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Research Article

Mixed Mortar Element Method for P_1^{NC}/P_0 Element and Its Multigrid Method for the Incompressible Stokes Problem

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We discuss a mortar-type P_1^{NC}/P_0 element method for the incompressible Stokes problem. We prove the inf-sup condition and obtain the optimal error estimate. Meanwhile, we propose a $\mathcal{W}$-cycle multigrid for solving this discrete problem and prove the optimal convergence of the multigrid method, that is, the convergence rate is independent of the mesh size and mesh level. Finally, numerical experiments are presented to confirm our theoretical results.

1. Introduction

As we all know, the application of viscous incompressible flows is of considerable interest. For example, the design of hydraulic turbines, or rheologically complex flows appears in many processes which are involved in plastics and molten metals. Therefore, in recent decades, many engineers and mathematicians have concentrated their efforts on the Stokes problem, especially the problem that can be handled by the finite element methods. In [1], Girault and Raviart provided a fairly comprehensive treatment of the most recent development in the finite-element method. Some new divergence-free elements were proposed to solve Stokes problem recently (see [2, 3] and others). Due to this development in the finite-element theory, many numerical algorithms were established to solve the Stokes equations. Among these algorithms, multigrid methods and domain decomposition methods for the Stokes equations are very prevalent. In [4], the authors constructed an efficient smoother. Based on the smoother, the multigrid methods have been greatly developed (see [5, 6]). Meanwhile, a FETI-DP method was extended to the incompressible Stokes equations in [7, 8], a BDDC algorithm for this problem was developed too in [9] and others.

In the last twenty years, mortar element methods have attracted much attention and it was first introduced in [10]. This method is a nonconforming domain decomposition method with nonoverlapping subdomains. In mortar finite-element methods, the meshes on adjacent subdomains may not match with each other across the interfaces of the subdomains. The coupling of the finite-element functions on adjacent meshes is done by enforcing the so-called mortar condition across the interfaces (see [10] for details). There have been considerable researches on the mortar element methods (see [11–13] and others).

In [12], the author discussed the mortar-type conforming element (P_2/P_1 element) method for the Stokes problem, and then Chen and Huang proposed the mortar-type nonconforming element (Q_1^{rot}/Q_0 element) method for the problem in [5]. It is well known that the rotated Q_1 element is a rectangle element, and it is not a flexible finite element since it is only suitable for the rectangular or L-shape-bounded domain. Moreover, the rotated Q_1 element is a quadratic element and is not as convenient as the linear elements in calculating.

In this paper, we apply the mortar element method coupling with P_1 nonconforming finite element to the incompressible Stokes problem. The P_1 nonconforming finite element is a triangular element and it is suitable for more extensive polygonal domain than the rotated Q_1 element. Moreover, owing to its linearity, the computational work is less than the rotated Q_1 element. We prove the so-called inf-sup condition and obtain the optimal error estimate. When solving the discrete problem, we also present a $\mathcal{W}$-cycle multigrid algorithm, but the analysis about the convergence of the multigrid is different from [5]. We only prove that the prolong operator satisfies the criterion which proposed in [14] and we obtain the optional convergence with simpler analysis than that in [5]. Meanwhile, we do some numerical experiments which were realized in [5]. From numerical results, we note that the number of iterations is less than the rotated Q_1 element method when achieving the same relative error.

The rest of this paper is organized as follows. In Section 2, we review the Stokes problem and introduce the mortar element method for P_1 nonconforming element. Section 3 gives verification of the inf-sup condition and error estimate. The multigrid algorithm and the convergence analysis are given in Sections 4 and 5, respectively. The last section presents some numerical experiments. Throughout this paper, we denote by “C” a universal constant which is independent from the mesh size and level, whose values can differ from place to place.

2. Preliminaries

We only consider the incompressible flow problem, the steady-state Stokes problem, so that we can compare the results with those in [5].

The partial differential equations of the model problem is

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= f \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where Ω is bounded convex polygonal domain in R^2, $\mathbf{u}$ represents the velocity of fluid, p is pressure, and $\mathbf{f}$ is external force. Define

$$L_0^2(\Omega) = \left\{ q \in L^2(\Omega) \mid \int_\Omega q\, dx = 0 \right\}. \tag{2.2}$$

The mixed variational formulation of problem (2.1) is to find $(\mathbf{u}, p) \in (H_0^1(\Omega))^2 \times L_0^2(\Omega)$ such that

$$\begin{gathered} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = \langle \mathbf{f}, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in \left(H_0^1(\Omega) \right)^2, \\ b(\mathbf{u}, q) = 0, \quad \forall q \in L_0^2(\Omega), \end{gathered} \tag{2.3}$$

where the bilinear formulations $a(\cdot,\cdot)$ on $(H_0^1(\Omega))^2 \times (H_0^1(\Omega))^2$, $b(\cdot,\cdot)$ on $(H_0^1(\Omega))^2 \times L_0^2(\Omega)$ and the dual parity $\langle \cdot,\cdot \rangle$ on $(L^2(\Omega))^2 \times (L^2(\Omega))^2$ are given, respectively, by

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v}\, dx, \qquad b(\mathbf{v}, q) = -\int_\Omega \operatorname{div}\ \mathbf{v} q\, dx, \qquad \langle \mathbf{f}, \mathbf{v} \rangle = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, dx. \tag{2.4}$$

It is well known that the bilinear form $b(\cdot,\cdot)$ satisfies the inf-sup condition, that is, there exists a positive constant β for any $q \in L_0^2(\Omega)$ such that

$$\sup_{\mathbf{v} \in \left(H_0^1(\Omega)\right)^2} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{(H^1(\Omega))^2}} \geq \beta \|q\|_{L^2(\Omega)}. \tag{2.5}$$

According to the assumption on Ω and the saddle point theory in [15], we know that if $\mathbf{f} \in (L^2(\Omega))^2$, then there exists a unique solution $(\mathbf{u}, p) \in (H_0^1(\Omega) \bigcap H^2(\Omega))^2 \times (L_0^2(\Omega) \bigcap H^1(\Omega))$ satisfying

$$\|\mathbf{u}\|_{(H^2(\Omega))^2} + \|p\|_{H^1(\Omega)} \leq C \|\mathbf{f}\|_{(L^2(\Omega))^2}. \tag{2.6}$$

We now introduce a mortar finite-element method for solving problem (2.3). First, we partition Ω into nonoverlapping polygonal subdomains such that

$$\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega}_i, \quad \Omega_i \bigcap \Omega_j = \phi \quad \text{if } i \neq j. \tag{2.7}$$

They are arranged, so that the intersection of $\Omega_i \bigcap \Omega_j$ for $i \neq j$ is an empty set or an edge, or a vertex; that is, the partition is geometrically conforming. Denote by γ_m the common open edge to Ω_i and Ω_j, then the interface $\Gamma = \bigcup_{i=1}^{N} \partial\Omega_i \setminus \partial\Omega$ is broken into a set of disjoint open straight segments γ_m $(1 \leq m \leq M)$, that is,

$$\Gamma = \bigcup_{m=1}^{M} \overline{\gamma}_m, \quad \gamma_m \bigcap \gamma_n = \phi \quad \text{if } m \neq n. \tag{2.8}$$

By $\gamma_{m(i)}$ we denote an edge of Ω_i called mortar and by $\delta_{m(j)}$ an edge of Ω_j that geometrically occupies the same place called nonmortar.

