Mathematical Problems in Engineering
Volume 6, pp. 225–265
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the Gordon and Breach Science
Publishers imprint.
Printed in Singapore.

Miss Analysis in Lambert Interceptions with Application to a New Guidance Law

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(Received 10 September 1999)

Devoted to the 60th birthday of Prof. Robert E. Skelton

For an interceptor that follows a Keplerian trajectory, we have obtained a closed-form linear expression for the miss distance in terms of the perturbations of the booster cut-off conditions, where the miss distance reflects the predicted miss at the Point of Closest Approach (PCA) between the interceptor and the target. We use this analysis result to develop a new guidance law which, in the absence of gravity, ensures (1) that the magnitude of the predicted PCA miss decays exponentially, and (2) that the magnitude of the relative velocity is constant. The same guidance law has been applied to interceptors flying in a gravity field. In the presence of random navigation errors in the new guidance law, the numerically simulated results show that increasing the guidance law gain increases the rms of the predicted PCA miss, which results in a degradation of the interception performance. A trade-off in gain magnitude is required to prevent this degradation.

Keywords: Miss distance; Guidance law; Lambert guidance; Cut-off conditions

1 INTRODUCTION

The classical Lambert Problem of astrodynamics is to determine an orbit that connects two given points in a given transfer time under the

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Keplerian assumptions of the restricted two-body problem [1–9]. Whereas this problem was originally stated for stationary endpoints, it has received substantial attention in the literature because of its potential application to the interception of celestial bodies and ballistic missiles. In such applications, an intercept vehicle is to follow a Keplerian trajectory between a departure point and an arrival point in a given transfer time; however the arrival point must coincide with the position of a given moving target. Hence we will use the name Lambert Interception for these applications.

In ballistic guidance, one typically determines nominal cut-off conditions and a nominal trajectory that connect two given points by a Keplerian orbit. However any perturbation in cut-off conditions (due for instance to navigation errors) will cause the interceptor to be on a perturbed trajectory and miss the target in general. In order to successfully guide the interceptor, it is therefore necessary to evaluate the miss distance correctly, taking into account variations in time of flight. Indeed, in missions of interception, what really matters in the closest approach, that is the minimum value of the distance between the interceptor and the target. This is in general not achieved at the time of nominal interception. This distinction between time of nominal interception and time of closest approach was recognized in [6] where the notion of *aiming vector* is introduced to quantify the miss distance in general interception problems. The results presented here are based on the same idea, but applied to the classical Lambert problem.

Some standard references [1–9] discuss the restricted two-body problem and Lambert problem. Various methods for solving Lambert's problem have been suggested in [10–18]. Alternative approaches for solving Lambert's problem were introduced by Battin *et al.* [10–12] and Nelson and Zarchan [13] and Brand [14]. Gooding [15] provided a procedure and an universal solution for Lambert's problem. Prussing [16] presented a simple geometric interpretation of the angles α and β in Lambert's equation, and their analogues for hyperbolic orbits. Sun *et al.* [17] analytically studied the characteristic features of the solutions of Lambert's problem under various terminal conditions, in the light of the extended form of Godal's time equation. White [18] introduced modified Lambert targeting, which is applicable to multi-stage rocket systems that lack a velocity control capability. For the extension of Lambert's theorem, Lancaster and Blanchard [19] presented a unified form of Lambert's theorem that is valid for elliptic, hyperbolic, and parabolic orbits. Lancaster *et al.* [20] combined all the various cases of Lambert's theorem into a single form for numerical calculations.

Throughout this paper we work under the assumptions of the restricted two-body problem, that is, the interceptor and target are particles of negligible mass above the sensible atmosphere. And it is assumed that the attracting central body has a spherically symmetric gravitational potential. We neglect the interaction with other celestial bodies. The position and velocity of the target will be assumed completely known. For an interceptor that follows a Keplerian trajectory and nominally intercepts a moving target, we obtain a closed form linear approximation for the miss distance in terms of the perturbations of the orbital elements, where the miss distance reflects the closest approach of the interceptor and target. This is accomplished by first determining the position and velocity of the interceptor and target at a common time, then computing the miss distance under the assumption of rectilinear uniform motion. The results can be used to compute the expected value of the miss distance, given the statistical properties of the cut-off errors.

This paper also presents a new guidance law that is developed using the Point of Closest Approach (PCA) miss vector. In the absence of gravity, the guidance law ensures that the magnitude of the PCA miss decays exponentially, and that the magnitude of the relative velocity is constant. The same guidance law will be applied to interceptors in a gravity field under the assumption that the gravitational acceleration is much smaller than a characteristic acceleration. We also assume that the interception engagement, in the presence of gravity, has relatively short duration, and uses relatively low thrust so that the velocities of the interceptor and target do not change substantially. This justifies the assumption of quasi-rectilinear motion.

The performance of the guidance law depends primarily on the measured data, which is the relative position and velocity of the missile and target. We will investigate the performance degradation, in terms of the rms of the predicted PCA miss, due to random navigation errors in implementing the guidance law. In addition we study how the guidance gain affects the accuracy on the interception engagement perturbed by random sensor noise.

The content of the paper is as follows. In Section 2, we show how to compute the PCA miss of two particles under the assumption of rectilinear uniform motion. We also state the well-known trajectory equation for the restricted two-body problem. In Section 3 we compute the PCA miss vector in the case of a stationary target. In Section 4, we compute the time at which the closest approach is achieved. Section 5 presents the predicted PCA miss vector in the case of a moving target, as a linear function of the cut-off errors. In Section 6, we develop a new guidance law to ensure that, in the absence of gravity, the magnitude of the PCA miss decays exponentially and that the magnitude of the velocity remains constant. Section 7 presents the application of the guidance law to interceptions in the presence of gravity. Section 8 studies the effect of the guidance law gain on the engagement duration, the miss distance, and the interceptor velocity. In Section 9, we investigate the performance degradation in terms of the rms of the predicted PCA miss, due to random navigation errors in implementing the guidance, through numerical simulations. We conclude in Section 10.

2 PRELIMINARY RESULTS

In this section, two preliminary results are introduced. The first result quantifies the PCA miss vector of two particles undergoing rectilinear uniform motion. This results will be used to obtain a linear approximation of the PCA miss, once the position and velocity of the interceptor and target are known at a common time.

Consider two particles in rectilinear, not necessarily coplanar motion, as shown in Fig. 1. Suppose that the velocities of the particles are constant and denoted by the vectors $\mathbf{V}_a$ and $\mathbf{V}_b$. Let $\mathbf{X}_a(t_0)$ and $\mathbf{X}_b(t_0)$ be the position of the particles at some time t_0. Let

$$\mathbf{R}(t) = \mathbf{X}_a(t) - \mathbf{X}_b(t), \tag{2.1}$$

and

$$\mathbf{V} = \mathbf{V}_a - \mathbf{V}_b, \tag{2.2}$$

be the position and velocity of particle a relative to particle b, respectively. Let t^* be the time of closest approach, that is $|\mathbf{R}(t^*)| \leq |\mathbf{R}(t)|$, presented a unified form of Lambert's theorem that is valid for elliptic, hyperbolic, and parabolic orbits. Lancaster *et al.* [20] combined all the various cases of Lambert's theorem into a single form for numerical calculations.

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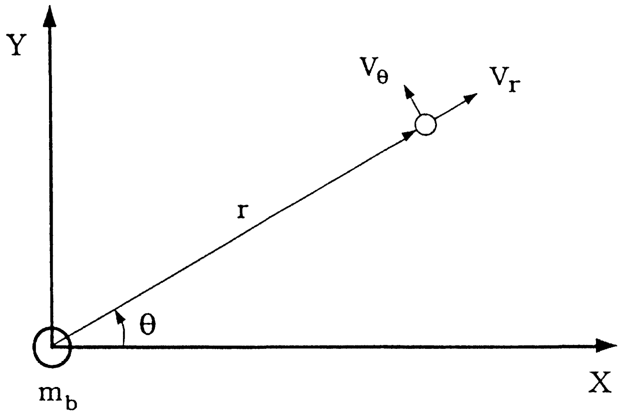

FIGURE 2 Geometry of two-body problem.

particle has polar position and velocity components r_1, θ_1, v_{r_1} and v_{θ_1}. Let

$$k = m_b \mu, \tag{2.6}$$

$$C_1(r_1, \theta_1, v_{r_1}, v_{\theta_1}) = \left[\left(\frac{1}{r_1} - \frac{k}{(r_1 v_{\theta_1})^2} \right) \cos(\theta_1) + \frac{v_{r_1}}{r_1 v_{\theta_1}} \sin(\theta_1) \right], \tag{2.7}$$

$$C_2(r_1, \theta_1, v_{r_1}, v_{\theta_1}) = \left[\left(\frac{1}{r_1} - \frac{k}{(r_1 v_{\theta_1})^2} \right) \sin(\theta_1) - \frac{v_{r_1}}{r_1 v_{\theta_1}} \cos(\theta_1) \right], \tag{2.8}$$

and

$$h(r, \theta; r_1, \theta_1, v_{r_1}, v_{\theta_1}) = \frac{1}{r} - C_1(r_1, \theta_1, v_{r_1}, v_{\theta_1}) \cos(\theta) - C_2(r_1, \theta_1, v_{r_1}, v_{\theta_1}) \sin(\theta) - \frac{k}{(r_1 v_{\theta_1})^2}. \tag{2.9}$$

Then the motion of the particle satisfies

$$h(r(t), \theta(t); r_1, \theta_1, v_{r_1}, v_{\theta_1}) \equiv 0, \quad \forall t \in \mathfrak{R}. \tag{2.10}$$

3 CASE OF A STATIONARY TARGET

We derive a linear approximation of the PCA miss vector under the assumption that the target is stationary at the point of nominal intercept. Under the assumptions of the restricted two-body problem, let $h(r(t), \theta(t); r_1, \theta_1, v_{r_1}, v_{\theta_1})$ be defined as in Eq. (2.10), where r and θ are the coordinates of an arbitrary point along the perturbed trajectory, and let (r_{T^o}, θ_{T^o}) be the coordinates of some fixed point in the plane of motion. Let $m^o(r, \theta)$ denote the distance from a point on the perturbed trajectory to the point (r_{T^o}, θ_{T^o}), that is

$$m^o(r, \theta) = \sqrt{(r \sin\theta - r_{T^o} \sin\theta_{T^o})^2 + (r\cos\theta - r_{T^o}\cos\theta_{T^o})^2}. \tag{3.1}$$

Consider all points (r, θ) that satisfy $h(r(t), \theta(t); r_1, \theta_1, v_{r_1}, v_{\theta_1}) = 0$. Equation (2.10) may be expressed in the form

$$\frac{1}{r} + \sigma(\theta) = 0, \tag{3.2}$$

where the definition of $\sigma(\theta)$ is evident. By implicitly differentiating Eq. (3.2) with respect to θ and solving for $\mathrm{d}r/\mathrm{d}\theta$, we obtain that

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = r^2 \frac{\mathrm{d}\sigma}{\mathrm{d}\theta} \tag{3.3}$$

$$= r^2 \frac{\mathrm{d}h}{\mathrm{d}\theta}. \tag{3.4}$$

Now suppose that (r^*, θ^*) satisfies $h(r^*, \theta^*; r_1, \theta_1, v_{r_1}, v_{\theta_1}) = 0$ and minimizes $m^o(r, \theta)$. Then clearly θ^* minimizes

$$m^o\left(-\frac{1}{\sigma(\theta)}, \theta\right) := \tilde{m}^o(\theta). \tag{3.5}$$

