Robust Stability of Nonlinear Piecewise Deterministic Systems Under Matching Conditions

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This paper deals with the robustness of the class of nonlinear piecewise deterministic systems with unknown but bounded uncertainties. Under the assumption that all the modes of the markovian jump process (disturbance) communicate, the complete access to the system state and the boundedness of the uncertainties, a sufficient condition for stochastic stability of this class of systems is given. An example is presented to validate the proposed results.

Keywords: nonlinear piecewise deterministic system; stochastic stability; Markov process; matching conditions

1 INTRODUCTION

It is well known that the physical systems are nonlinear in nature. Sometimes it is possible to describe these physical systems by a set of linear ordinary differential equations and consequently the well established results of linear systems can be used to analyze and design them, this will be true if their mode of operation does not deviate too much from the normal set of operating conditions. But in practice, one often encounters

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situations where the linearized model is inadequate or inaccurate and the use of a nonlinear model is then more suitable to describe the physical system.

Many approaches have been proposed to analyze and design this class of systems. However the problem concerning the robustness of this class of systems is still an open question. Some papers dealing with this question have been published, among them, we quote the paper of Barmish et al. [1].

Robust control offers the advantage to design a controller which enable us to cope with the uncertainties which appeared in the more realistic models. This problem has been investigated in the linear case by several authors, see Dorato et al. [10]. The nonlinear class of systems we considered in this paper belongs to the class of piecewise deterministic systems. This class has some particularities. In fact, the system state vector has two components, the first one is continuous in nature, the second one is discrete and random, taking values in a finite set and referred to as the mode. For the research papers in this area, see, for example, the works of Sworder [16], Wonham [19], Rishel [15], Davis [8], Vermes [18], Mariton and Bertrand [14], Boukas and Haurie [4], Boukas [3], Ji and Chizek [12], Boukas and Mignanego [2] and Boukas et al. [6].

Few results concerning the robustness of this class of systems have been reported. Boukas [5] considers the robustness of a class of linear piecewise deterministic systems whose uncertainties are upper bounded. A sufficient condition for stochastic stability of this class of systems is given. Under the complete access to the state variables and the mode, the H^∞ control problem has been investigated by De Souza and Fragoso [9].

In this paper, we consider a nonlinear piecewise deterministic systems. The sufficient conditions which guarantee the stability of a class of piecewise deterministic nonlinear systems are described. The paper is organized as follows: In section 2, we give a brief description of the class of piecewise deterministic nonlinear systems and give some assumptions. In section 3, we construct the controller and recall the definition of the stochastic stability of this class of systems. In section 4, the sufficient condition for the robust stability of the piecewise deterministic systems under matching conditions is given. In section 5, an example is presented to illustrate these results.

2 SYSTEM AND ASSUMPTIONS

We consider an uncertain piecewise deterministic nonlinear system described by:

$$\dot{x}(t) = A(x(t), \xi(t), t) + \delta A(x(t), \xi(t), a, t) + [B(x(t), \xi(t), t)$$
$$+ \ \delta B(x(t), \xi(t), a, t)]u(t) \tag{1}$$

where the n-dimensional vector $x(t) \in \mathbb{R}^n$ stands for the state of the system and the m-dimensional vector $u(t) \in \mathbb{R}^m$ is the control. $A(x, \xi, t)$, $\delta A(x, \xi, a, t)$, $B(x, \xi, t)$ and $\delta B(x(t), \xi(t), a, t)$ are matrices of appropriate dimensions. $\xi(t)$ represents a continuous discrete-state Markov process taking values in a finite set $B = \{1, 2,..., s\}$ with transition probability $Pr\{\xi(t + \delta t) = \beta | \xi(t) = \alpha\}$ given by:

$$Pr\{\xi(t + \delta t) = \beta | \xi(t) = \alpha\} = \begin{cases} q_{\alpha\beta}\delta t + o(\delta t), & \text{if } \alpha \neq \beta \\ 1 + q_{\alpha\alpha}\delta t + o(\delta t), & \text{if } \alpha = \beta \end{cases} \tag{2}$$

In this relation, $q_{\alpha\beta}$ stands for the transition probability rate from state α to state β and satisfies the following relations:

$$q_{\alpha\beta} \geq 0 \tag{3}$$

$$q_\alpha = -q_{\alpha\alpha} = \sum_{\beta \in B, \alpha \neq \beta} q_{\alpha\beta} \tag{4}$$

For each $\alpha \in \beta$, $\delta A(x(t), \xi(t), a, t)$ and $\delta B(x(t), \xi(t), a, t)$ represent the system's uncertainties. The vector parameter a lies within a specified bounded and connected set $\mathfrak{A} \subset \mathbb{R}^p$.