With each Ω_i, we associate a quasiuniform triangulation $\mathcal{T}_h(\Omega_i)$ made of elements that are triangles. The mesh size h_i is the diameter of largest element in $\mathcal{T}_h(\Omega_i)$. We define $h = \max_{1\le i\le N} h_i$, $\mathcal{T}_h = \bigcup_{i=1}^{N} \mathcal{T}_h(\Omega_i)$. Let CR nodal points be the nonconforming nodal points, that is, the midpoints of the edges of the elements in $\mathcal{T}_h(\Omega_i)$. Denote the set of CR nodal points belonging to $\overline{\Omega}_i$, $\partial\Omega_i$ and $\partial\Omega$ by $\Omega_{ih}^{\mathrm{CR}}$, $\partial\Omega_{ih}^{\mathrm{CR}}$ and $\partial\Omega_h^{\mathrm{CR}}$, respectively.

For each triangulation $\mathcal{T}_h(\Omega_i)$ on Ω_i, the P_1 nonconforming element velocity space and piecewise constant pressure space are defined, respectively, as follows:

$$\begin{aligned} X_h(\Omega_i) = \Big\{ &\mathbf{v}_i \in \left(L^2(\Omega_i)\right)^2 |\mathbf{v}_i|_\tau \text{ is linear } \forall \tau \in \mathcal{T}_h(\Omega_i), \\ &\mathbf{v}_i \text{ is continuous at midpoint of } \tau, \mathbf{v}(m_i) = \mathbf{0}\ \forall m_i \in \partial\Omega_h^{\mathrm{CR}} \Big\}, \\ Q_h(\Omega_i) = \Big\{ &q_i \in L^2(\Omega_i)\ \left|q_i\right|_\tau \text{ is a constant for } \tau \in \mathcal{T}_h(\Omega_i) \Big\}. \end{aligned} \tag{2.9}$$

Then the product space $\widetilde{X}_h(\Omega) = \prod_{i=1}^{N} X_h(\Omega_i)$ is a global P_1 nonconforming element space for $\mathcal{T}_h$ on Ω.

For any interface $\gamma_m = \gamma_{m(i)} = \delta_{m(j)}$ $(1 \le m \le M)$, there are two different and independent triangulations $\mathcal{T}_h(\gamma_{m(i)})$ and $\mathcal{T}_h(\delta_{m(j)})$, which produce two sets of CR nodes belonging to γ_m: the midpoints of the elements belonging to $\mathcal{T}_h(\gamma_{m(i)})$ and $\mathcal{T}_h(\delta_{m(j)})$ denoted by $\gamma_{m(i)}^{\mathrm{CR}}$ and $\delta_{m(j)}^{\mathrm{CR}}$, respectively.

In order to introduce the mortar condition across the interfaces γ_m, we need the auxiliary test space $S_h(\delta_{m(j)})$ which is defined by

$$\begin{aligned} &S_h(\delta_{m(j)}) \\ &\quad = \left\{ \mathbf{v} \in \left(L^2(\delta_{m(j)})\right)^2 | \, \mathbf{v} \text{ is piecewise constant on elements of triangulation } \mathcal{T}_h(\delta_{m(j)}) \right\}. \end{aligned} \tag{2.10}$$

For each nonmortar edge $\delta_{m(j)}$, define the L^2-projection operator: $Q_{h,\delta_{m(j)}} : (L^2(\gamma_m))^2 \to S_h(\delta_{m(j)})$ by

$$\left(Q_{h,\delta_{m(j)}}\mathbf{v}, \mathbf{w}\right)_{L^2(\delta_{m(j)})} = (\mathbf{v}, \mathbf{w})_{L^2(\delta_{m(j)})}, \quad \forall \mathbf{w} \in S_h(\delta_{m(j)}). \tag{2.11}$$

Now we can define the mortar-type P_1 nonconforming element space as follows:

$$\begin{aligned} X_h(\Omega) = \Big\{ &\mathbf{v} \in \widetilde{X}_h(\Omega) \mid \mathbf{v}|_{\Omega_i} \in X_h(\Omega_i),\ Q_{h,\delta_{m(j)}}\left(\mathbf{v}|_{\delta_{m(j)}}\right) = Q_{h,\delta_{m(j)}}\left(\mathbf{v}|_{\gamma_{m(i)}}\right), \\ &\forall \gamma_m = \gamma_{m(i)} = \delta_{m(j)} \subset \Gamma \Big\}, \end{aligned} \tag{2.12}$$

the condition of the equality in (2.12) which the velocity function $\mathbf{v}$ satisfies is called mortar condition.

The global P_0 element pressure space on Ω is defined by

$$Q_h(\Omega) = \left\{ q \in L_0^2(\Omega) \mid q\big|_{\Omega_i} \in Q_h(\Omega_i) \right\}. \tag{2.13}$$

We now establish the discrete system for problem (2.3) based on the mixed finite-element spaces $X_h(\Omega) \times Q_h(\Omega)$.

We first define the following formulations:

$$\begin{aligned} a_{h_i}\left(\mathbf{u}_h^i, \mathbf{v}_h^i\right) &= \sum_{\tau \in \mathcal{T}_h(\Omega_i)} \int_\tau \nabla \mathbf{u}_h^i \cdot \nabla \mathbf{v}_h^i \, dx, \quad \forall \mathbf{u}_h^i, \mathbf{v}_h^i \in X_h(\Omega_i), \\ b_{h_i}\left(\mathbf{v}_h^i, p_h^i\right) &= -\sum_{\tau \in \mathcal{T}_h(\Omega_i)} \int_\tau \operatorname{div} \mathbf{v}_h^i \cdot p_h^i \, dx, \quad \forall \mathbf{v}_h^i \in X_h(\Omega_i),\ \forall p_h^i \in Q_h(\Omega_i). \end{aligned} \tag{2.14}$$

Let

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = \sum_{i=1}^N a_{h_i}(\mathbf{u}_h, \mathbf{v}_h), \qquad b_h(\mathbf{v}_h, p_h) = \sum_{i=1}^N b_{h_i}(\mathbf{v}_h, p_h). \tag{2.15}$$

Then the discrete approximation of problem (2.3) is to find $(\mathbf{u}_h, p_h) \in X_h(\Omega) \times Q_h(\Omega)$ such that

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) &= \langle \mathbf{f}, \mathbf{v}_h \rangle, \quad \forall \mathbf{v}_h \in X_h(\Omega), \\ b_h(\mathbf{u}_h, q_h) &= 0, \quad \forall q_h \in Q_h(\Omega). \end{aligned} \tag{2.16}$$

In the next section, we prove that the discrete problem (2.16) has a unique solution and we obtain error estimate.

