique. There are several interesting problems of the controller design for the same stochastic nonholonomic mobile robot, for example, the tracking control and

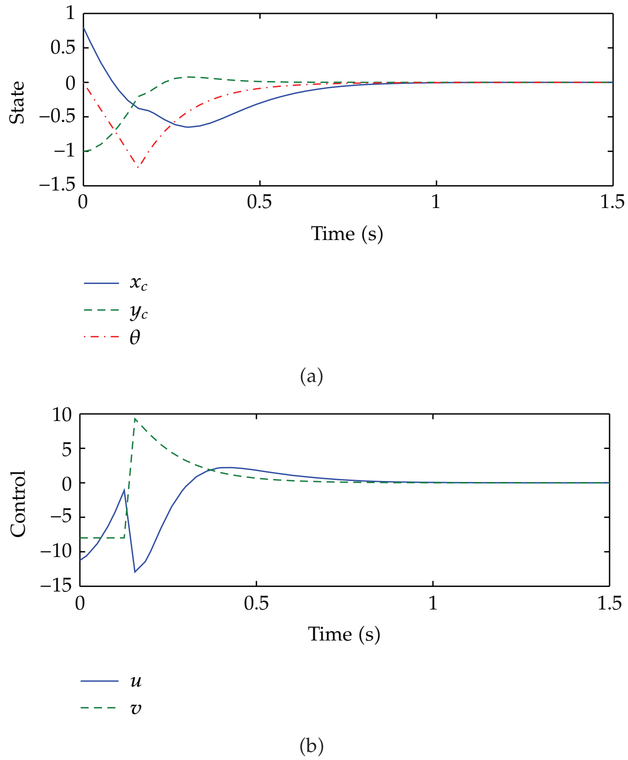


Figure 5: Responses of closed-loop system with zero initial heading angle by using switching (7.2).

the adaptive control, and the further extensions to more general chained-form nonholonomic systems with stochastic disturbance. These directions are all under the current research.

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