In the rest of this section, we will give some assumptions which will allow us to state the sufficient conditions for the robustness of the stochastic stabilizability of the class of piecewise deterministic nonlinear systems. Our first assumption is introduced to guarantee the existence of solutions of the state equations.

ASSUMPTION 1   *For each $\alpha \in B$, $A(.)$, $B(.)$, $\delta A(.)$ and $\delta B(.)$ are Lipchitz continuous in (x, a, t).*

Next assumption is called matching conditions. In deterministic case, the matching conditions are properties of the system's structure only. They guarantee that the uncertainty vector does not influence the dynamics more than the control vector u (See Gutman [11], Leitmann [13], or Thorp and Barmish [17]). In our case, we will assume that the matching conditions hold for each mode. Since we assume the complete access to the system's state and its mode, the assumption is reasonable too.

ASSUMPTION 2   *For each $\alpha \in B$, there are mappings $D(x, \alpha, a, t)$ and $E(x, \alpha, a, t)$ such that*

$$\delta A(x, \alpha, a, t) \quad = \quad B(x, \alpha, t)D(x, \alpha, a, t) \tag{5}$$

$$\delta B(x, \alpha, a, t) \quad = \quad B(x, \alpha, t)E(x, \alpha, a, t) \tag{6}$$

$$\|E(x, \alpha, a, t)\| \quad < \quad 1 \tag{7}$$

where $D(.)$ and $E(.)$ are matrices of appriorate dimensions which are continuous in (x, a, t) for each $\alpha \in B$.
Let the stochastic Lyapunov function $V(\alpha, x)$ be defined by

$$V(\alpha, x) = x'P(\alpha)x \tag{8}$$

where $P(\alpha)$ is a symmetric positive definite matrix.

ASSUMPTION 3   *Let $A_0V(\alpha, x)$ be defined by:*

$$A_0V(\alpha, x) = \nabla_x'V(\alpha, x)A(x, \alpha, t) + \sum_{\beta \in B} q_{\alpha\beta}V(\beta, x) \tag{9}$$

where ∇_x' denotes the transpose of gradient operation. We also require that there is a constant $\gamma_1 > 0$ such that

$$A_0V(\alpha, x) \leq -\gamma_1V(\alpha, x) \tag{10}$$

3 CONTROLLER CONSTRUCTION AND THE CONCEPT OF STABILITY

In this section, we proceed to construct a control law $u^*(.)$ which will later be shown to stabilize the class of systems under study. The first step in the construction of the control law is to select two functions $\Delta_1(.)$ and $\Delta_2(.)$ such that

$$\max_{a \in \mathfrak{A}} \|D(x, \alpha, a, t)\| \leq \Delta_1(x, \alpha, t) \tag{11}$$

$$\max_{a \in \mathfrak{A}} \|E(x, \alpha, a, t)\| \leq \Delta_2(x, \alpha, t) \tag{12}$$

The standing assumptions 1–3 assure that there is a $\Delta_2(x, \alpha, t)$ such that $\Delta_2(x, \alpha, t) < 1$ for each $\alpha \in \mathfrak{B}$ and $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, and we can choose $\Delta_1(.)$ and $\Delta_2(.)$ to be continuous in (x, t) for each $\alpha \in \mathfrak{B}$.

Now, for each $\alpha \in \mathfrak{B}$ we choose any nonnegative function $\gamma(x, \alpha, t)$ which is continuous in (x, t) and satisfying the following inequality:

$$\gamma(x, \alpha, t) \geq -\frac{\Delta_1^2(x, \alpha, t)}{4C_1[A_0 V(\alpha, x) + 1][1 - \Delta_2(x, \alpha, t)]} \tag{13}$$

where $0 < C_1 < 1$.