3. Existence, Uniqueness, and Error Estimate of the Discrete Solution

According to the Brezzi theory, the well-posedness of problem (2.16) depends closely on the characteristics of both bilinear forms $a_h(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$. We equip the space $X_h(\Omega)$ with the following norm:

$$\|\mathbf{v}\|_h^2 := \sum_{i=1}^N \|\mathbf{v}\|_{h,i}^2, \qquad \|\mathbf{v}\|_{h,i}^2 := a_{h_i}(\mathbf{v}, \mathbf{v}). \tag{3.1}$$

We can find in [1] that the local space family $\{X_h^0(\Omega_i), Q_h^0(\Omega_i)\}$ is div-stable; that is, there exists a constant $\widetilde{\beta}$ independent of h_i such that

$$\sup_{\widetilde{\mathbf{v}}_h \in X_h^0(\Omega_i)} \frac{b_h(\widetilde{\mathbf{v}}_h, \widetilde{q}_h)}{\|\widetilde{\mathbf{v}}_h\|_{h,i}} \geq \widetilde{\beta} \|\widetilde{q}_h\|_{L^2(\Omega_i)}, \quad \forall \widetilde{q}_h \in Q_h^0(\Omega_i), \tag{3.2}$$

where $X_h^0(\Omega_i) = \{\mathbf{v} \in X_h(\Omega_i) \mid \mathbf{v}(m_i) = \mathbf{0}, \forall m_i \in \partial\Omega_{i,h}^{\mathrm{CR}}\}, Q_h^0(\Omega_i) = Q_h(\Omega_i) \bigcap L_0^2(\Omega_i)$.

In order to prove that the global space family $X_h(\Omega) \times Q_h(\Omega)$ is div-stable, it is necessary to define the global spaces as

$$\check{Q}_h(\Omega) = \left\{ \check{q} = \prod_{i=1}^{N} \check{q}_i \in R^N, (\check{q}, 1) = \sum_{i=1}^{N} \check{q}_i |\Omega_i| = 0 \right\}. \tag{3.3}$$

We first prove that the family $\{X_h(\Omega), \check{Q}_h(\Omega)\}$ is div-stable.

Lemma 3.1. *The following inf-sup condition holds:*

$$\sup_{\mathbf{v}_h \in X_h(\Omega)} \frac{b_h(\mathbf{v}_h, \check{q})}{\|\mathbf{v}_h\|_h} \geq \check{\beta} \|\check{q}\|_{L^2(\Omega)} \quad \forall \check{q} \in \check{Q}_h(\Omega), \tag{3.4}$$

where the constant $\check{\beta}$ does not depend on h.

Proof. We decompose the space $(H_0^1(\Omega))^2$ by $(H_0^1(\Omega))^2 = \prod_{i=1}^N V(\Omega_i)(V(\Omega_i) = (H_0^1(\Omega))^2|_{\Omega_i})$ and define a local interpolation operator π_i: $V(\Omega_i) \to X_h(\Omega_i)$ as

$$\pi_i \mathbf{v}(m_i) = \frac{1}{|e_i|} \int_{e_i} \mathbf{v}\, ds, \tag{3.5}$$

where e_i is an edge of $\tau \in \mathcal{T}_h(\Omega_i)$, m_i is the midpoint of e_i. Then we can define a global interpolation operator $\pi : (H_0^1(\Omega))^2 \to \widetilde{X}_h(\Omega)$ as follows:

$$\pi \mathbf{v} = (\pi_1 \mathbf{v}_1, \pi_2 \mathbf{v}_2, \ldots, \pi_N \mathbf{v}_N), \quad \mathbf{v}_i = \mathbf{v}|_{\Omega_i}, \quad \forall \mathbf{v} \in \left(H_0^1(\Omega)\right)^2. \tag{3.6}$$

Define the operator $\Xi_{h,\delta_{m(j)}}$: $\widetilde{X}_h(\Omega) \to \widetilde{X}_h(\Omega)$ by

$$\left(\Xi_{h,\delta_{m(j)}} \mathbf{v}\right)(m_i) = \begin{cases} Q_{h,\delta_{m(j)}}\left(\mathbf{v}|_{\gamma_{m(i)}} - \mathbf{v}|_{\delta_{m(j)}}\right)(m_i), & m_i \in \delta_{m(j)}^{\mathrm{CR}}, \\ 0, & \text{otherwise}. \end{cases} \tag{3.7}$$

We can deduce that for any $\mathbf{v} \in (H_0^1(\Omega))^2$, there exists a $\mathbf{v}_h^* \in X_h(\Omega)$ satisfying

$$b\left(\mathbf{v} - \mathbf{v}_h^*, \check{q}\right) = 0. \tag{3.8}$$

In fact, we can set $\mathbf{v}_h^* = \pi\mathbf{v} + \sum_{m=1}^{M} \Xi_{h,\delta_{m(j)}}(\pi\mathbf{v})$. Obviously $\mathbf{v}_h^* \in X_h(\Omega)$ and

$$\begin{aligned}
b\left(\mathbf{v} - \mathbf{v}_h^*, \check{q}\right) &= -\sum_{i=1}^{N} \sum_{\tau\in\mathcal{T}_h(\Omega_i)} \int_\tau \operatorname{div}\left(\mathbf{v} - \mathbf{v}_h^*\right)\check{q}\, dx = -\sum_{i=1}^{N} \sum_{\tau\in\mathcal{T}_h(\Omega_i)} \int_{\partial\tau} \left(\mathbf{v} - \mathbf{v}_h^*\right) \cdot \mathbf{n}\check{q}\, ds \\
&= -\sum_{\tau\in\mathcal{T}_h} \int_{\partial\tau} (\mathbf{v} - \pi\mathbf{v}) \cdot \mathbf{n}\check{q} ds + \sum_{\tau\in\mathcal{T}_h} \int_{\partial\tau} \sum_{m=1}^{M} \Xi_{h,\delta_{m(j)}}(\pi\mathbf{v}) \cdot \mathbf{n}\check{q}\, ds \\
&= \sum_{j=1}^{M} \int_{\delta_{m(j)}} Q_{h,\delta_{m(j)}}\left((\pi\mathbf{v})|_{\gamma_{m(i)}} - (\pi\mathbf{v})|_{\delta_{m(j)}} \right) \cdot \mathbf{n}\, \check{q}_j\, ds \\
&= \sum_{j=1}^{M} \int_{\delta_{m(j)}} \left((\pi\mathbf{v})|_{\gamma_{m(i)}} - (\pi\mathbf{v})|_{\delta_{m(j)}} \right) \cdot \mathbf{n}\check{q}_j\, ds \\
&= \sum_{j=1}^{M} \int_{\delta_{m(j)}} \left(\mathbf{v}|_{\gamma_{m(i)}} - \mathbf{v}|_{\delta_{m(j)}} \right) \cdot \mathbf{n}\check{q}_j\, ds \\
&= 0.
\end{aligned} \tag{3.9}$$

On the other hand

$$\|\mathbf{v}^*\|_h \le \|\pi\mathbf{v}\|_h + \left\|\Xi_{h,\delta_{m(j)}}\pi\mathbf{v}\right\|_h, \tag{3.10}$$

by norm equivalence we have

$$\begin{aligned}
\|\pi\mathbf{v}\|_h^2 &= \sum_\tau |\pi\mathbf{v}|_{H^1(\tau)}^2 \le C\sum_\tau \left(\pi\mathbf{v}(m_i) - \pi\mathbf{v}(m_j)\right)^2 \\
&= C\sum_\tau \left(\frac{1}{|e_i|} \int_{e_i} \mathbf{v} ds - \frac{1}{|e_j|} \int_{e_j} \mathbf{v} ds \right)^2 \\
&= C\sum_\tau \left(\frac{1}{|e_i|} \int_{e_i} (\mathbf{v} - \overline{\mathbf{v}}) ds - \frac{1}{|e_j|} \int_{e_j} (\mathbf{v} - \overline{\mathbf{v}}) ds \right)^2 \\
&\le C\sum_\tau \left(\frac{1}{|e_i|^2} \left(\int_{e_i} (\mathbf{v} - \overline{\mathbf{v}}) ds \right)^2 + \frac{1}{|e_j|^2} \left(\int_{e_j} (\mathbf{v} - \overline{\mathbf{v}}) ds \right)^2 \right),
\end{aligned} \tag{3.11}$$