The right-hand side of Eq. (3.5) is a differentiable function defined on an open set and is therefore minimized only if

$$\frac{\mathrm{d}\tilde{m}^o}{\mathrm{d}\theta}(\theta^*) = 0. \tag{3.6}$$

The derivative of $\tilde{m}^{\mathrm{o}}$ with respect to θ becomes

$$\frac{\mathrm{d}\tilde{m}^{\mathrm{o}}}{\mathrm{d}\theta} = \frac{\partial m^{\mathrm{o}}}{\partial \theta} + \frac{\partial m^{\mathrm{o}}}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}\theta} \tag{3.7}$$

$$= \frac{\partial m^{\mathrm{o}}}{\partial \theta} + r^2 \frac{\partial m^{\mathrm{o}}}{\partial r}\left(\frac{\partial h}{\partial \theta}\right) \tag{3.8}$$

$$= g(r, \theta). \tag{3.9}$$

Hence it follows that if (r^*, θ^*) minimizes $m^{\mathrm{o}}(r, \theta)$ and satisfies

$$h(r(t), \theta(t); r_1, \theta_1, v_{r_1}, v_{\theta_1}) = 0, \tag{3.10}$$

then it must also satisfy

$$g(r^*, \theta^*) = 0. \tag{3.11}$$

Based upon the above characterization of the closest approach, we can now derive a linear approximation of the PCA miss vector against a stationary target. Suppose the mission objective is to intercept a fixed target at some nominal time t_f^{o}, where the target position vector is given by $\mathbf{R}_{\mathrm{T^o}}$, with polar coordinates $(r_{\mathrm{T^o}}, \theta_{\mathrm{T^o}})$. Also assume nominal initial conditions $r_1^{\mathrm{o}}, \theta_1^{\mathrm{o}}, v_{r_1}^{\mathrm{o}}$ and $v_{\theta_1}^{\mathrm{o}}$ are known, such that the mission objectives are nominally achieved.

Differentiating Eqs. (3.10) and (3.11) with respect to parameters, we have

$$\mathrm{d}h = \frac{\partial h}{\partial r}\,\mathrm{d}r + \frac{\partial h}{\partial \theta}\,\mathrm{d}\theta + \frac{\partial h}{\partial r_1}\,\mathrm{d}r_1 + \frac{\partial h}{\partial \theta_1}\,\mathrm{d}\theta_1 + \frac{\partial h}{\partial v_{r_1}}\,\mathrm{d}v_{r_1} + \frac{\partial h}{\partial v_{\theta_1}}\,\mathrm{d}v_{\theta_1} \tag{3.12}$$

$$= 0, \tag{3.13}$$

and

$$\mathrm{d}g = \frac{\partial g}{\partial r}\,\mathrm{d}r + \frac{\partial g}{\partial \theta}\,\mathrm{d}\theta = 0. \tag{3.14}$$

Equation (3.14) implies that

$$\delta r^{\mathrm{o}} = -\left(\frac{\partial g}{\partial \theta} \Big/ \frac{\partial g}{\partial r}\right)_{\mathrm{o}} \delta\theta^{\mathrm{o}}, \tag{3.15}$$

where the subscript $\circ$ denotes that the derivatives are to be evaluated at $(r_{T^\circ}, \theta_{T^\circ})$. Substituting Eq. (3.15) into Eq. (3.12), we have the relation

$$\left(\frac{\partial h}{\partial \theta} - \frac{\partial h}{\partial r}\left(\frac{\partial g}{\partial \theta}\Big/\frac{\partial g}{\partial r}\right)\right)_\circ \delta\theta^\circ = -\left[\frac{\partial h}{\partial r_1}\right]_\circ \delta r_1 - \left[\frac{\partial h}{\partial \theta_1}\right]_\circ \delta\theta_1 - \left[\frac{\partial h}{\partial v_{r_1}}\right]_\circ \delta v_{r_1} - \left[\frac{\partial h}{\partial v_{\theta_1}}\right]_\circ \delta v_{\theta_1}. \quad (3.16)$$

Now suppose that the initial conditions are perturbed, and consider the resulting trajectory. Let r_1, θ_1, v_{r_1} and v_{θ_1} be the actual initial conditions and define the perturbation vector as

$$\Delta = [r_1 - r_1^\circ, \theta_1 - \theta_1^\circ, v_{r_1} - v_{r_1}^\circ, v_{\theta_1} - v_{\theta_1}^\circ]^T. \quad (3.17)$$

Define the following useful matrices

$$H = \left[\frac{\partial h}{\partial r}, \frac{\partial h}{\partial \theta}\right]_\circ^T, \quad (3.18)$$

$$H_1 = \left[\frac{\partial h}{\partial r_1}, \frac{\partial h}{\partial \theta_1}, \frac{\partial h}{\partial v_{r_1}}, \frac{\partial h}{\partial v_{\theta_1}}\right]_\circ^T, \quad (3.19)$$

$$G = \left[-\frac{\partial g}{\partial \theta}, \frac{\partial g}{\partial r}\right]_\circ^T, \quad (3.20)$$

where the subscript $\circ$ denotes that the derivatives are evaluated at the nominal initial conditions and the nominal target position. Let $\mathbf{R}^*(r^*, \theta^*)$ denotes the position of the intercept vehicle along the perturbed trajectory when $m^\circ(r, \theta)$ is minimized. That is,

$$m^\circ(r^*, \theta^*) \leq m^\circ(r, \theta), \quad (3.21)$$

for all r and θ such that $h(r(t), \theta(t); r_1, \theta_1, v_{r_1}, v_{\theta_1}) = 0$.

Let

$$\delta r^\circ = r^* - r_{T^\circ}, \quad (3.22)$$

$$\delta\theta^\circ = \theta^* - \theta_{T^\circ}, \quad (3.23)$$

and

$$M^\circ = [\delta r^\circ, \delta\theta^\circ]^T. \quad (3.24)$$

The PCA miss vector against a stationary target is then linearly approximated by

$$M^{\circ} = \begin{bmatrix} \delta r^{\circ} \\ \delta \theta^{\circ} \end{bmatrix} \tag{3.25}$$

$$= \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(\frac{\partial g}{\partial \theta} \Big/ \frac{\partial g}{\partial r} \right) \right)_{\circ} \delta\theta^{\circ} \tag{3.26}$$

$$= - \left[\frac{\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(\frac{\partial g}{\partial \theta} \Big/ \frac{\partial g}{\partial r} \right)}{\frac{\partial h}{\partial \theta} - \frac{\partial h}{\partial r} \left(\frac{\partial g}{\partial \theta} \Big/ \frac{\partial g}{\partial r} \right)} \right]_{\circ} H_1^{\mathrm{T}} \Delta \tag{3.27}$$

$$= \left[\frac{-G H_1^{\mathrm{T}}}{H^{\mathrm{T}} G} \right]_{\circ} \Delta. \tag{3.28}$$

4 DETERMINATION OF TIME DIFFERENCE

This section presents a linear approximation of the time difference between the closest approach and the nominal interception, under the assumption of a stationary target. The difficulty of this derivation is that the characterization of closest approach (3.28) is based upon the trajectory equation (2.10) where time has been eliminated. To recover information about elapsed time, we have to evaluate how some integrals of motion are perturbed. Considering the case of hyperbolic nominal motion, the perturbed semi-major axis and some variables in Lambert's equation are evaluated in terms of the booster cut-off perturbation vector. Finally from these results, the elapsed time between nominal intercept and closest approach has a linearly approximated expression.

As shown in Fig. 3, radius vectors $\mathbf{r}_1$ and $\mathbf{r}$ are the position vectors of points P_1 and P. The endpoints are separated by the transfer angle ω and the chord c. Then, from Lambert's theorem, the transfer time between specified endpoints is determined by the semi-major axis of transfer orbit a, the chord length c and the sum of the radii from focus to points P_1 and P.

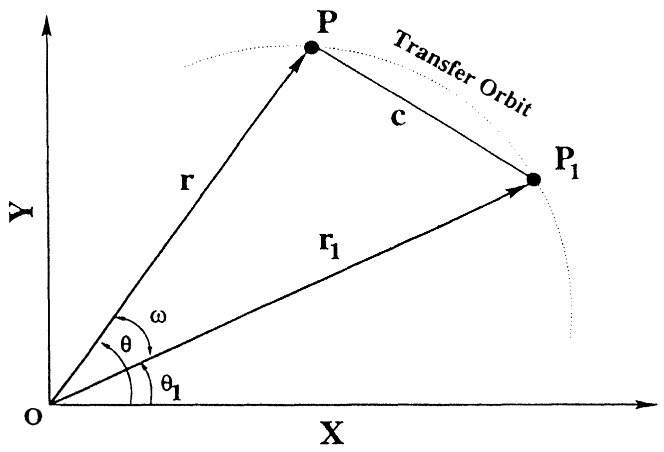

FIGURE 3 Geometry of Lambert's theorem.

Consider the off-nominal launch of an intercept vehicle moving toward a fixed target with polar coordinates given by (r_{T^o}, θ_{T^o}). Let a^o and a be the semi-major axis for the nominal and perturbed trajectories, respectively. Since the semi-major axis satisfies

$$\frac{-k}{2a} = \frac{v^2}{2} - \frac{k}{r}, \tag{4.1}$$

where $v^2 = v_r^2 + v_\theta^2$, the difference between the actual and nominal semi-major axis, $\delta a = a - a^o$, is linearly approximated by

$$\delta a \cong \frac{2a^{o^2}}{k} \left[\frac{k}{r_1^{o2}} \;\; 0 \;\; V_{r_1}^o \;\; V_{\theta_1}^o \right] \Delta =: A^T \Delta, \tag{4.2}$$

where the definition of A is evident. It should be noticed that the semi-major axis a is negative in the case of hyperbolic transfer.