Remark 3.1   Notice that $\Delta_1(.)$ and $\Delta_2(.)$ are the upper bound functions of the uncertainties functions $\Delta A(.)$ and $\Delta B(.)$. In practice, it is all the time possible to get the expression of these two functions.

We define the controller by the following expression:

$$u^*(x, \alpha, t) := -\gamma(x, \alpha, t)B'(x, \alpha, t)\nabla_x V(\alpha, x) \tag{14}$$

Remark 3.2   Since we assume the complete access to the state and the mode of the system, it is then always possible to compute the control law defined by Eq. (14).

The aim of the control system is to stabilize the system under consideration. In the deterministic case, this problem have received much attention from the researchers of the control community. In the stochastic case, particularly for the case of the class of systems under consideration, there is in the literature few results regarding the stabilizability or the stability

concepts. In the rest of this paper the definition 3.1 will be used to define the stabilizability of our class of systems.

DEFINITION 3.1   The system (1)–(2) is said to be stochastically stabilizable if, for all finite $x_0 \in \mathbb{R}^n$ and $\alpha \in \mathcal{B}$, there exists a state feedback control, i.e. $u = u^*(x, \alpha, t) := -\gamma(x, \alpha, t)B'(x, \alpha, t)\nabla_x V(\alpha, x)$, such that there exists a symmetric positive definite matrix $\tilde{P}$ satisfying:

$$\lim_{T \to \infty} E_{u(.)} \left\{ \int_0^T x'(t, x_0, \alpha, a, u)x(t, x_0, \alpha, a, u)dt \big| x_0, \alpha \right\} \leq x_0' \tilde{P} x_0 \quad (15)$$

where $x(t, x_0, \alpha, a, u)$ represents the corresponding solution of system (1) at time t when the control $u(.)$ is used and the initial conditions are respectively x_0 and α.

4 MAIN RESULT

In this section, we first consider the stochastic stabilizability of certain nonlinear piecewise deterministic systems, then proof the same result for the uncertain nonlinear piecewise deterministic system.

4.1 Stochastic Stabilizability of Certain Nonlinear Piecewise Deterministic Systems

We first consider the certain piecewise deterministic nonlinear systems described by

$$\dot{x}(t) = A(x(t), \xi(t), t) + B(x(t), \xi(t), t)u(t) \quad (16)$$

Then we have the following result:

THEOREM 4.1   *Subject to assumptions 1 and 3, and the use of the controller constructed in section 3, (in this case γ can be any positive constant) the certain piecewise deterministic nonlinear system (16) and (2) is stochastically stabilizable.*

Proof   Consider the weak infinitesimal operator $\tilde{A}$ of the process $\{\xi, x(t), t \in [0, T]\}$, which is given by (see C.E.De Souza and M.D. Fragoso [9]):

$$\tilde{A}V(\alpha, x) = \nabla'_x V(\alpha, x)\{A(x, \alpha, t) + B(x, \alpha, t)u^*\} + \sum_{\beta \in \beta} q_{\alpha\beta} V(\beta, x)$$

$$= A_0 V(\alpha, x) - \nabla'_x V(\alpha, x)B(x, \alpha, t)\gamma(x, \alpha, t)B'(x, \alpha, t)\nabla_x V(\alpha, x)$$

$$= A_0 V(\alpha, x) - \gamma(x, \alpha, t)\|B'(x, \alpha, t)\nabla_x V(\alpha, x)\|^2$$

Therefore, we have

$$\tilde{A}V(\alpha, x) \leq A_0 V(\alpha, x) \leq -\gamma_1 V(\alpha, x) \tag{17}$$

Then by Dynkin's formula and the Gronwall–Bellman lemma, we have for all $\alpha \in \mathscr{B}$,

$$E[V(\alpha, x)] \leq \exp(-\gamma_1 t)V(\alpha, x) \tag{18}$$

Therefore

$$E[V(\alpha, x)|x_0, \xi(0) = \alpha] = E[x'P(\alpha)x|x_0, \xi(0) = \alpha] \leq \exp$$
$$(-\gamma_1 t) x'_0 P(\alpha)x_0 \tag{19}$$