where m_i, m_j are the midpoints of the edges of τ, and $\overline{\mathbf{v}}$ is the integral average of $\mathbf{v}$ in τ, by Hölder inequality, trace theorem, and Friedrichs' inequality we can get

$$\begin{aligned}\frac{1}{|e_i|^2}\left(\int_{e_i}(\mathbf{v}-\overline{\mathbf{v}})ds\right)^2 &\le \frac{1}{|e_i|}\int_{e_i}(\mathbf{v}-\overline{\mathbf{v}})^2 ds \le Ch^{-1}\int_{\partial\tau}(\mathbf{v}-\overline{\mathbf{v}})^2 ds \\ &\le C\left(h^{-2}\int_{\tau}(\mathbf{v}-\overline{\mathbf{v}})^2 dx + |\mathbf{v}-\overline{\mathbf{v}}|^2_{H^1(\tau)}\right) \\ &\le C|\mathbf{v}|^2_{H^1(\tau)},\end{aligned} \tag{3.12}$$

and combining (3.11), we obtain

$$\|\pi\mathbf{v}\|_h \le C\|\mathbf{v}\|_h. \tag{3.13}$$

Using norm equivalence we derive

$$\begin{aligned}\left\|\Xi_{h,\delta_{m(j)}}\pi\mathbf{v}\right\|_h^2 &\le C\sum_{m_i\in\delta^{CR}_{m(j)}}\left(\Xi_{h,\delta_{m(j)}}\pi\mathbf{v}(m_i)\right)^2 \\ &= C\sum_{m_i\in\delta^{CR}_{m(j)}}\left(Q_{h,\delta_{m(j)}}\left((\pi\mathbf{v})|_{\gamma_{m(i)}} - (\pi\mathbf{v})|_{\delta_{m(j)}}\right)(m_i)\right)^2 \\ &\le Ch^{-1}\left\|Q_{h,\delta_{m(j)}}\left((\pi\mathbf{v})|_{\gamma_{m(i)}} - (\pi\mathbf{v})|_{\delta_{m(j)}}\right)(m_i)\right\|^2_{0,\gamma_m} \\ &\le Ch^{-1}\left\|(\pi\mathbf{v})|_{\gamma_{m(i)}} - (\pi\mathbf{v})|_{\delta_{m(j)}}\right\|^2_{0,\gamma_m} \\ &\le Ch^{-1}\left(\left\|(\pi\mathbf{v})|_{\gamma_{m(i)}} - \mathbf{v}|_{\delta_{m(j)}}\right\|^2_{0,\gamma_m} + \left\|\mathbf{v}|_{\delta_{m(j)}} - (\pi\mathbf{v})|_{\delta_{m(j)}}\right\|^2_{0,\gamma_m}\right) \\ &:= Ch^{-1}(K_1+K_2).\end{aligned} \tag{3.14}$$

From trace theorem and (3.13), it follows that

$$K_2 \le Ch\|\mathbf{v}\|^2_{h,j}. \tag{3.15}$$

So we only need to estimate K_1. Owing to $\mathbf{v}\in(H_0^1(\Omega))^2$, we then obtain

$$\left\|(\pi\mathbf{v})|_{\gamma_{m(i)}} - \mathbf{v}|_{\delta_{m(j)}}\right\|^2_{0,\gamma_m} = \left\|(\pi\mathbf{v})|_{\gamma_{m(i)}} - \mathbf{v}|_{\gamma_{m(i)}}\right\|^2_{0,\gamma_m} \le Ch\|\mathbf{v}\|^2_{h,i}. \tag{3.16}$$

The bounds in (3.15) and (3.16) lead to

$$\left\|\Xi_{h,\delta_{m(j)}}\pi\mathbf{v}\right\|_h^2 \le C\left(\|\mathbf{v}\|^2_{h,i} + \|\mathbf{v}\|^2_{h,j}\right), \tag{3.17}$$

which together with (3.13) and (3.17) give

$$\|\mathbf{v}^*\|_h \le C\|\mathbf{v}\|_{(H^1(\Omega))^2}. \tag{3.18}$$

Since $\{(H_0^1(\Omega))^2, L_0^2(\Omega)\}$ is div-stable, following (3.8) and (3.18), by Fortin rules, we have completed the proof of Lemma 3.1 □

Now we recall the following Brezzi theory about the existence, uniqueness, and error estimate for the discrete solution.

Theorem 3.2. *The bilinear forms $a_h(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$ have the following properties:*

(i) *$a_h(\cdot,\cdot)$ is continuous and uniformly elliptic on the mortar-type P_1 nonconforming space $X_h(\Omega)$, that is,*

$$\begin{aligned} &a_h(\mathbf{u}_h,\mathbf{v}_h) \le \|\mathbf{u}_h\|_h\|\mathbf{v}_h\|_h, \quad \forall \mathbf{u}_h,\mathbf{v}_h \in X_h(\Omega),\\ &a_h(\mathbf{v}_h,\mathbf{v}_h) \ge C\|\mathbf{v}_h\|_h^2, \quad \forall \mathbf{v}_h \in X_h(\Omega); \end{aligned} \tag{3.19}$$

(ii) *$b_h(\cdot,\cdot)$ is also continuous on the space family $X_h(\Omega)\times Q_h(\Omega)$, that is,*

$$b_h(\mathbf{v}_h,q) \le \|\mathbf{v}_h\|_h\|q\|_{L^2(\Omega)}, \quad \forall \mathbf{v}_h \in X_h(\Omega),\ q \in Q_h(\Omega); \tag{3.20}$$

(iii) *the family $\{X_h(\Omega), Q_h(\Omega)\}$ satisfies the inf-sup condition, that is, there exists a constant β that does not depend on h of triangulation such that*

$$\sup_{\mathbf{v}\in X_h(\Omega)} \frac{b_h(\mathbf{v},q)}{\|\mathbf{v}\|_h} \ge \beta\|q\|_{L^2(\Omega)}, \quad \forall q \in Q_h(\Omega), \tag{3.21}$$

so the problem (2.16) *has a unique solution, and if one lets $(\mathbf{u},p)$, $(\mathbf{u}_h,p_h)$ be the solution of* (2.3) *and* (2.16), *respectively, where $(\mathbf{u},p) \in (H_0^1(\Omega))^2 \times L_0^2(\Omega)$, $\mathbf{u}|_{\Omega_k} \in (H^2(\Omega_k))^2$, $p|_{\Omega_k} \in H^1(\Omega_k)$, then*

$$\|\mathbf{u}-\mathbf{u}_h\|_h + \|p-p_h\|_{L^2(\Omega)} \le C\sum_{k=1}^N h_k\left(\|\mathbf{u}\|_{(H^2(\Omega_k))^2} + \|p\|_{H^1(\Omega_k)}\right). \tag{3.22}$$

Proof. The statements of Brezzi theory are that the properties (3.19)–(3.21) lead to the existence, uniqueness, and error estimate of the discrete solution. In [16], it is proven that $a_h(\cdot,\cdot)$ is continuous on $X_h(\Omega)$ and is elliptic with a constant uniformly bounded. Furthermore, it is straightforward that $b_h(\cdot,\cdot)$ is continuous on $X_h(\Omega)\times Q_h(\Omega)$. The point that needs verification is a uniform inf-sup condition (3.21), or equivalently that the family $\{X_h(\Omega)\times Q_h(\Omega)\}$ is div-stable.