Let $\delta \mathbf{r} = \mathbf{r} - \mathbf{r}_{T^o}$ and $\delta \mathbf{r}_1 = \mathbf{r}_1 - \mathbf{r}_1^o$. Then

$$\delta \mathbf{r} = M^o \cong \left[\frac{-GH_1^T}{H^T G} \right] \Delta, \tag{4.3}$$

$$\delta \mathbf{r}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \Delta. \tag{4.4}$$

The perturbed radii from focus to points P_1 and P are evaluated as

$$\delta r = \lambda_r^{\mathrm{T}}\Delta, \quad \delta r_1 = \lambda_{r_1}^{\mathrm{T}}\Delta, \quad \delta\omega = \lambda_\omega^{\mathrm{T}}\Delta, \tag{4.5}$$

where

$$\lambda_r = \left[[1\ 0]\begin{bmatrix} -GH_1^{\mathrm{T}} \\ \overline{H^{\mathrm{T}}G} \end{bmatrix}\right]^{\mathrm{T}}, \tag{4.6}$$

$$\lambda_{r_1} = [1\ 0\ 0\ 0]^{\mathrm{T}}, \tag{4.7}$$

$$\lambda_\omega = \left[[0\ 1]\left[\begin{bmatrix} -GH_1^{\mathrm{T}} \\ \overline{H^{\mathrm{T}}G} \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\right]\right]^{\mathrm{T}}. \tag{4.8}$$

The chord length c can be expressed in terms of radii r, r_1 and the transfer angle ω,

$$c = (r^2 + r_1^2 - 2r_1 r\cos\omega)^{1/2} =: f(r, r_1, \omega). \tag{4.9}$$

We define

$$s = \frac{(r + r_1 + c)}{2}, \tag{4.10}$$

$$q = \frac{\sqrt{r_1 r}}{s}\cos\frac{\omega}{2} =: q(r, r_1, \omega, s), \tag{4.11}$$

$$\gamma_{\mathrm{h}} = \cosh^{-1}\left(1 - \frac{s}{a}\right), \tag{4.12}$$

$$\zeta_{\mathrm{h}} = 2\cosh^{-1}\left(q\sinh\left(\frac{\gamma_{\mathrm{h}}}{2}\right)\right). \tag{4.13}$$

Then the perturbations in $c, s, q, \gamma_{\mathrm{h}}$ and ζ_{h} are linearly approximated by

$$\delta c = \left[\frac{\partial f}{\partial r_1}\right]_{\mathrm{o}}\delta r_1 + \left[\frac{\partial f}{\partial r}\right]_{\mathrm{o}}\delta r + \left[\frac{\partial f}{\partial \omega}\right]_{\mathrm{o}}\delta\omega =: C^{\mathrm{T}}\Delta, \tag{4.14}$$

$$\delta s = \tfrac{1}{2}(\delta r_1 + \delta r + \delta c) =: \tfrac{1}{2}S^{\mathrm{T}}\Delta, \tag{4.15}$$

$$\delta q = \left[\frac{\partial q}{\partial r_1}\right]_{\mathrm{o}}\delta r_1 + \left[\frac{\partial q}{\partial r}\right]_{\mathrm{o}}\delta r + \left[\frac{\partial q}{\partial \omega}\right]_{\mathrm{o}}\delta\omega + \left[\frac{\partial q}{\partial s}\right]_{\mathrm{o}}\delta s =: \tfrac{1}{2}Q^{\mathrm{T}}\Delta, \tag{4.16}$$

$$\delta\gamma_{\mathrm{h}} = \frac{1}{a^{\mathrm{o}}\sinh\gamma_{\mathrm{h}}^{\mathrm{o}}}\left(\frac{s^{\mathrm{o}}}{a^{\mathrm{o}}}\delta a - \delta s\right) =: \Phi_{\mathrm{h}}^{\mathrm{T}}\Delta, \tag{4.17}$$

and

$$\delta\zeta_{\rm h} = \frac{1}{\cosh(\zeta_{\rm h}^{\rm o}/2)}\left[2\sinh\left(\frac{\gamma_{\rm h}^{\rm o}}{2}\right)\delta q + q^{\rm o}\cosh\left(\frac{\gamma_{\rm h}^{\rm o}}{2}\right)\delta\gamma_{\rm h}\right] =: Z_{\rm h}^{\rm T}\Delta, \tag{4.18}$$

where the subscript $\circ$ denotes that the derivatives are evaluated at the nominal conditions.

The nominal intercept time is given by

$$t_f^{\rm o} = \sqrt{\frac{(-a^{\rm o})^3}{k}}[(\sinh\gamma^{\rm o} - \sinh\zeta^{\rm o}) - (\gamma^{\rm o} - \zeta^{\rm o})]. \tag{4.19}$$

Then the time at PCA can be expressed as

$$t_f = t_f^{\rm o} + \delta t \tag{4.20}$$

$$= \sqrt{\frac{(-a^{\rm o} - \delta a)^3}{k}}[(\sinh(\gamma^{\rm o} + \delta\gamma) - \sinh(\zeta^{\rm o} + \delta\zeta)) - ((\gamma^{\rm o} + \delta\gamma) - (\zeta^{\rm o} + \delta\zeta))]. \tag{4.21}$$

Then the time difference between nominal intercept and closest approach, $\delta t = t_f - t_f^{\rm o}$, is linearly approximated by

$$\delta t = \frac{3t_f^{\rm o}}{2a^{\rm o}}\delta a + 2\sqrt{\frac{(-a^{\rm o})^3}{k}}\left[\sinh^2\frac{\gamma^{\rm o}}{2}\delta\gamma - \sinh^2\frac{\zeta^{\rm o}}{2}\delta\zeta\right] \tag{4.22}$$

$$= \left[\frac{3t_f^{\rm o}}{2a^{\rm o}}A^{\rm T} + 2\sqrt{\frac{(-a^{\rm o})^3}{k}}\left[\sinh^2\left(\frac{\gamma_{\rm h}^{\rm o}}{2}\right)\Phi_{\rm h}^{\rm T} - \sinh^2\left(\frac{\zeta_{\rm h}^{\rm o}}{2}\right)Z_{\rm h}^{\rm T}\right]\right]\Delta \tag{4.23}$$

$$= \Gamma_{\rm h}^{\rm T}\Delta, \tag{4.24}$$

where the definition of $\Gamma_{\rm h}$ is evident and subscript h pertains to a hyperbolic trajectory.

We have determined the PCA time for an interceptor on a hyperbolic trajectory and a fixed target. We now treat the case of an interceptor on an elliptic trajectory in a similar way. Let

$$\gamma_e = \arccos\left(1 - \frac{s}{a}\right), \tag{4.25}$$

$$\zeta_e = 2\arcsin\left(q\sin\left(\frac{\gamma_e}{2}\right)\right), \tag{4.26}$$

$$\delta\gamma_e = -\frac{1}{a^o \sin\gamma_e^o}\left(\frac{s^o}{a^o}\delta a - \delta s\right) =: \Phi_e^T\Delta, \tag{4.27}$$

$$\delta\zeta_e = \frac{1}{\cos(\zeta_e^o/2)}\left[2\sin\left(\frac{\gamma_e^o}{2}\right)\delta q + q^o\cos\left(\frac{\gamma_e^o}{2}\right)\delta\gamma_e\right] =: Z_e^T\Delta. \tag{4.28}$$

The nominal intercept time is given by

$$t_f^o = \sqrt{\frac{(a^o)^3}{k}}[(\gamma^o - \zeta^o) - (\sin\gamma^o - \sin\zeta^o)]. \tag{4.29}$$

Then the time at PCA can be expressed as

$$t_f = t_f^o + \delta t \tag{4.30}$$

$$= \sqrt{\frac{(a^o + \delta a)^3}{k}}[((\gamma^o + \delta\gamma) - (\zeta^o + \delta\zeta)) - (\sin(\gamma^o + \delta\gamma) - \sin(\zeta^o + \delta\zeta))]. \tag{4.31}$$

The time difference between nominal intercept and closest approach, $\delta t = t_f - t_f^o$, is linearly approximated by

$$\delta t = \frac{3t_f^o}{2a^o}\delta a + 2\sqrt{\frac{(a^o)^3}{k}}\left[\sin^2\frac{\gamma^o}{2}\delta\gamma - \sin^2\frac{\zeta^o}{2}\delta\zeta\right] \tag{4.32}$$

$$= \left[\frac{3t_f^o}{2a^o}A^T + 2\sqrt{\frac{(a^o)^3}{k}}\left[\sin^2\left(\frac{\gamma_e^o}{2}\right)\Phi_e^T - \sin^2\left(\frac{\zeta_e^o}{2}\right)Z_e^T\right]\right]\Delta \tag{4.33}$$

$$= \Gamma_e^T\Delta, \tag{4.34}$$

where the definition of Γ_e is evident and subscript e pertains to an elliptic trajectory.

5 CASE OF A MOVING TARGET

To determine the PCA miss of a ballistic interceptor against a moving target, we first compute the position and velocity vectors of the interceptor and target at a common time. This common time is chosen as the time at which the interceptor, flying along the perturbed trajectory, is closest to the point of nominal intercept. Consider a particle moving due to the gravitational forces of an attracting body under the assumptions of the restricted two-body problem. Suppose that at some time t_f^{o} the polar components of the particle position and velocity are given by $r_{\mathrm{T^o}}$, $\theta_{\mathrm{T^o}}$, $v_{r_{\mathrm{T^o}}}$ and $v_{\theta_{\mathrm{T^o}}}$. Let δt be a small time increment. Then at time $t_f^{\mathrm{o}} + \delta t$ the particle position and velocity are linearly approximated by

$$r_{\mathrm{T}} = r_{\mathrm{T^o}} + v_{r_{\mathrm{T^o}}}\delta t, \tag{5.1}$$

$$\theta_{\mathrm{T}} = \theta_{\mathrm{T^o}} + \frac{v_{\theta_{\mathrm{T^o}}}}{r_{\mathrm{T^o}}}\delta t, \tag{5.2}$$

$$v_{r_{\mathrm{T}}} = v_{r_{\mathrm{T^o}}} + \left(\frac{v_{\theta_{\mathrm{T^o}}}^2}{r_{\mathrm{T^o}}} - \frac{k}{r_{\mathrm{T^o}}^2}\right)\delta t, \tag{5.3}$$

$$v_{\theta_{\mathrm{T}}} = v_{\theta_{\mathrm{T^o}}} - \frac{v_{r_{\mathrm{T^o}}} v_{\theta_{\mathrm{T^o}}}}{r_{\mathrm{T^o}}}\delta t. \tag{5.4}$$

Since the position of the intercept vehicle at the common time has been obtained, its velocity is evaluated at that time. Suppose that at some initial time t_1 the polar coordinates of the particle position and velocity are given by r_1, θ_1, v_{r_1} and v_{θ_1}.

Recall that

$$h(r(t), \theta(t); r_1, \theta_1, v_{r_1}, v_{\theta_1}) = 0 \tag{5.5}$$

for all $r(t)$ and $\theta(t)$ along the trajectory. The total derivative of $h(t)$ with respect time t is

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\partial h}{\partial r}\frac{\mathrm{d}r}{\mathrm{d}t} + \frac{\partial h}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{5.6}$$

$$= 0. \tag{5.7}$$

Since

$$\frac{\mathrm{d}r}{\mathrm{d}t} = v_r, \tag{5.8}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{v_\theta}{r}, \tag{5.9}$$

Equation (5.6) becomes

$$v_\theta = -rv_r\left(\frac{\partial h}{\partial r}\Big/\frac{\partial h}{\partial \theta}\right)_t \tag{5.10}$$

$$= -rv_r\rho \tag{5.11}$$

$$= V_\theta(r(t),\theta(t);r_1,\theta_1,v_{r_1},v_{\theta_1}), \tag{5.12}$$

where the definitions of ρ and V_θ are evident, and the subscript t denotes that the derivatives are evaluated at time t.

For the determination of v_r, we apply the energy conservation law along the trajectory,

$$\frac{k}{r_1}+\frac{1}{2}(v_{r_1}^2+v_{\theta_1}^2)=\frac{k}{r}+\frac{1}{2}(v_r^2+v_\theta^2), \tag{5.13}$$

for all r and θ, satisfying

$$h(r(t),\theta(t);r_1,\theta_1,v_{r_1},v_{\theta_1})=0. \tag{5.14}$$

Replacing v_θ in Eq. (5.13) by Eq. (5.11), we have

$$v_r^2=\frac{k/r_1+(1/2)(v_{r_1}^2+v_{\theta_1}^2)-k/r}{1/2(1+r^2\rho^2)} \tag{5.15}$$

$$= V_r(r(t),\theta(t);r_1,\theta_1,v_{r_1},v_{\theta_1}), \tag{5.16}$$

where the definition of V_r is evident.