Thus we have,

$$E\left\{\int_0^T x'(t)P(\alpha)x(t)dt\Big|x_0, \alpha\right\} \leq \left(\int_0^T \exp(-\gamma_1 t)dt\right)x'_0 P(\alpha)x_0$$

$$\leq -\frac{1}{\gamma_1}[\exp(-\gamma_1 T) - 1]x'_0 P(\alpha)x_0$$

$$\tag{20}$$

Let $T \to \infty$ we have,

$$\lim_{T\to\infty} E_{u(.)}\left\{\int_0^T x'(t)P(\alpha)x(t)dt\Big|x_0, \alpha\right\} \leq \frac{1}{\gamma_1} x'_0 P(\alpha)x_0 \tag{21}$$

Let

$$\tilde{P} = \max_{\alpha \in B} \frac{P(\alpha)}{\gamma_1 \|P(\alpha)\|} \tag{22}$$

we have

$$\lim_{T\to\infty} E_{u(.)}\left\{\int_0^T x'(t)x(t)dt\Big|x_0, \alpha\right\} \le x_0'\tilde{P}x_0 \qquad (23)$$

This completes the proof of the theorem.   ◇

In the following we will show that the controller constructed in section 3 also stabilizes the uncertain systems which guarantees that the nonlinear piecewise deterministic systems' stochastic stability has robust property.

4.2 Stochastic Stability of Uncertain Nonlinear Piecewise Deterministic Systems

The following result states the sufficient conditions which guarantee the stability of the class of piecewise deterministic nonlinear system (1)–(2).

THEOREM 4.2 *Subject to assumptions 1–3, and use the controller constructed in section 3, the uncertain piecewise deterministic nonlinear system (1)–(2) is stochastically stabilizable.*

Proof Again, we consider the weak infinitesimal operator $\tilde{A}$ of the process $\{\xi, x(t), t \in [0, T]\}$, which is given by:

$$\tilde{A}V(\alpha, x) = \nabla_x'V(\alpha, x)\{A(x, \alpha, t) + \delta A(x, \alpha, a, t)$$
$$+ [B(x, \alpha, t) + \delta B(x, \alpha, a, t)]u^*\} + \sum_{\beta\in B} q_{\alpha\beta}V(\beta, x)$$
$$= A_0V(\alpha, x) + \nabla_x'V(\alpha, x)\delta A(x, \alpha, a, t)$$
$$- \nabla_x'V(\alpha, x)[B(x, \alpha, t)$$
$$+ \delta B(x, \alpha, a, t)]\gamma(x, \alpha, t)B'(x, \alpha, t)\nabla_x V(\alpha, x)$$
$$= A_0V(\alpha, x) - \gamma(x, \alpha, t)\|B'(x, \alpha, t)\nabla_x V(\alpha, x)\|^2$$
$$+ \nabla_x'V(\alpha, x)B(x, \alpha, t)[D(x, \alpha, a, t)$$
$$- \gamma(x, \alpha, t)E(x, \alpha, a, t)B'(x, \alpha, t)\nabla_x V(\alpha, x)]$$

let

$$\phi(x, \alpha, t) = B'(x, \alpha, t)\nabla_x V(\alpha, x), \tag{24}$$

and recalling the definition of $\Delta_1(.)$ and $\Delta_2(.)$, we have

$$\tilde{A}V(\alpha, x) \leq A_0 V(\alpha, x) + \Delta_1(x, \alpha, t)\|\phi(x, \alpha, t)\|$$
$$- \gamma(x, \alpha, t)\|\phi(x, \alpha, t)\|^2[1 - \Delta_2(x, \alpha, t)] \tag{25}$$