Using local inf-sup condition (3.2) and the above lemma, arguing as the proof in Proposition 5.1 of [12], we have the global inf-sup condition (3.21). □

4. Numerical Algorithm

In this section, we present a numerical algorithm, that is, the $\mathcal{W}$-cycle multigrid method for the discrete system (2.16), and we prove the optional convergence of the multigrid method. We use a simpler and more convenient analysis method than that in [5].

In order to set the multigrid algorithm, we need only to change the index h of the partition $\mathcal{T}_h$ in Section 2 to be k, and let $\mathcal{T}_1$ be the coarsest partition. By connecting the opposite midpoints of the edges of the triangle, we split each triangle of $\mathcal{T}_1$ into four triangles and we refine the partition $\mathcal{T}_1$ into T_2. The partition $\mathcal{T}_2$ is quasi-uniform of size $h_2 = h_1/2$. Repeating this process, we get a sequence of the partition $\mathcal{T}_k(k = 1, 2, \ldots, L)$, each quasi-uniform of size $h_k = h_1/2^{k-1}$.

As in Section 2, with the partition $\mathcal{T}_k$, we define the mortar P_1 nonconforming element velocity space and P_0 element pressure space as X_k and Q_k, respectively. We can see that X_k $(k = 1, 2, \ldots, L)$ are nonnested, and Q_k $(k = 1, 2, \ldots, L)$ are nested. Furthermore, we denote the P_1 nonconforming element product space on Ω by $\widetilde{X}_k$.

Let $\{\varphi_k^i\}$ be the basis of X_k, and let $\{\psi_k^i\}$ be the basis of Q_k. For any $\mathbf{v}_k \in X_k$, $q_k \in Q_k$, we have the corresponding vector $\underline{v}_k = (\underline{v}_{k,i})$ and $\underline{q}_k = (\underline{q}_{k,i})$. We introduce the matrice A_k, B_k, and $\underline{f}_k$ having the entries $a_{k,ij} = a(\varphi_k^i, \varphi_k^j)$, $b_{k,ij} = b(\varphi_k^i, \psi_k^j)$, and $\underline{f}_{k,i} = (\mathbf{f}, \varphi_k^i)$, respectively. Then at level k, the problem (2.16) is equivalent to

$$\begin{pmatrix} A_k & B_k^T \\ B_k & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_k \\ \underline{p}_k \end{pmatrix} = \begin{pmatrix} \underline{f}_k \\ 0 \end{pmatrix}. \tag{4.1}$$

In the following of this section, we introduce our multigrid method; the key of this method is the intergrid transfer operator.

We first define the intergrid transfer operator on the product space, $L_{k-1}^k : \widetilde{X}_{k-1} \to \widetilde{X}_k$

$$L_{k-1}^k \mathbf{v}(m_i) = \begin{cases} \mathbf{v}(m_i), & m_i \in \kappa, \kappa \in T_{k-1}, \\ \dfrac{1}{2}\left(\mathbf{v}|_{\kappa_1}(m_i) + \mathbf{v}|_{\kappa_2}(m_i) \right), & m_i \in \partial\kappa_1 \bigcap \partial\kappa_2, \kappa_1, \kappa_2 \in T_k, \\ 0, & m_i \in \partial\Omega, \end{cases} \tag{4.2}$$

where κ, κ_i $(i = 1, 2)$ is the partition of $\mathcal{T}_{k-1}$, $\mathcal{T}_k$ respectively, $m_i \in \Omega_{k,i}^{\mathrm{CR}}$ $(1 \le i \le N)$.

Then we define the intergrid operator on the mortar P_1 nonconforming element velocity space, R_{k-1}^k:$X_{k-1} \to X_k$

$$R_{k-1}^k \mathbf{v} = L_{k-1}^k \mathbf{v} + \sum_{m=1}^{M} \Xi_{k,\delta_{m(j)}} L_{k-1}^k \mathbf{v}, \tag{4.3}$$

where $\Xi_{k,\delta_{m(j)}}$ is defined as (3.7).

On the P_0 element pressure space, we apply the natural injection operator J_{k-1}^k: $Q_{k-1} \to Q_k$, that is,

$$J_{k-1}^k = I. \tag{4.4}$$

Therefore, our prolongation operator on velocity space and pressure space can be written as

$$I_{k-1}^{k} = \left[R_{k-1}^{k}, J_{k-1}^{k}\right]. \tag{4.5}$$

Multigrid Algorithm

If $k = 1$, compute the $(\mathbf{u}_1, p_1)$ directly. If $k \geq 2$, do the following three steps.

Step 1. Presmoothing: for $j = 0, 1, \ldots, m_1 - 1$, solving the following problem:

$$\begin{aligned}\begin{pmatrix} \underline{u}_k^{j+1} \\ \underline{p}_k^{j+1} \end{pmatrix} &= \begin{pmatrix} \underline{u}_k^{j} \\ \underline{p}_k^{j} \end{pmatrix} - \begin{pmatrix} \alpha_k I_k & B_k^T \\ B_k & 0 \end{pmatrix}^{-1} \\ &\quad \times \left\{ \begin{pmatrix} A_k & B_k^T \\ B_k & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_k^{j} \\ \underline{p}_k^{j} \end{pmatrix} - \begin{pmatrix} \underline{f}_k \\ 0 \end{pmatrix} \right\},\end{aligned} \tag{4.6}$$

where α_k is a real number which is not smaller than the maximal eigenvalue of A_k.

Step 2. Coarse grid correction: find $(\tilde{\mathbf{u}}_{k-1}, \tilde{p}_{k-1}) \in X_{k-1} \times Q_{k-1}$, such that

$$\begin{aligned} &a_{k-1}(\tilde{\mathbf{u}}_{k-1}, \mathbf{v}_{k-1}) + b_{k-1}(\mathbf{v}_{k-1}, \tilde{p}_{k-1}) \\ &\quad = \left\langle \mathbf{f}, R_{k-1}^{k}\mathbf{v}_{k-1} \right\rangle - a_k\left(\mathbf{u}_k^{m_1}, R_{k-1}^{k}\mathbf{v}_{k-1}\right) - b_k\left(R_{k-1}^{k}\mathbf{v}_{k-1}, p_k^{m_1}\right), \quad \forall \mathbf{v}_{k-1} \in X_{k-1}, \\ &b_{k-1}(\tilde{\mathbf{u}}_{k-1}, q_{k-1}) = 0, \quad \forall q_{k-1} \in Q_{k-1}. \end{aligned} \tag{4.7}$$

Compute the approximation $(\mathbf{u}_{k-1}^{*}, p_{k-1}^{*})$ by applying $\mu \geq 2$ iteration steps of the multigrid algorithm applied to the above equations on level $k-1$ with zero starting value. Set

$$\mathbf{u}_k^{m_1+1} = \mathbf{u}_k^{m_1} + R_{k-1}^{k}\mathbf{u}_{k-1}^{*}, \qquad p_k^{m_1+1} = p_k^{m_1} + p_{k-1}^{*}. \tag{4.8}$$

Step 3. Postsmoothing: for $j = 0, 1, \ldots, m_2 - 1$ solving following problem:

$$\begin{aligned}\begin{pmatrix} \underline{u}_k^{m_1+j+2} \\ \underline{p}_k^{m_1+j+2} \end{pmatrix} &= \begin{pmatrix} \underline{u}_k^{m_1+j+1} \\ \underline{p}_k^{m_1+j+1} \end{pmatrix} - \begin{pmatrix} \alpha_k I_k & B_k^T \\ B_k & 0 \end{pmatrix}^{-1} \\ &\quad \times \left\{ \begin{pmatrix} A_k & B_k^T \\ B_k & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_k^{m_1+j+1} \\ \underline{p}_k^{m_1+j+1} \end{pmatrix} - \begin{pmatrix} \underline{f}_k \\ 0 \end{pmatrix} \right\},\end{aligned} \tag{4.9}$$

then, $(\mathbf{u}_k^{m_1+m_2+1}, p_k^{m_1+m_2+1})$ is the result of one iteration step.