Let $V_\theta(r(t),\theta(t);r_1,\theta_1,v_{r_1},v_{\theta_1})$ and $V_r(r(t),\theta(t);r_1,\theta_1,v_{r_1},v_{\theta_1})$ be defined as in Eqs. (5.12) and (5.16), respectively. Let $\delta v_{r_\mathrm{m}} = v_{r_\mathrm{m}} - v_{r_\mathrm{m}}^\mathrm{o}$ and $\delta v_{\theta_\mathrm{m}} = v_{\theta_\mathrm{m}} - v_{\theta_\mathrm{m}}^\mathrm{o}$, where $(v_{r_\mathrm{m}}^\mathrm{o}, v_{\theta_\mathrm{m}}^\mathrm{o})$ are the polar components of the intercept vehicle velocity at nominal intercept and $(v_{r_\mathrm{m}}, v_{\theta_\mathrm{m}})$ are the polar components of the intercept vehicle velocity at the time of closest approach to the target. Then δv_r and δv_θ can be linearly approximated by

$$\delta v_{r_\mathrm{m}} = \frac{1}{2v_{r_\mathrm{m}}^\mathrm{o}}\left[\left[\frac{\partial V_r}{\partial r},\frac{\partial V_r}{\partial \theta}\right]_\mathrm{o}\left[\frac{-GH_1^\mathrm{T}}{H^\mathrm{T}G}\right]+\left[\frac{\partial V_r}{\partial r_1},\frac{\partial V_r}{\partial \theta_1},\frac{\partial V_r}{\partial v_{r_1}},\frac{\partial V_r}{\partial v_{\theta_1}}\right]_\mathrm{o}\right]\Delta, \tag{5.17}$$

$$\delta v_{\theta_\mathrm{m}} = \left[\left[\frac{\partial V_\theta}{\partial r},\frac{\partial V_\theta}{\partial \theta}\right]_\mathrm{o}\left[\frac{-GH_1^\mathrm{T}}{H^\mathrm{T}G}\right]+\left[\frac{\partial V_\theta}{\partial r_1},\frac{\partial V_\theta}{\partial \theta_1},\frac{\partial V_\theta}{\partial vr_1},\frac{\partial V_\theta}{\partial v_{\theta_1}}\right]_\mathrm{o}\right]\Delta \tag{5.18}$$

where Δ is defined in Eq. (3.17), H, H_1, and G are defined in Eqs. (3.18)–(3.20), respectively, and the subscript $\circ$ denotes that the derivatives are evaluated at the nominal intercept. Denote this approximation by

$$\begin{bmatrix} \delta v_{r_m} \\ \delta v_{\theta_m} \end{bmatrix} =: V^T \Delta. \tag{5.19}$$

Hence the position and velocity of the interceptor at the point of closest approach are expressed in terms of the perturbation vector:

$$\begin{bmatrix} r_m \\ \theta_m \end{bmatrix} = \begin{bmatrix} r_{T^\circ} \\ \theta_{T^\circ} \end{bmatrix} + \begin{bmatrix} \delta r_m^\circ \\ \delta \theta_m^\circ \end{bmatrix}, \tag{5.20}$$

$$= \begin{bmatrix} r_{T^\circ} \\ \theta_{T^\circ} \end{bmatrix} + \begin{bmatrix} -GH_1^T \\ \hline H^T G \end{bmatrix} \Delta, \tag{5.21}$$

$$\begin{bmatrix} v_{r_m} \\ v_{\theta_m} \end{bmatrix} = \begin{bmatrix} v_{r_m}^\circ \\ v_{\theta_m}^\circ \end{bmatrix} + \begin{bmatrix} \delta v_{r_m} \\ \delta v_{\theta_m} \end{bmatrix}, \tag{5.22}$$

$$= \begin{bmatrix} v_{r_m}^\circ \\ v_{\theta_m}^\circ \end{bmatrix} + V^T \Delta. \tag{5.23}$$

Similarly, the position and velocity of the target at the PCA time is expressed in terms of the perturbation vector:

$$\begin{bmatrix} r_T \\ \theta_T \end{bmatrix} = \begin{bmatrix} r_{T^\circ} \\ \theta_{T^\circ} \end{bmatrix} + \begin{bmatrix} v_{r_{T^\circ}} \\ \dfrac{v_{\theta_{T^\circ}}}{r_{T^\circ}} \end{bmatrix} \delta t, \tag{5.24}$$

$$= \begin{bmatrix} r_{T^\circ} \\ \theta_{T^\circ} \end{bmatrix} + \begin{bmatrix} v_{r_{T^\circ}} \\ \dfrac{v_{\theta_{T^\circ}}}{r_{T^\circ}} \end{bmatrix} \Gamma^T \Delta, \tag{5.25}$$

$$\begin{bmatrix} v_{r_T} \\ v_{\theta_T} \end{bmatrix} = \begin{bmatrix} v_{r_T^\circ} \\ v_{\theta_T^\circ} \end{bmatrix} + \begin{bmatrix} \dfrac{v_{\theta_{T^\circ}}^2}{r_{T^\circ}} - \dfrac{k}{r_{T^\circ}^2} \\ -\dfrac{v_{r_{T^\circ}} v_{\theta_{T^\circ}}}{r_{T^\circ}} \end{bmatrix} \delta t, \tag{5.26}$$

$$= \begin{bmatrix} v_{r_T^\circ} \\ v_{\theta_T^\circ} \end{bmatrix} + \begin{bmatrix} \dfrac{v_{\theta_{T^\circ}}^2}{r_{T^\circ}} - \dfrac{k}{r_{T^\circ}^2} \\ -\dfrac{v_{r_{T^\circ}} v_{\theta_{T^\circ}}}{r_{T^\circ}} \end{bmatrix} \Gamma^T \Delta. \tag{5.27}$$

For the determination of the PCA miss, we transform polar components of the relative position and velocity to a Cartesian reference frame. The position of the interceptor relative to the target in a polar coordinate is given by

$$\begin{bmatrix} \delta r \\ \delta\theta \end{bmatrix} = \begin{bmatrix} r_{\mathrm{m}} \\ \theta_{\mathrm{m}} \end{bmatrix} - \begin{bmatrix} r_{\mathrm{T}} \\ \theta_{\mathrm{T}} \end{bmatrix}, \tag{5.28}$$

$$= \left(\begin{bmatrix} -GH_1^{\mathrm{T}} \\ \hline H^{\mathrm{T}}G \end{bmatrix} - \begin{bmatrix} v_{r_{\mathrm{T}^{\mathrm{o}}}} \\ \dfrac{v_{\theta_{\mathrm{T}^{\mathrm{o}}}}}{r_{\mathrm{T}^{\mathrm{o}}}} \end{bmatrix} \Gamma^{\mathrm{T}} \right) \Delta. \tag{5.29}$$

The position of the interceptor relative to the target in a Cartesian frame is linearly approximated by

$$\begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = \begin{bmatrix} \cos\theta_{\mathrm{T}^{\mathrm{o}}} & -r_{\mathrm{T}^{\mathrm{o}}} \sin\theta_{\mathrm{T}^{\mathrm{o}}} \\ \sin\theta_{\mathrm{T}^{\mathrm{o}}} & r_{\mathrm{T}^{\mathrm{o}}} \cos\theta_{\mathrm{T}^{\mathrm{o}}} \end{bmatrix} \left(\begin{bmatrix} -GH_1^{\mathrm{T}} \\ \hline H^{\mathrm{T}}G \end{bmatrix} - \begin{bmatrix} v_{r_{\mathrm{T}^{\mathrm{o}}}} \\ \dfrac{v_{\theta_{\mathrm{T}^{\mathrm{o}}}}}{r_{\mathrm{T}^{\mathrm{o}}}} \end{bmatrix} \Gamma^{\mathrm{T}} \right) \Delta, \tag{5.30}$$

$$:= P^{\mathrm{T}}\Delta. \tag{5.31}$$

The velocity of the interceptor in Cartesian coordinates can be expressed by

$$\begin{aligned} \begin{bmatrix} v_{x_{\mathrm{m}}} \\ v_{y_{\mathrm{m}}} \end{bmatrix} &= \begin{bmatrix} \cos\theta_{\mathrm{T}^{\mathrm{o}}} & -r_{\mathrm{T}^{\mathrm{o}}} \sin\theta_{\mathrm{T}^{\mathrm{o}}} \\ \sin\theta_{\mathrm{T}^{\mathrm{o}}} & r_{\mathrm{T}^{\mathrm{o}}} \cos\theta_{\mathrm{T}^{\mathrm{o}}} \end{bmatrix} \begin{bmatrix} v^{\mathrm{o}}_{r_{\mathrm{m}}} \\ v^{\mathrm{o}}_{\theta_{\mathrm{m}}} \end{bmatrix} \\ &\quad + \begin{bmatrix} \cos\theta_{\mathrm{T}^{\mathrm{o}}} & -r_{\mathrm{T}^{\mathrm{o}}} \sin\theta_{\mathrm{T}^{\mathrm{o}}} \\ \sin\theta_{\mathrm{T}^{\mathrm{o}}} & r_{\mathrm{T}^{\mathrm{o}}} \cos\theta_{\mathrm{T}^{\mathrm{o}}} \end{bmatrix} V^{\mathrm{T}}\Delta \\ &\quad - \begin{bmatrix} v^{\mathrm{o}}_{r_{\mathrm{m}}} \sin\theta_{\mathrm{T}^{\mathrm{o}}} + v^{\mathrm{o}}_{\theta_{\mathrm{m}}} \cos\theta_{\mathrm{T}^{\mathrm{o}}} \\ -v^{\mathrm{o}}_{r_{\mathrm{m}}} \cos\theta_{\mathrm{T}^{\mathrm{o}}} + v^{\mathrm{o}}_{\theta_{\mathrm{m}}} \sin\theta_{\mathrm{T}^{\mathrm{o}}} \end{bmatrix} \delta\theta^{\mathrm{o}} \\ &:= V^{\mathrm{o}}_{\mathrm{m}} + V^{\mathrm{T}}_{\mathrm{m}}\Delta. \end{aligned} \tag{5.32}$$

Similarly, the velocity vector of the target in a Cartesian coordinate has the form

$$\begin{bmatrix} v_{x_T} \\ v_{y_T} \end{bmatrix} = \begin{bmatrix} \cos\theta_{T^o} & -r_{T^o}\sin\theta_{T^o} \\ \sin\theta_{T^o} & r_{T^o}\cos\theta_{T^o} \end{bmatrix} \begin{bmatrix} v_{r_T^o} \\ v_{\theta_T^o} \end{bmatrix}$$

$$+ \begin{bmatrix} \cos\theta_{T^o} & -r_{T^o}\sin\theta_{T^o} \\ \sin\theta_{T^o} & r_{T^o}\cos\theta_{T^o} \end{bmatrix} \begin{bmatrix} \dfrac{v_{\theta_{T^o}}^2}{r_{T^o}} - \dfrac{k}{r_{T^o}^2} \\ -\dfrac{v_{r_{T^o}} v_{\theta_{T^o}}}{r_{T^o}} \end{bmatrix} \Gamma^T \Delta$$

$$- \begin{bmatrix} v_{r_T^o}\sin\theta_{T^o} + v_{\theta_T^o}\cos\theta_{T^o} \\ -v_{r_T^o}\cos\theta_{T^o} + v_{\theta_T^o}\sin\theta_{T^o} \end{bmatrix} \frac{v_{\theta_{T^o}}}{r_{T^o}} \Gamma^T \Delta$$

$$:= V_T^o + V_T^T \Delta. \tag{5.33}$$

Define the velocity of the interceptor relative to the target,

$$\begin{bmatrix} V_{x_{m/T}} \\ V_{y_{m/T}} \end{bmatrix} = V_m^o + V_m^T \Delta - (V_T^o + V_T^T \Delta) \tag{5.34}$$

$$:= V_{m/T}^o + V_{m/T}^T \Delta. \tag{5.35}$$

Applying the expression of PCA miss vector, Eq. (2.4), we obtain:

$$\mathbf{M} = P^T\Delta - \frac{(P^T\Delta)^T (V_{m/T}^o + V_{m/T}^T\Delta)}{(V_{m/T}^o + V_{m/T}^T\Delta)^T (V_{m/T}^o + V_{m/T}^T\Delta)} (V_{m/T}^o + V_{m/T}^T\Delta). \tag{5.36}$$