If $\Delta_1(x, \alpha, t) = 0$, then

$$\tilde{A}V(\alpha, x) \leq A_0 V(\alpha, x) \tag{26}$$

If $\Delta_1(x, \alpha, t) \neq 0$, then $\gamma(x, \alpha, t) > 0$ and

$$\tilde{A}V(\alpha, x) \leq A_0 V(\alpha, x) + \Delta_1(x, \alpha, t)\|\phi(x, \alpha, t)\|$$
$$- \gamma(x, \alpha, t)\|\phi(x, \alpha, t)\|^2[1 - \Delta_2(x, \alpha, t)]$$
$$= A_0 V(\alpha, x) + \frac{\Delta_1^2(x, \alpha, t)}{4\gamma(x, \alpha, t)[1 - \Delta_2(x, \alpha, t)]}$$
$$- \frac{[1 - \Delta_1(x, \alpha, t)]\gamma(x, \alpha, t)}{\Delta_1^2(x, \alpha, t)}\left[\Delta_1(x, \alpha, t)\|\phi(x, \alpha, t)\|\right.$$
$$\left.- \frac{\Delta_1^2(x, \alpha, t)}{2\gamma(x, \alpha, t)[1 - \Delta_2(x, \alpha, t)]}\right]^2$$
$$\leq A_0 V(\alpha, x) + \frac{\Delta_1^2(x, \alpha, t)}{4\gamma(x, \alpha, t)[1 - \Delta_2(x, \alpha, t)]}$$
$$\leq (1 - C_1)A_0 V(\alpha, x) \tag{27}$$

Combining cases $\Delta_1(.) = 0$ and $\Delta_1(.) \neq 0$, and noting that $C_1 < 1$, we have

$$\tilde{A}V(\alpha, x) \leq A_0 V(\alpha, x) \leq -\gamma_1 V(\alpha, x) \tag{28}$$

Then by the same argument with theorem 4.1, we have

$$\lim_{T \to \infty} E_{u(.)} \left\{ \int_0^T x'(t)x(t)dt | x_0, \alpha \right\} \leq x_0' \tilde{P} x_0 \qquad (29)$$

This completes the proof of the theorem.   ◇

5 ILLUSTRATIVE EXAMPLE

To illustrate the proposed results, let us consider the following piecewise deterministic system:

• mode 1:

$$\dot{x}_1(t) = -x_1(t)$$

$$\dot{x}_2(t) = -x_2(t) + u - \frac{a_1(t)\sin x_1(t)}{l_1}$$

where l_1 is a positive constant and $a_1(t)$ is the uncertainty.

• mode 2:

$$\dot{x}_1(t) = -x_1(t)$$

$$\dot{x}_2(t) = -x_2(t) + u - \frac{a_2(t)\sin x_1(t)}{l_2}$$

where l_2 is a positive constant and $a_2(t)$ is the uncertainty.

Let the jump Markov process be described by the following transition matrix:

$$\Lambda = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$$

Let

$$P(\alpha) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

for all $\alpha \in \beta$. In this example, for both mode 1 and mode 2,

$$A = \begin{bmatrix} -x_1(t) \\ -x_2(t) \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\delta B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

for mode 1,

$$\delta A = \begin{bmatrix} 0 \\ -\dfrac{a_1(t)\sin x_1(t)}{l_1} \end{bmatrix} \tag{30}$$

for mode 2,

$$\delta A = \begin{bmatrix} 0 \\ -\dfrac{a_2(t)\sin x_1(t)}{l_2} \end{bmatrix} \tag{31}$$

and for both mode 1 and mode 2 we have,

$$A_0 V(1, x) = A_0 V(2, x) = -2x_1^2 - 2x_2^2 \tag{32}$$

Therefore, we can choose $\gamma_1 = 1$, $\Delta_1 = \tau|\sin x_1(t)|$, where τ is a constant, such that $\max\left(\dfrac{|a_1(t)|}{l_1}, \dfrac{|a_2(t)|}{l_2}\right) \le \tau$, $\Delta_2 = 0$. Let $C_1 = \dfrac{1}{2}$ and γ satisfying

$$\gamma \ge \frac{\tau^2 \sin^2 x_1}{4(x_1^2 + x_2^2)} \tag{33}$$

Then all the assumptions of theorem 4.2 are satisfied, and the system is stochastically stabilizable.

6 CONCLUSION

In this paper, we have dealt with the class of uncertain piecewise deterministic nonlinear systems. Under some appropriate assumptions, a robust controller design approach has been presented for the class of nonlinear uncertain piecewise deterministic systems using the matching conditions. These results can be easily extended to other type of uncertainties.

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