For convenience, at level k the problem (2.16) can be written as followes: find $(\mathbf{u}_k, p_k) \in X_k \times Q_k$ such that

$$L_{h,k}((\mathbf{u}_k, p_k); (\mathbf{v}_k, q_k)) = F_k((\mathbf{v}_k, q_k)), \quad \forall (\mathbf{v}_k, q_k) \in X_k \times Q_k. \tag{4.10}$$

Since $L_{h,k}((\mathbf{u}_k, p_k); (\mathbf{v}_k, q_k))$ is a symmetric bilinear form on $X_k \times Q_k$, there is a complete set of eigenfunctions (ϕ_k^j, ψ_k^j), which satisfy

$$L_{h,k}\left((\mathbf{u}_k, p_k); (\mathbf{v}_k, q_k)\right) = \lambda_j\left[\left(\phi_k^j, \mathbf{v}_k\right)_0 + h^2\left(\psi_k^j, q_k\right)_0\right], \quad \forall(\mathbf{v}_k, q_k) \in X_k \times Q_k,$$
$$(\mathbf{v}_k, q_k) = \sum_j c_j\left(\phi_k^j, \psi_k^j\right). \tag{4.11}$$

In order to verify that our multigrid algorithm is optimal, we need to define a set of mesh-dependent norms. For each $k \geq 0$ we equip $X_k \times Q_k$ with the norm

$$|||(\mathbf{v}, q)|||_{0,k} = \|(\mathbf{v}, q)\|_{0,k} = \left(\|\mathbf{v}\|_{L^2(\Omega)}^2 + h_k^2\|q\|_{L^2(\Omega)}^2\right)^{1/2} = \left((\mathbf{v}, \mathbf{v})_k + h_k^2(q, q)_k\right)^{1/2}, \tag{4.12}$$

and define

$$|||(\mathbf{v}_k, q_k)|||_{s,k} = \left\{\sum_j |\lambda_j|^s |c_j|^2\right\}^{1/2}, \qquad \|\mathbf{v}\|_k^2 = \sum_\tau (\nabla\mathbf{v}, \nabla\mathbf{v})_k. \tag{4.13}$$

For our multigrid algorithm, we have the following optional convergence conclusion.

Theorem 4.1. *If $(\mathbf{u}, p)$ and $(\mathbf{u}_h^i, p_h^i)$ $(0 \leq i \leq m+1)$ are the solutions of problems* (2.16) *and* (4.10), *respectively, then there exists a constant $0 < \gamma < 1$ and positive integer m, all are independent of the level number k, such that*

$$\left|\left|\left|(\mathbf{u}, p) - \left(\mathbf{u}_k^{m+1}, p_k^{m+1}\right)\right|\right|\right|_{0,k} \leq \gamma |||(\mathbf{u}, p) - (\mathbf{u}_k^0, p_k^0)|||_{0,k}. \tag{4.14}$$

To prove this theorem, we give in the next section two basic properties for convergence analysis of the multigrid, that is, the smoothing property and approximation property.

5. Proof of Theorem 4.1

From the standard multigrid theory, the $\mathcal{W}$-cycle yields a h-independent convergence rate based on the following two basic properties.

We first show the smoothing property. By [[12] Theorem 5.1], we have the following.

Lemma 5.1 (smoothing property). *Assume that $\lambda_{\max}(A_k) \leq \alpha_k \leq C\lambda_{\max}(A_k)$, if the number of smoothing steps is m, then*

$$|||(\mathbf{u}_h^m - \mathbf{u}_h, p_h^m - p_h)|||_{2,k} \leq \frac{Ch^{-2}}{m}\left\|\mathbf{u}_h^0 - \mathbf{u}_h\right\|_{L^2(\Omega)}. \tag{5.1}$$

The property has been proved in [11].

Next, we prove the approximation property. We just apply the following conclusion in [14], which can simplify the complexity of theoretical analysis.

Lemma 5.2. *If the prolongation operator I_{k-1}^k defined in (4.5) satisfies the following criterion, Then, the approximation property in multigrid method holds and the multigrid algorithm converges optimally.*

(A.1) $\|\mathbf{v} - R_{k-1}^k \mathbf{v}\|_{L^2(\Omega)} \le C h_k \|\mathbf{v}\|_{k-1}, \ \forall \mathbf{v} \in X_{k-1}$,

(A.2) $\|J_{k-1}^k q\|_{L^2(\Omega)} \le C\|q\|_{L^2(\Omega)}, \ \forall q \in Q_{k-1}$,

(A.3) $|||(\mathbf{u}_k, p_k) - I_{k-1}^k(\mathbf{u}_{k-1}, p_{k-1})|||_{0,k} \le C h_k^2 (\|\mathbf{u}\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)})$.

where $(\mathbf{u}, p) \in (H_0^1(\Omega) \cap H^2(\Omega))^2 \times (L_0^2(\Omega) \cap H^1(\Omega))$ *is the solution of* (2.3) *with the force term* $\mathbf{f} \in (L^2(\Omega))^2$ *and* $(\mathbf{u}_{k-1}, p_{k-1})$, $(\mathbf{u}_k, p_k)$ *are the mixed finite element approximation of* $(\mathbf{u}, p)$ *at levels* $k-1$ *and* k, *respectively.*

This lemma has been proved in [14].

Lemma 5.3 (approximation property). *Let* $(I_{k-1}^k)^* : X_k \times Q_k \longrightarrow X_{k-1} \times Q_{k-1} (k \ge 1)$ *be defined as follows:*

$$L_{k-1}\Big(\Big(I_{k-1}^k\Big)^*(\mathbf{v}_k, q_k), (\mathbf{v}_{k-1}, q_{k-1})\Big), \\ = L_k\Big((\mathbf{v}_k, q_k), I_{k-1}^k(\mathbf{v}_{k-1}, q_{k-1})\Big), \forall (\mathbf{v}_{k-1}, q_{k-1}) \in X_{k-1} \times Q_{k-1}, (\mathbf{v}_k, q_k) \in X_k \times Q_k. \tag{5.2}$$

Then one has

$$\Big|\Big|\Big|(\mathbf{v}, q) - I_{k-1}^k\Big(I_{k-1}^k\Big)^*(\mathbf{v}, q)\Big|\Big|\Big|_{0,k} \le C h_k^2 |||(\mathbf{v}, q)|||_{2,k}, \quad \forall (\mathbf{v}, q) \in X_k \times Q_k. \tag{5.3}$$

Proof. By Lemma 5.2, we only need to prove our prolongation operator I_{k-1}^k that satisfies (A.1), (A.2), and (A.3).