Finally, the PCA miss vector of a ballistic interceptor against a moving target can be approximated linearly. Let M^o be defined as in Eq. (3.28). Let Γ be defined as in Eqs. (4.24) or (4.34), respectively, depending on whether the nominal trajectory of the interceptor is hyperbolic or elliptic. Let (r_{T^o}, θ_{T^o}) be polar coordinates of the nominal intercept point. Let $(v_{r_m}^o, v_{\theta_m}^o)$ and $(v_{r_{T^o}}, v_{\theta_{T^o}})$ be the polar components of the velocities of the intercept vehicle and the target at nominal intercept, respectively. Then the linear approximation of the PCA miss vector is

$$\mathbf{M} \cong \left(I - \frac{V_{m/T}^o (V_{m/T}^o)^T}{(V_{m/T}^o)^T V_{m/T}^o} \right) P^T \Delta, \tag{5.37}$$

where

$$P^{\mathrm{T}}\Delta := \begin{bmatrix} \cos\theta_{\mathrm{T^o}} & -r_{\mathrm{T^o}}\sin\theta_{\mathrm{T^o}} \\ \sin\theta_{\mathrm{T^o}} & r_{\mathrm{T^o}}\cos\theta_{\mathrm{T^o}} \end{bmatrix} \left(M^{\mathrm{o}} - \begin{bmatrix} v_{r_{\mathrm{T^o}}} \\ \dfrac{v_{\theta_{\mathrm{T^o}}}}{r_{\mathrm{T^o}}} \end{bmatrix} \Gamma^{\mathrm{T}}\Delta \right), \tag{5.38}$$

and

$$V^{\mathrm{o}}_{\mathrm{m/T}} := \begin{bmatrix} \cos\theta_{\mathrm{T^o}} & -\sin\theta_{\mathrm{T^o}} \\ \sin\theta_{\mathrm{T^o}} & \cos\theta_{\mathrm{T^o}} \end{bmatrix} \left(\begin{bmatrix} v^{\mathrm{o}}_{r_{\mathrm{m}}} \\ v^{\mathrm{o}}_{\theta_{\mathrm{m}}} \end{bmatrix} - \begin{bmatrix} v_{r_{\mathrm{T^o}}} \\ v_{\theta_{\mathrm{T^o}}} \end{bmatrix} \right). \tag{5.39}$$

We can use the expression of the PCA miss vector in Eq. (5.37) to evaluate the statistical properties of the miss distance, for given statistical properties of the cut-off errors. Denote

$$\mathbf{M} \cong \left(I - \frac{V^{\mathrm{o}}_{\mathrm{m/T}}(V^{\mathrm{o}}_{\mathrm{m/T}})^{\mathrm{T}}}{(V^{\mathrm{o}}_{\mathrm{m/T}})^{\mathrm{T}} V^{\mathrm{o}}_{\mathrm{m/T}}} \right) P^{\mathrm{T}}\Delta \tag{5.40}$$

$$=: \Psi^{\mathrm{T}}\Delta, \tag{5.41}$$

and suppose the perturbation in cut-off parameters has a normal distribution, with mean $\bar{\Delta}$ and covariance matrix $\mathbf{P}_\Delta$, that is

$$\Delta \sim \mathbf{N}(\bar{\Delta}, \mathbf{P}_\Delta). \tag{5.42}$$

Then the miss vector $\mathbf{M}$ is also normal with mean $\Psi^{\mathrm{T}}\bar{\Delta}$ and covariance matrix $\Psi^{\mathrm{T}}\mathbf{P}_\Delta\Psi$,

$$\mathbf{M} \sim \mathbf{N}(\Psi^{\mathrm{T}}\bar{\Delta}, \Psi^{\mathrm{T}}\mathbf{P}_\Delta\Psi). \tag{5.43}$$

Furthermore, the expected square miss distance (i.e., the expected value of the square of the miss distance) is given by

$$\mathbf{E}[\|\mathbf{M}\|_2^2] = tr(\Psi^{\mathrm{T}}(\mathbf{P}_\Delta + \bar{\Delta}\bar{\Delta})\Psi), \tag{5.44}$$

and can be used as a figure of merit for guidance.

A closed form linear expression has been presented for the predicted PCA miss vector in terms of perturbations of the cut-off conditions. Of course, the value of this result must be judged by how well it approximates the actual PCA miss in practice. While such an error analysis is difficult to make in general, we have substantial numerical evidence to suggest that this process works well for small perturbations in cut-off conditions. Roughly speaking, when the perturbations in cut-off conditions are of the order of a few percents, the difference between the actual PCA miss and that predicted by the linear approximation will be of the order of a few percents also. Accordingly, this section presents a typical example where this comparison is made. Suppose that the interceptor and target are above the sensible atmosphere, and the earth is a perfectly non-rotating homogeneous sphere. Also suppose that there is no interaction with other celestial bodies. Assume that the mission of the interceptor, in hyperbolic motion, is to intercept the target at the given nominal point $(r_{T^o}, \theta_{T^o}) = (9099\,\text{km}, 0.7854\,\text{rad})$ and the nominal cut-off conditions of the intercept vehicle for the mission are $r_1^o = 6877.2024\,\text{km}$, $\theta_1^o = 0\,\text{rad}$, $v_{r_1}^o = 0.2383\,\text{km/s}$, $v_{\theta_1}^o = 17.0224\,\text{km/s}$. The radius of the earth is 6378.135 km. The k is $3.98601 \times 10^5\,\text{km}^3/\text{s}$. The nominal semi-major axes of the missile and target are given by $a^o = -2292.14\,\text{km}$ and $a_T = -14864.04\,\text{km}$, respectively.

Consider the different cases of small perturbations in cut-off condition as shown in Table I. $\|\mathbf{M}\|_{\text{actual}}$ and $\|\mathbf{M}\|_{\text{linearized}}$ in Table I represent the magnitude of the exact numerical calculation and the linear approximation of the PCA miss developed in this paper, respectively, and the numbers in parentheses are the perturbations relative to nominal. As seen in Table I, the closed form expression for the predicted PCA miss satisfactorily approximates the actual PCA miss, and the approximation is better for smaller perturbations in cut-off conditions.

TABLE I Comparison of magnitudes of linearly approximated PCA miss vector and actual PCA miss vector

Case	0	1	2	3	4	5
r_1 (km)	6877.20 (0%)	6882.70 (0.08%)	6885.45 (0.12%)	6888.20 (0.16%)	6890.95 (0.20%)	6893.70 (0.24%)
θ_1 (rad)	0 (0%)	0.0012 (0.12%)	0.0018 (0.18%)	0.0024 (0.24%)	0.0030 (0.30%)	0.0036 (0.36%)
v_{r_1} (km/s)	0.2383 (0%)	0.2387 (0.16%)	0.2389 (0.23%)	0.2391 (0.34%)	0.2393 (0.42%)	0.2395 (0.50%)
v_{θ_1} (km/s)	17.0224 (0%)	17.0560 (0.20%)	17.0730 (0.30%)	17.090 (0.40%)	17.1070 (0.50%)	17.1240 (0.60%)
$\|\mathbf{M}_{\text{actual}}\|$ (km)	0	4.5100	6.7075	8.8892	11.0444	13.1782
$\|\mathbf{M}_{\text{predicted}}\|$ (km)	0	4.5020	6.7529	9.0094	11.2603	13.5114
Difference (km)	0 (0%)	0.0161 (0.35%)	0.0552 (0.82%)	0.1325 (1.49%)	0.2329 (2.10%)	0.3561 (2.71%)

6 GUIDANCE LAW IN THE ABSENCE OF GRAVITY

In the absence of gravity, the equations of relative motion between interceptor and target are

$$\dot{\mathbf{R}} = \mathbf{V}, \tag{6.1}$$

$$\dot{\mathbf{V}} = \mathbf{U}, \tag{6.2}$$

where **R**, **V** and **U** are relative position vector, relative velocity vector and guidance acceleration vector of the interceptor, respectively. The PCA miss is then evaluated by Eq. (2.4), that is, if at time t_o we turn off the boosters (i.e. $\mathbf{U}=0$, $t > t_o$), then the resulting miss $\mathbf{M}(t_o)$ will be given by Eq. (2.4). Let us postulate a guidance law of the form

$$\mathbf{U} = \alpha\mathbf{R} + \beta\mathbf{V}, \tag{6.3}$$

where α and β are parameters that we choose to ensure

$$\frac{\mathrm{d}(\mathbf{M}\cdot\mathbf{M})}{\mathrm{d}t} < 0. \tag{6.4}$$

Taking the derivative of Eq. (2.4) with respect to time, we have

$$\dot{\mathbf{M}} = \frac{2(\mathbf{R}\cdot\mathbf{V})(\mathbf{V}\cdot\dot{\mathbf{V}})}{(\mathbf{V}\cdot\mathbf{V})^2}\mathbf{V} - \frac{(\mathbf{R}\cdot\dot{\mathbf{V}})}{(\mathbf{V}\cdot\mathbf{V})}\mathbf{V} - \frac{(\mathbf{R}\cdot\mathbf{V})}{(\mathbf{V}\cdot\mathbf{V})}\dot{\mathbf{V}}. \tag{6.5}$$

By substituting $\dot{\mathbf{V}} = \mathbf{U} = \alpha\mathbf{R} + \beta\mathbf{V}$, we obtain

$$\dot{\mathbf{M}} = \alpha\left[\frac{2(\mathbf{R}\cdot\mathbf{V})^2}{(\mathbf{V}\cdot\mathbf{V})^2} - \frac{(\mathbf{R}\cdot\mathbf{R})}{(\mathbf{V}\cdot\mathbf{V})}\right]\mathbf{V} - \alpha\frac{(\mathbf{R}\cdot\mathbf{V})}{(\mathbf{V}\cdot\mathbf{V})}\mathbf{R}, \tag{6.6}$$

which, remarkably, is independent of β. Hence

$$\frac{\mathrm{d}(\mathbf{M}\cdot\mathbf{M})}{\mathrm{d}t} = 2(\mathbf{M}\cdot\dot{\mathbf{M}}) \tag{6.7}$$

$$= 2\alpha\frac{(\mathbf{R}\cdot\mathbf{V})}{(\mathbf{V}\cdot\mathbf{V})}\left[\frac{(\mathbf{R}\cdot\mathbf{V})^2}{(\mathbf{V}\cdot\mathbf{V})} - (\mathbf{R}\cdot\mathbf{R})\right] \tag{6.8}$$

$$= -2\alpha\frac{(\mathbf{R}\cdot\mathbf{V})}{(\mathbf{V}\cdot\mathbf{V})}(\mathbf{M}\cdot\mathbf{M}), \tag{6.9}$$

where we have used the identity

$$\mathbf{M} \cdot \mathbf{M} = (\mathbf{R} \cdot \mathbf{R}) - \frac{(\mathbf{R} \cdot \mathbf{V})^2}{(\mathbf{V} \cdot \mathbf{V})}. \tag{6.10}$$