For any $\mathbf{v} \in X_{k-1}$, the inequality (A.1) holds. In fact

$$\Big\|\mathbf{v} - R_{k-1}^k \mathbf{v}\Big\|_{L^2(\Omega)} \le \Big\|\mathbf{v} - L_{k-1}^k \mathbf{v}\Big\|_{L^2(\Omega)} + \Bigg\|\sum_{m=1}^{M} \Xi_{k,\delta_{m(j)}} L_{k-1}^k \mathbf{v}\Bigg\|_{L^2(\Omega)}, \tag{5.4}$$

by Lemma 5.2 in [14], we can get

$$\Big\|\mathbf{v} - L_{k-1}^k \mathbf{v}\Big\|_{L^2(\Omega)} \le C h_k \|\mathbf{v}\|_{k-1}, \tag{5.5}$$

by norm equivalence, we deduce

$$\begin{aligned}
\left\|\Xi_{k,\delta_{m(j)}} L_{k-1}^k \mathbf{v}\right\|_{L^2(\Omega)}^2 &\le h_k^2 \sum_{m_i^k \in \delta_{k,m(j)}^{\mathrm{CR}}} \left(\Xi_{k,\delta_{m(j)}} L_{k-1}^k \mathbf{v}\right)^2 \left(m_i^k\right) \\
&= h_k^2 \sum_{m_i^k \in \delta_{k,m(j)}^{\mathrm{CR}}} Q_{k,\delta_{m(j)}} \left(\left(L_{k-1}^k \mathbf{v}\right)|_{\gamma_{m(i)}} - \left(L_{k-1}^k \mathbf{v}\right)|_{\delta_{m(j)}}\right)^2 \left(m_i^k\right) \\
&\le C h_k \left\| Q_{k,\delta_{m(j)}} \left(\left(L_{k-1}^k \mathbf{v}\right)|_{\gamma_{m(i)}} - \left(L_{k-1}^k \mathbf{v}\right)|_{\delta_{m(j)}}\right)\right\|_{0,\gamma_m}^2 \\
&\le C h_k \left\|\left(L_{k-1}^k \mathbf{v}\right)|_{\gamma_{m(i)}} - \left(L_{k-1}^k \mathbf{v}\right)|_{\delta_{m(j)}}\right\|_{0,\gamma_m}^2 \\
&\le C h_k \left(\left\|\left(L_{k-1}^k \mathbf{v}\right)|_{\gamma_{m(i)}} - \mathbf{v}|_{\delta_{m(j)}}\right\|_{0,\gamma_m}^2 + \left\|\mathbf{v}|_{\delta_{m(j)}} - \left(L_{k-1}^k \mathbf{v}\right)|_{\delta_{m(j)}}\right\|_{0,\gamma_m}^{2.0}\right) \\
&= C h_k (K_1 + K_2).
\end{aligned} \tag{5.6}$$

Using trace theorem and (5.5), we have

$$K_2 \le C h_k \|\mathbf{v}\|_{k-1,j}^2. \tag{5.7}$$

Owing to $\mathbf{v} \in X_{k-1}$, then

$$\begin{aligned}
\left\|\left(L_{k-1}^k \mathbf{v}\right)|_{\gamma_{m(i)}} - \mathbf{v}|_{\delta_{m(j)}}\right\|_{0,\delta_{m(j)}}^2 &\le 2\left\|\left(L_{k-1}^k \mathbf{v}\right)|_{\gamma_{m(i)}} - Q_{k-1,\delta_{m(j)}}\left(\mathbf{v}|_{\gamma_{m(i)}}\right)\right\|_{0,\gamma_{m(i)}}^2 \\
&\quad + 2\left\|Q_{k-1,\delta_{m(j)}}\left(\mathbf{v}|_{\delta_{m(j)}}\right) - \mathbf{v}|_{\delta_{m(j)}}\right\|_{0,\delta_{m(j)}}^2.
\end{aligned} \tag{5.8}$$

The second term of the above inequality can be estimated as follows:

$$\left\|Q_{k-1,\delta_{m(j)}}\left(\mathbf{v}|_{\delta_{m(j)}}\right) - \mathbf{v}|_{\delta_{m(j)}}\right\|_{0,\gamma_m}^2 = \sum_{e \in T_{k-1}(\delta_{m(j)})} \int_e (\mathbf{v} - Q_e \mathbf{v})^2 ds, \tag{5.9}$$

where Q_e is the L^2 orthogonal projection onto one-dimensional space which consists of constant functions on an element e, and e is an edge of E which is in the triangulation T_{k-1}. Using the scaling argument in [17], for any constant c we have

$$\begin{aligned}
\int_e (\mathbf{v} - Q_e \mathbf{v})^2 ds &\le \int_e (\mathbf{v} - c)^2 ds \le C h_k \int_{\hat{e}} (\hat{\mathbf{v}} - c)^2 d\hat{s} \le C h_k \|\hat{\mathbf{v}} - c\|_{1,\hat{E}}^2 \\
&\le C h_k |\hat{\mathbf{v}}|_{1,\hat{E}}^2 \le C h_k |\mathbf{v}|_{1,E}^2,
\end{aligned} \tag{5.10}$$

which combining with (5.9) gives

$$\left\|Q_{\mathrm{k}-1,\delta_{m(j)}}\left(\mathbf{v}|_{\delta_{m(j)}}\right)-\mathbf{v}|_{\delta_{m(j)}}\right\|_{0,\gamma_m}\leq Ch_k^{1/2}\|\mathbf{v}\|_{k,j}. \tag{5.11}$$

For the first term of the right side of (5.8), we have

$$\begin{aligned}
&\left\|\left(L_{k-1}^{k}\mathbf{v}\right)|_{\gamma_{m(i)}}-Q_{k-1,\delta_{m(j)}}\left(\mathbf{v}|_{\gamma_{m(i)}}\right)\right\|_{0,\gamma_m}^{2}\\
&\quad=\left\|\left(L_{k-1}^{k}\mathbf{v}\right)|_{\gamma_{m(i)}}-\mathbf{v}|_{\gamma_{m(i)}}+\mathbf{v}|_{\gamma_{m(i)}}+Q_{k-1,\delta_{m(j)}}\left(\mathbf{v}|_{\gamma_{m(i)}}\right)\right\|_{0,\gamma_m}^{2}\\
&\quad\leq 2\left\|\left(L_{k-1}^{k}\mathbf{v}-\mathbf{v}\right)|_{\gamma_{m(i)}}\right\|_{0,\gamma_{m(i)}}^{2}+2\left\|\mathbf{v}|_{\gamma_{m(i)}}-Q_{k-1,\delta_{m(j)}}\left(\mathbf{v}|_{\gamma_{m(i)}}\right)\right\|_{0,\gamma_{m(i)}}^{2}\\
&\quad=F_1+F_2.
\end{aligned} \tag{5.12}$$

Trace theorem and (5.5) give

$$F_1\leq Ch_k\left\|L_{k-1}^{k}\mathbf{v}\right\|_{k,i}^{2}\leq Ch_k\|\mathbf{v}\|_{k-1,i}^{2}. \tag{5.13}$$

For F_2, by trace theorem and the approximation of the operator $Q_{k-1,\delta_{m(j)}}$, we have

$$F_2\leq Ch_k\|\mathbf{v}\|_{k-1,i}^{2}, \tag{5.14}$$

which together with (5.4)–(5.13), gives (A.1).