Since Eq. (6.9) is independent of β, β does not contribute to the time derivative of the magnitude of the miss vector. For a chosen time constant τ, we let

$$\frac{1}{\tau} = 2\alpha \frac{(\mathbf{R} \cdot \mathbf{V})}{(\mathbf{V} \cdot \mathbf{V})}, \tag{6.11}$$

which can be solved for α as

$$\alpha = \frac{(\mathbf{V} \cdot \mathbf{V})}{2\tau(\mathbf{R} \cdot \mathbf{V})}. \tag{6.12}$$

Although the parameter β has no influence on the reduction of the miss distance, it can be used to prevent the magnitude of the relative velocity vector from increasing. Set the time history of the magnitude of relative velocity constant:

$$\|\mathbf{V}(t)\| = \text{constant}. \tag{6.13}$$

Differentiation of the magnitude of the relative velocity with respect to time yields

$$\frac{\mathrm{d}(\mathbf{V} \cdot \mathbf{V})}{\mathrm{d}t} = 2\mathrm{V} \cdot \dot{\mathbf{V}} \tag{6.14}$$

$$= 0. \tag{6.15}$$

By substituting $\dot{\mathbf{V}} = \mathbf{U} = \alpha\mathbf{R} + \beta\mathbf{V}$ in Eq. (6.3) into Eq. (6.15), we have

$$\mathbf{V} \cdot \dot{\mathbf{V}} = \mathbf{V} \cdot \mathbf{U} \tag{6.16}$$

$$= \mathbf{V} \cdot (\alpha\mathbf{R} + \beta\mathbf{V}) \tag{6.17}$$

$$= \alpha(\mathbf{R} \cdot \mathbf{V}) + \beta(\mathbf{V} \cdot \mathbf{V}) \tag{6.18}$$

$$= 0. \tag{6.19}$$

Solving for the parameter β in Eq. (6.18), we obtain

$$\beta = -\alpha \frac{\mathbf{R} \cdot \mathbf{V}}{\mathbf{V} \cdot \mathbf{V}}. \tag{6.20}$$

Replacing α in Eq. (6.20) by Eq. (6.12), β is determined by

$$\beta = -\frac{1}{2\tau}. \tag{6.21}$$

From the above derivations, we conclude that, in the absence of gravity, the guidance law

$$\mathbf{U} = \frac{(\mathbf{V}\cdot\mathbf{V})}{2\tau(\mathbf{R}\cdot\mathbf{V})}\mathbf{R} - \frac{1}{2\tau}\mathbf{V}, \tag{6.22}$$

ensures that the magnitude of the PCA miss decays exponentially according to

$$\frac{\mathrm{d}(\mathbf{M}\cdot\mathbf{M})}{\mathrm{d}t} = -\frac{1}{\tau}(\mathbf{M}\cdot\mathbf{M}), \tag{6.23}$$

and simultaneously ensures a constant magnitude of the relative velocity.

7 APPLICATION TO INTERCEPTIONS IN THE GRAVITY FIELD

In the presence of gravity, consider the equations of motion for a missile and a target,

$$\dot{\mathbf{r}}_{\mathrm{m}} = \mathbf{v}_{\mathrm{m}}, \tag{7.1}$$

$$\dot{\mathbf{v}}_{\mathrm{m}} = -\frac{k}{r_{\mathrm{m}}^3}\mathbf{r}_{\mathrm{m}} + \mathbf{U}, \tag{7.2}$$

$$\dot{\mathbf{r}}_{\mathrm{T}} = \mathbf{v}_{\mathrm{T}}, \tag{7.3}$$

$$\dot{\mathbf{v}}_{\mathrm{T}} = -\frac{k}{r_{\mathrm{T}}^3}\mathbf{r}_{\mathrm{T}}, \tag{7.4}$$

respectively, where $\mathbf{U}$ is the guidance acceleration as defined in Eq. (6.22).

The equations of relative motion of the interceptor and target become

$$\dot{\mathbf{R}} = \mathbf{V}, \tag{7.5}$$

$$\dot{\mathbf{V}} = \mathbf{A} + \mathbf{U}, \tag{7.6}$$

where

$$\mathbf{A} := \frac{k}{r_{\mathrm{T}}^3}\mathbf{r}_{\mathrm{T}} - \frac{k}{r_{\mathrm{m}}^3}\mathbf{r}_{\mathrm{m}}. \tag{7.7}$$

Originally, in Section 2, the PCA miss vector was computed as a projection of the relative position onto the relative velocity for two particles undergoing rectilinear uniform motion. In the absence of gravity, the magnitude of the PCA miss vector is constant with respect to time. In the presence of gravity, we still use the expression (5.37) for the miss vector, under the condition that the engagement is relatively short so that the velocities of the interceptor and target do not change much, which justifies the assumption of quasi-rectilinear motion. We will call the expression (5.37), in the presence of gravity, the Predicted Point of Closest Approach (PPCA) miss vector. The time derivative of the square of the magnitude of the PPCA miss vector $\mathbf{M}$, over the relatively short engagement, has the form

$$\frac{\mathrm{d}(\mathbf{M}\cdot\mathbf{M})}{\mathrm{d}t} = 2(\mathbf{M}\cdot\mathbf{M}) \tag{7.8}$$

$$= -2\alpha\frac{(\mathbf{R}\cdot\mathbf{V})}{(\mathbf{V}\cdot\mathbf{V})}(\mathbf{M}\cdot\mathbf{M}) - 2\frac{(\mathbf{R}\cdot\mathbf{V})}{(\mathbf{V}\cdot\mathbf{V})}\left[(\mathbf{R}\cdot\mathbf{A}) - \frac{(\mathbf{R}\cdot\mathbf{V})}{(\mathbf{V}\cdot\mathbf{V})}(\mathbf{V}\cdot\mathbf{A})\right]. \tag{7.9}$$

Let

$$\frac{1}{\tau} = 2\alpha\frac{(\mathbf{R}\cdot\mathbf{V})}{(\mathbf{V}\cdot\mathbf{V})}. \tag{7.10}$$

Equation (7.9) then becomes

$$\frac{\mathrm{d}(\mathbf{M}\cdot\mathbf{M})}{\mathrm{d}t} = -\frac{1}{\tau}(\mathbf{M}\cdot\mathbf{M}) - \frac{1}{\tau}\left[\frac{1}{\alpha}(\mathbf{R}\cdot\mathbf{A}) - \frac{1}{\alpha}\frac{(\mathbf{R}\cdot\mathbf{V})}{(\mathbf{V}\cdot\mathbf{V})}(\mathbf{V}\cdot\mathbf{A})\right]. \tag{7.11}$$

Since the missile is close to the target around the PCA, Eq. (7.7) can be approximated by

$$\mathbf{A} \sim \frac{k}{r_{\mathrm{m}}^3}(\mathbf{r}_{\mathrm{T}} - \mathbf{r}_{\mathrm{m}}) = \frac{k}{r_{\mathrm{m}}^3}\mathbf{R}. \tag{7.12}$$

Substituting Eq. (7.12) into (7.11), we have

$$\frac{\mathrm{d}(\mathbf{M}\cdot\mathbf{M})}{\mathrm{d}t} \sim -\frac{1}{\tau}(\mathbf{M}\cdot\mathbf{M}) - \frac{1}{\tau}\left[\frac{k}{\alpha r_{\mathrm{m}}^3}\left[-(\mathbf{R}\cdot\mathbf{R}) + \frac{(\mathbf{R}\cdot\mathbf{V})^2}{(\mathbf{V}\cdot\mathbf{V})}\right]\right] \tag{7.13}$$

$$= -\frac{1}{\tau}(\mathbf{M}\cdot\mathbf{M}) + \frac{1}{\tau}\left[\frac{k}{\alpha r_{\mathrm{m}}^3}(\mathbf{M}\cdot\mathbf{M})\right] \tag{7.14}$$

$$= -\frac{1}{\tau}(\mathbf{M}\cdot\mathbf{M})\left[1 - \frac{k}{\alpha r_{\mathrm{m}}^3}\right]. \tag{7.15}$$

Equation (7.15) implies that if the ratio of gravitational acceleration to α is much less than the unity, then the guidance law (6.22) causes the miss to decay approximately exponentially, that is,

$$\frac{|k/r_{\mathrm{m}}^3|}{|\alpha|} \ll 1 \Rightarrow \frac{\mathrm{d}(\mathbf{M}\cdot\mathbf{M})}{\mathrm{d}t} \sim -\frac{1}{\tau}(\mathbf{M}\cdot\mathbf{M}). \tag{7.16}$$

Let us now analyze the effect of the guidance law (6.22) on the magnitude of the relative velocity in the presence of gravity.

Since

$$\dot{\mathbf{V}} = \mathbf{A} + \mathbf{U}, \tag{7.17}$$

$$= \mathbf{A} + \alpha\mathbf{R} + \beta\mathbf{V}, \tag{7.18}$$

the time derivative of relative velocity becomes

$$\mathbf{V}\cdot\dot{\mathbf{V}} = \mathbf{V}\cdot(\mathbf{A} + \alpha\mathbf{R} + \beta\mathbf{V}), \tag{7.19}$$

$$= \mathbf{A}\cdot\mathbf{V} + \alpha(\mathbf{R}\cdot\mathbf{V}) + \beta(\mathbf{V}\cdot\mathbf{V}). \tag{7.20}$$

Replacing **A** by Eq. (7.12), α by Eq. (6.12) and β by Eq. (6.21), Eq. (7.20) is approximated by

$$\mathbf{V}\cdot\dot{\mathbf{V}} \sim \frac{k}{r_{\mathrm{m}}^3}(\mathbf{R}\cdot\mathbf{V}) + \frac{(\mathbf{V}\cdot\mathbf{V})(\mathbf{R}\cdot\mathbf{V})}{2\tau(\mathbf{R}\cdot\mathbf{V})} - \frac{(\mathbf{V}\cdot\mathbf{V})}{2\tau}, \tag{7.21}$$

$$= \frac{k}{r_{\mathrm{m}}^3}(\mathbf{R}\cdot\mathbf{V}). \tag{7.22}$$

Hence

$$\frac{d/dt(\mathbf{V}\cdot\mathbf{V})}{(\mathbf{V}\cdot\mathbf{V})} = \frac{2\mathbf{V}\cdot\dot{\mathbf{V}}}{\mathbf{V}\cdot\mathbf{V}} \tag{7.23}$$

$$\sim \frac{2k}{r_{\mathrm{m}}^3}\frac{(\mathbf{R}\cdot\mathbf{V})}{\mathbf{V}\cdot\mathbf{V}} \tag{7.24}$$

$$= \frac{k}{r_{\mathrm{m}}^3 \alpha\tau}. \tag{7.25}$$

Equation (7.25) implies that if the ratio of gravitational acceleration to α is small, then the guidance law (6.22) will keep the magnitude of the velocity approximately constant, that is

$$\frac{|k/r_{\mathrm{m}}^3|}{|\alpha|} \ll 1 \Rightarrow (\mathbf{V}\cdot\mathbf{V}) \sim \text{constant}. \tag{7.26}$$

8 NUMERICAL EXAMPLE

In this section, we study the performance of the guidance law against a non-maneuvering target in the presence of gravity. The effect of the guidance law gain on the engagement duration, the miss distance and the interceptor velocity will be investigated. Assume that the interceptor moves in a gravitational inverse-square central force field above the sensible atmosphere. Suppose that there is no interaction with other celestial bodies. Also assume that the interceptor is on a hyperbolic orbit, and has small perturbations in cut-off conditions which cause a miss. Suppose that the implementation of the guidance law is not contaminated by navigation errors. Assume that the interceptor has initial conditions $x_{\mathrm{m}}(0) = 6566.69$ km, $y_{\mathrm{m}}(0) = 5679.22$ km, $v_{\mathrm{m}x}(0) = -2.08$ km/s, $v_{\mathrm{m}y} = 16.55$ km/s, and the target has initial conditions $x_{\mathrm{T}}(0) = 6613.32$ km, $y_{\mathrm{T}}(0) = 5827.58$ km, $v_{\mathrm{T}x}(0) = -2.94$ km/s, $v_{\mathrm{T}y} = 10.42$ km/s. The radius of Earth is 6378.135 km. The constant k is $3.98601 \times 10^5\ \mathrm{km}^3/\mathrm{s}$.

Figure 4 illustrates the time histories of the magnitudes of the PPCA miss distance vector near the PCA, corresponding to various time constants. The dash-dotted line in Fig. 4 represents the time history under the condition that the interceptor is moving on a perturbed hyperbolic trajectory toward the target without guidance law. Notice that the flat dash-dotted line states that the magnitude of the PPCA

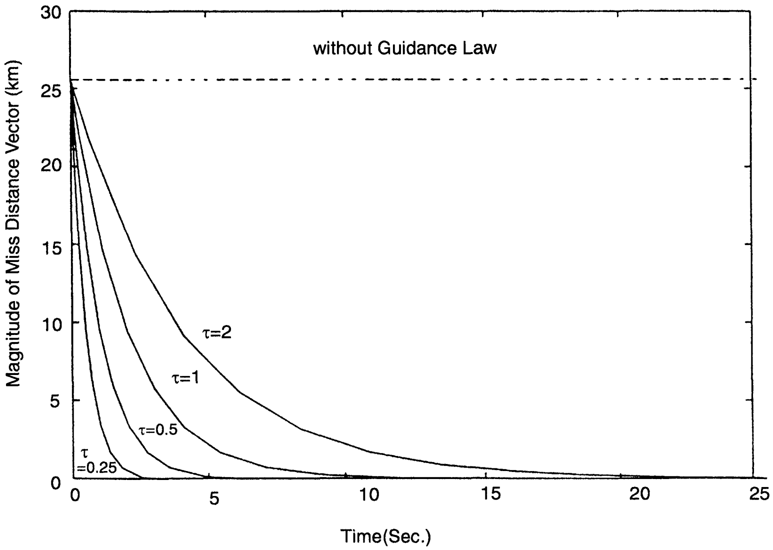

FIGURE 4 Time histories of the predicted PCA miss distance with respect to different guidance law gains in case 4 of Table I.

miss vector is constant, as expected from the definition of PPCA miss vector for the case of relatively short engagement. The solid lines show the time histories after the interceptor activates its guidance law. Parameter τ is a time constant in the PCA miss dynamics after the guidance law is activated. This parameter affects how fast the PCA miss decays exponentially, specifically a small τ, which increases the guidance law gain, reduces the PCA miss rapidly. The results show that the magnitude of the miss vector decreases quasi-exponentially with respect to time in the presence of gravity under the condition that the ratio of gravitational acceleration to α is much less than unity, as illustrated in Fig. 5.

Figure 6 shows the engagement duration as a function of the guidance law gain $1/2\tau$ in the absence of measurement errors. As expected, reducing τ, which increases the guidance law gain, shortens the PCA time. This yields a tighter engagement as seen in Fig. 7.

Figure 8 illustrates the time histories of the relative velocity after the guidance law is implemented. As shown in Fig. 8, the relative velocity remains approximately constant, corresponding to various guidance

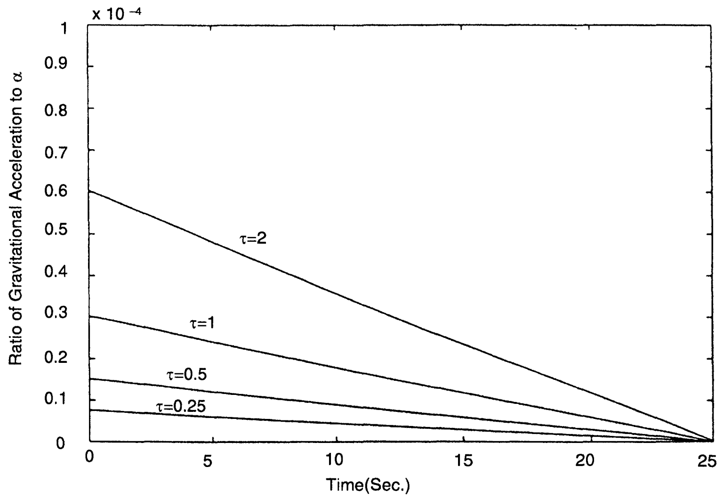

FIGURE 5 Time histories of the ratio of gravitational acceleration to α.

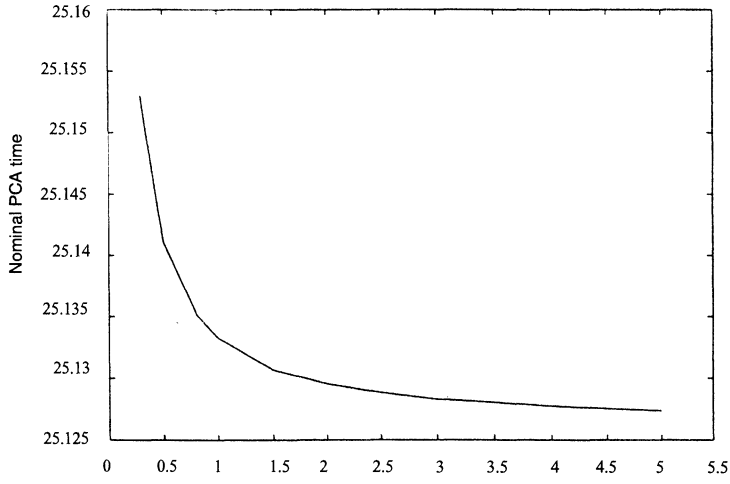

FIGURE 6 Duration of intercept engagement as a function of guidance law gain.

gains applied. This implies that an interception with high guidance gains does not require an increase in the interceptor velocity.

From these simulations, an interception with small time constant yields a tighter engagement. On the contrary, an interception with

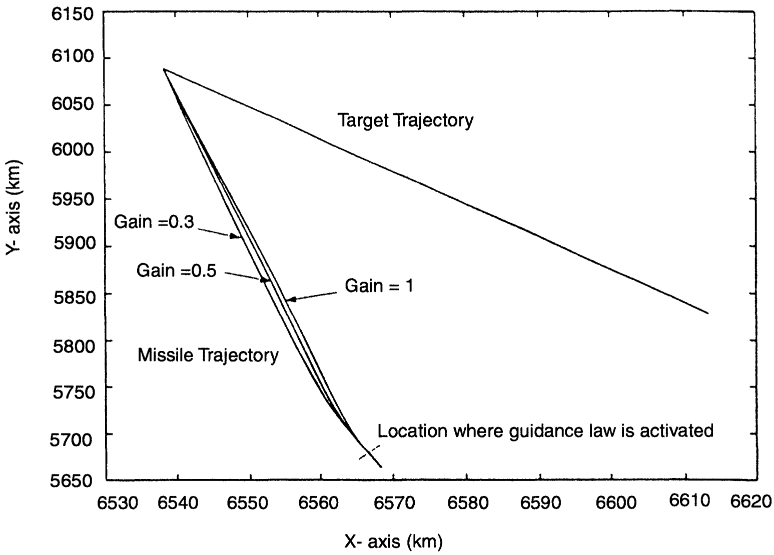

FIGURE 7 Intercept trajectories with respect to different guidance law gains in case 4 of Table I.

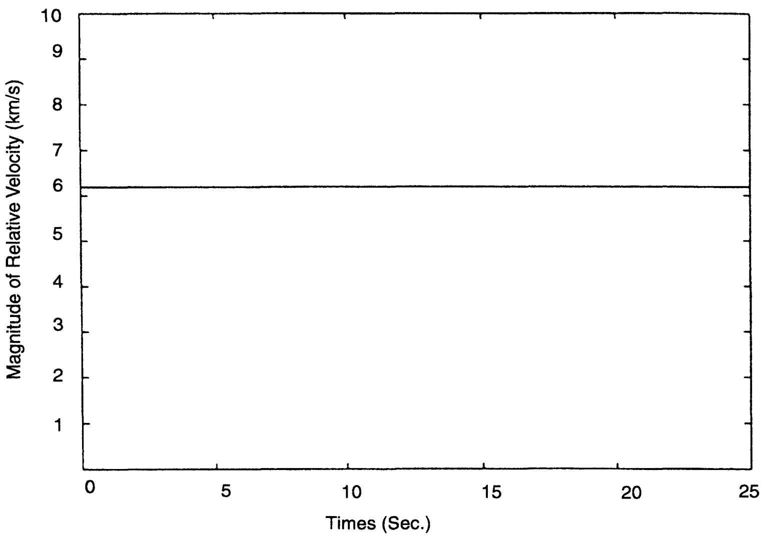

FIGURE 8 Time histories of interceptor velocity with respect to different guidance law gains in case 4 of Table I.

large time constant lasts longer. However, in either case, the interception in the presence of gravity requires approximately constant velocity of the interceptor.

9 RMS OF MISS DUE TO NAVIGATION ERRORS IN GUIDANCE LAW

We have developed a new guidance law that causes the magnitude of the PPCA miss vector to decrease exponentially or quasi-exponentially, depending on the relative magnitude of the gravity field. The performance of the guidance law depends primarily on the measured data, which is the relative position and velocity of the missile and target. We will investigate the performance degradation in terms of the rms of the PPCA miss, due to random navigation errors in implementing the guidance. In addition we will study how the guidance gain affects the accuracy on the interception when the engagement is perturbed by sensor noise.