Obviously, (A.2) naturally holds so we only need to prove (A.3).

By proof of Lemma 5.2 in [14], we can see that

$$\begin{aligned}
&\left\|\left|(\mathbf{u}_k,p_k)-I_{k-1}^{k}(\mathbf{u}_{k-1},p_{k-1})\right|\right\|_{0,k}\\
&\quad\leq\left\|\mathbf{u}_k-L_{k-1}^{k}\mathbf{u}_{k-1}\right\|_{0,k}+\left\|\sum_{m=1}^{M}\Xi_{k,\delta_{m(j)}}L_{k-1}^{k}\mathbf{u}_{k-1}\right\|_{0,k}+h_k^2\left\|p_k-J_{k-1}^{k}p_{k-1}\right\|_{0,k}\\
&\quad\leq Ch_k^2\left(\|\mathbf{u}\|_{H^2(\Omega)}+\|p\|_{H^1(\Omega)}\right)+\left\|\sum_{m=1}^{M}\Xi_{k,\delta_{m(j)}}L_{k-1}^{k}\mathbf{u}_{k-1}\right\|_{0,k}.
\end{aligned} \tag{5.15}$$

Arguing as (5.6), we obtain

$$\begin{aligned}
&\left\| \Xi_{k,\delta_{m(j)}} L_{k-1}^{k} \mathbf{u}_{k-1} \right\|_{0,k}^{2} \\
&\leq h_k^2 \sum_{m_i^k \in \delta_{k,m(j)}^{\mathrm{CR}}} \Xi_{k,\delta_{m(j)}} \left(L_{k-1}^{k} \mathbf{u}_{k-1} \right)^2 \left(m_i^k \right) \\
&= h_k^2 \sum_{m_i^k \in \delta_{k,m(j)}^{\mathrm{CR}}} \left(Q_{k,\delta_{m(j)}} \left(\left(L_{k-1}^{k} \mathbf{u}_{k-1} \right) |_{\gamma_{m(i)}} - \left(L_{k-1}^{k} \mathbf{u}_{k-1} \right) |_{\delta_{m(j)}} \right) \right)^2 \left(m_i^k \right) \\
&\leq C h_k \left\| Q_{k,\delta_{m(j)}} \left(\left(L_{k-1}^{k} \mathbf{u}_{k-1} \right) |_{\gamma_{m(i)}} - \mathbf{u}_k |_{\gamma_{m(i)}} + \mathbf{u}_k |_{\delta_{m(j)}} - \left(L_{k-1}^{k} \mathbf{u}_{k-1} \right) |_{\delta_{m(j)}} \right) \right\|_{0,k}^2 \\
&\leq C h_k \left(\left\| \left(L_{k-1}^{k} \mathbf{u}_{k-1} \right) |_{\gamma_{m(i)}} - \mathbf{u}_k |_{\gamma_{m(i)}} \right\|_{0,\gamma_m}^2 + \left\| \left(L_{k-1}^{k} \mathbf{u}_{k-1} \right) |_{\delta_{m(j)}} - \mathbf{u}_k |_{\delta_{m(j)}} \right\|_{0,\gamma_m}^2 \right) \\
&= C h_k (K_1 + K_2).
\end{aligned} \tag{5.16}$$

By (5.15) and trace theorem, we get that

$$K_1 \leq C h_k^3 \| \mathbf{u} \|_{H^2(\Omega_i)}^2, \qquad K_2 \leq C h_k^3 \| \mathbf{u} \|_{H^2(\Omega_j)}^2, \tag{5.17}$$

together with (5.15), (A.3) has been proved, and we have completed the proof of Lemma 5.3. □

6. Numerical Results

In this section, we present some numerical results to illustrate the theory developed in the earlier sections. The examples are as same as those in [5], so that we can compare the conclusion with the mortar rotated Q_1 element method.

Here we deal with $\Omega = (0,1)^2$. We choose the exact solution of (2.1) as

$$u_1 = 2x^2(1-x)^2 y(1-y)(1-2y), \qquad u_2 = -2x(1-x)(1-2x)y^2(1-y)^2, \tag{6.1}$$

for the velocity and $p = x^2 - y^2$ for the pressure.

For simplicity, we decompose Ω into two subdomains: $\Omega_1 = (0,1) \times (0,1/2)$ as nonmortar domain and $\Omega_2 = (0,1) \times (1/2,1)$ as mortar domain. The sizes of the coarsest grid are denoted by $h_{1,1}$ and $h_{1,2}$, respectively (see Figure 1). The test concerns the convergence of the $\mathcal{W}$-cycle multigrid algorithm. In what follows, k denotes the level, N_u and N_p are the number of the unknowns of the velocity and pressure, the norm $\| \cdot \|_{0,d}$ is the usual Euclidean norm of a vector which is equivalent to $\| \cdot \|_h$. $\mathrm{iter}_{(m_1,m_2)}$ denotes the number of iterations to achieve the relative error of residue less than 10^{-3}, where m_1 and m_2 are the presmoothing steps, and the postsmoothing steps respectively, and the initial approximative solution for the iteration is zero. The numerical results are presented in Tables 1 and 2.

From Table 1, we can see that the errors of the mortar element method for the velocity and the pressure are small, which demonstrates Theorem 3.2.

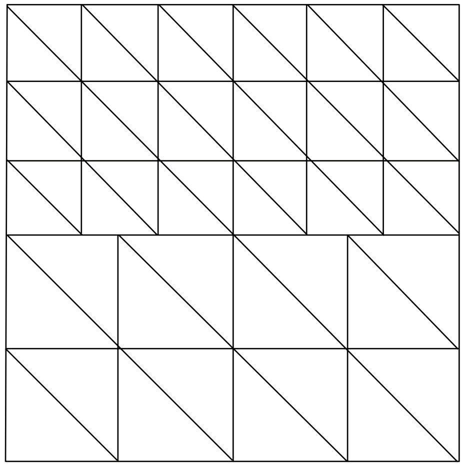

Figure 1: The coarsest mesh with $h_{1,1} = 1/4$ and $h_{1,2} = 1/6$.

Table 1: Error estimate for the mortar element method with $h_{1,1} = 1/4$ and $h_{1,2} = 1/6$.

k	N_u	N_p	$\|\underline{u} - \underline{u}_k\|_{0,d}$	$\|p - p_k\|_{L^2(\Omega)}$
1	178	51	0.0772974	0.164013
2	668	207	0.0455288	0.133103
3	2584	831	0.0242334	0.0733174
4	10160	3327	0.0123836	0.037679
5	40288	13311	0.00619081	0.0195834

Table 2: Iterative numbers for the $\mathcal{W}$-cycle with $h_{1,1} = 1/4$ and $h_{1,2} = 1/6$.

k	2	3	4	5
$\text{iter}_{(4,4)}$	9	8	8	9
$\text{iter}_{(5,5)}$	8	8	7	7

From Table 2, we can see that the convergence for the $\mathcal{W}$-cycle multigrid algorithm is optimal; that is, the number of iterations is independent of the level number k. Meanwhile, we note that the number of iterations is less than the rotated Q_1 element method in [5] when achieving the same relative error.

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