In the presence of gravitational forces, consider the equations of motion for the missile and target,

$$\dot{\mathbf{r}}_{\mathrm{m}} = \mathbf{v}_{\mathrm{m}}, \tag{9.1}$$

$$\dot{\mathbf{v}}_{\mathrm{m}} = \mathbf{g}_{\mathrm{m}} + \mathbf{U}, \tag{9.2}$$

and

$$\dot{\mathbf{r}}_{\mathrm{T}} = \mathbf{v}_{\mathrm{T}}, \tag{9.3}$$

$$\dot{\mathbf{v}}_{\mathrm{T}} = \mathbf{g}_{\mathrm{T}}, \tag{9.4}$$

where subscripts m and T represent missile and target, respectively,

$$\mathbf{U} = \frac{\mathbf{V}\cdot\mathbf{V}}{2\tau(\mathbf{R}\cdot\mathbf{V})}\mathbf{R} - \frac{1}{2\tau}\mathbf{V}, \tag{9.5}$$

$$\mathbf{R} = \mathbf{r}_{\mathrm{m}} - \mathbf{r}_{\mathrm{T}}, \tag{9.6}$$

$$\mathbf{V} = \mathbf{v}_{\mathrm{m}} - \mathbf{v}_{\mathrm{T}}, \tag{9.7}$$

$$\mathbf{g}_{\mathrm{m}} = -\frac{k}{r_{\mathrm{m}}}\mathbf{r}_{\mathrm{m}}, \tag{9.8}$$

$$\mathbf{g}_{\mathrm{T}} = -\frac{k}{r_{\mathrm{T}}}\mathbf{r}_{\mathrm{T}}. \tag{9.9}$$

Equations (9.1)–(9.4) yield, in expanded form

$$\dot{x}_{\mathrm{m}} = v_{\mathrm{m}x}, \tag{9.10}$$

$$\dot{y}_{\mathrm{m}} = v_{\mathrm{m}y}, \tag{9.11}$$

$$\begin{aligned}\dot{v}_{\mathrm{m}x} = &-\frac{k}{(x_{\mathrm{m}}^2 + y_{\mathrm{m}}^2)^{3/2}} x_{\mathrm{m}} \\ &+ \frac{(v_{\mathrm{m}x} - v_{\mathrm{T}x})^2 + (v_{\mathrm{m}y} - v_{\mathrm{T}y})^2}{2\tau[(x_{\mathrm{m}} - x_{\mathrm{T}})(v_{\mathrm{m}x} - v_{\mathrm{T}x}) + (y_{\mathrm{m}} - y_{\mathrm{T}})(v_{\mathrm{m}y} - v_{\mathrm{T}y})]}(x_{\mathrm{m}} - x_{\mathrm{T}}) \\ &- \frac{1}{2\tau}(v_{\mathrm{m}x} - v_{\mathrm{T}x}),\end{aligned} \tag{9.12}$$

$$\begin{aligned}\dot{v}_{\mathrm{m}y} = &-\frac{k}{(x_{\mathrm{m}}^2 + y_{\mathrm{m}}^2)^{3/2}} y_{\mathrm{m}} \\ &+ \frac{(v_{\mathrm{m}x} - v_{\mathrm{T}x})^2 + (v_{\mathrm{m}y} - v_{\mathrm{T}y})^2}{2\tau[(x_{\mathrm{m}} - x_{\mathrm{T}})(v_{\mathrm{m}x} - v_{\mathrm{T}x}) + (y_{\mathrm{m}} - y_{\mathrm{T}})(v_{\mathrm{m}y} - v_{\mathrm{T}y})]}(y_{\mathrm{m}} - y_{\mathrm{T}}) \\ &- \frac{1}{2\tau}(v_{\mathrm{m}y} - v_{\mathrm{T}y}),\end{aligned} \tag{9.13}$$

$$\dot{x}_{\mathrm{T}} - v_{\mathrm{T}x}, \tag{9.14}$$

$$\dot{y}_{\mathrm{T}} = v_{\mathrm{T}y}, \tag{9.15}$$

$$\dot{v}_{\mathrm{T}x} = -\frac{k}{(x_{\mathrm{T}}^2 + y_{\mathrm{T}}^2)^{3/2}} x_{\mathrm{T}}, \tag{9.16}$$

$$\dot{v}_{\mathrm{T}y} = -\frac{k}{(x_{\mathrm{T}}^2 + y_{\mathrm{T}}^2)^{3/2}} y_{\mathrm{T}}. \tag{9.17}$$

Let $x = [x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]^{\mathrm{T}} = [x_{\mathrm{m}}, y_{\mathrm{m}}, v_{\mathrm{m}x}, v_{\mathrm{m}y}, x_{\mathrm{T}}, y_{\mathrm{T}}, v_{\mathrm{T}x}, v_{\mathrm{T}y}]^{\mathrm{T}}$. Then we have a state-space representation of the motion:

$$\dot{x}_1 = x_3, \tag{9.18}$$

$$\dot{x}_2 = x_4, \tag{9.19}$$

$$\begin{aligned}\dot{x}_3 = &-\frac{k}{(x_1^2 + x_2^2)^{3/2}} x_1 \\ &+ \frac{(x_3 - x_7)^2 + (x_4 - x_8)^2}{2\tau[(x_1 - x_5)(x_3 - x_7) + (x_2 - x_6)(x_4 - x_8)]}(x_1 - x_5) \\ &- \frac{1}{2\tau}(x_3 - x_7),\end{aligned} \tag{9.20}$$

$$\dot{x}_4 = -\frac{k}{(x_1^2 + x_2^2)^{3/2}} x_2 + \frac{(x_3 - x_7)^2 + (x_4 - x_8)^2}{2\tau[(x_1 - x_5)(x_3 - x_7) + (x_2 - x_6)(x_4 - x_8)]}(x_2 - x_6) - \frac{1}{2\tau}(x_4 - x_8), \tag{9.21}$$

$$\dot{x}_5 = x_7, \tag{9.22}$$

$$\dot{x}_6 = x_8, \tag{9.23}$$

$$\dot{x}_7 = -\frac{k}{(x_5^2 + x_6^2)^{3/2}} x_5, \tag{9.24}$$

$$\dot{x}_8 = -\frac{k}{(x_5^2 + x_6^2)^{3/2}} x_6. \tag{9.25}$$

Now suppose that vectors **R** and **V** are corrupted by measurement errors and become

$$\mathbf{R} + v = \begin{bmatrix} R_x + v_x \\ R_y + v_y \end{bmatrix} \tag{9.26}$$

$$= \begin{bmatrix} x_{\mathrm{m}} - x_{\mathrm{T}} + v_x \\ y_{\mathrm{m}} - y_{\mathrm{T}} + v_y \end{bmatrix}, \tag{9.27}$$

$$\mathbf{V} + \omega = \begin{bmatrix} V_x + \omega_x \\ V_y + \omega_y \end{bmatrix} \tag{9.28}$$

$$= \begin{bmatrix} v_{\mathrm{m}x} - v_{\mathrm{T}x} + \omega_x \\ v_{\mathrm{m}y} - v_{\mathrm{T}y} + \omega_y \end{bmatrix}, \tag{9.29}$$

where $v = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$ and $\omega = \begin{bmatrix} \omega_x \\ \omega_y \end{bmatrix}$ represent errors in the measurement of relative position and velocity, respectively.

Then Eqs. (9.20) and (9.21), perturbed by sensor noise, become

$$\dot{x}_3 = -\frac{k}{(x_1^2 + x_2^2)^{3/2}} x_1 + \frac{[(x_3 - x_7 + \omega_x)^2 + (x_4 - x_8 + \omega_y)^2](x_1 - x_5 + v_x)}{2\tau[(x_1 - x_5 + v_x)(x_3 - x_7 + \omega_x) + (x_2 - x_6 + v_y)(x_4 - x_8 + \omega_y)]} - \frac{1}{2\tau}(x_3 - x_7 + \omega_x), \tag{9.30}$$

$$\dot{x}_4 = -\frac{k}{(x_1^2 + x_2^2)^{3/2}} x_2 + \frac{[(x_3 - x_7 + \omega_x)^2 + (x_4 - x_8 + \omega_y)^2](x_2 - x_6 + v_y)}{2\tau[(x_1 - x_5 + v_x)(x_3 - x_7 + \omega_x) + (x_2 - x_6 + v_y)(x_4 - x_8 + \omega_y)]} - \frac{1}{2\tau}(x_4 - x_8 + \omega_y). \tag{9.31}$$

Hence the complete system of equations of perturbed motion becomes Eqs. (9.18), (9.19), (9.30), (9.31) and (9.22)–(9.25).

If we define a miss vector as the relative position at nominal predicted PCA time, t_f^{o}, then

$$\mathbf{M} = \mathbf{R}(t_f^{\text{o}}) \tag{9.32}$$

$$= \begin{bmatrix} x_{\text{m}} - x_{\text{T}} \\ y_{\text{m}} - y_{\text{T}} \end{bmatrix} (t_f^{\text{o}}) \tag{9.33}$$

$$= \begin{bmatrix} x_1 - x_5 \\ x_2 - x_6 \end{bmatrix} (t_f^{\text{o}}). \tag{9.34}$$

Assume that the sensor noise processes v and ω are small. Linearization of the dynamic system about the nominal trajectory in the presence of sensor noise yields

$$\delta\dot{x} = \left(\frac{\partial f}{\partial x}\right)_{x^{\text{o}}}^{\text{T}} \delta x + \left(\frac{\partial f}{\partial v}\right)_{x^{\text{o}}}^{\text{T}} v + \left(\frac{\partial f}{\partial \omega}\right)_{x^{\text{o}}}^{\text{T}} \omega, \tag{9.35}$$

where the partial derivatives are evaluated along the nominal trajectory. Let us define

$$A(t) = \left(\frac{\partial f}{\partial x}\right)_{x^{\text{o}}}^{\text{T}}, \tag{9.36}$$

$$B(t) = \begin{bmatrix} \left(\frac{\partial f}{\partial v}\right)_{x^{\text{o}}}^{\text{T}} \\ \left(\frac{\partial f}{\partial \omega}\right)_{x^{\text{o}}}^{\text{T}} \end{bmatrix}, \tag{9.37}$$

$$\delta u(t) = \begin{bmatrix} v \\ \omega \end{bmatrix}. \tag{9.38}$$

Hence the equations of relative motion become a standard linear dynamic system:

$$\delta\dot{x}(t) = A(t)\delta x(t) + B(t)\delta u(t). \tag{9.39}$$

Explicitly, the elements of $A(t)$ and $B(t)$ can be evaluated as follows. We have

$$\delta\dot{x}_1 = \left(\frac{\partial f_1}{\partial x_3}\right)_{x^o} \delta x_3, \tag{9.40}$$

$$\delta\dot{x}_2 = \left(\frac{\partial f_1}{\partial x_4}\right)_{x^o} \delta x_4, \tag{9.41}$$

$$\begin{aligned}\delta\dot{x}_3 = {} & \left(\frac{\partial f_3}{\partial x_1}\right)_{x^o} \delta x_1 + \left(\frac{\partial f_3}{\partial x_2}\right)_{x^o} \delta x_2 + \left(\frac{\partial f_3}{\partial x_3}\right)_{x^o} \delta x_3 \\ & + \left(\frac{\partial f_3}{\partial x_4}\right)_{x^o} \delta x_4 + \left(\frac{\partial f_3}{\partial x_5}\right)_{x^o} \delta x_5 + \left(\frac{\partial f_3}{\partial x_6}\right)_{x^o} \delta x_6 \\ & + \left(\frac{\partial f_3}{\partial x_7}\right)_{x^o} \delta x_7 + \left(\frac{\partial f_3}{\partial x_8}\right)_{x^o} \delta x_8 + \left(\frac{\partial f_3}{\partial v_x}\right)_{x^o} \delta v_x \\ & + \left(\frac{\partial f_3}{\partial v_y}\right)_{x^o} \delta v_y + \left(\frac{\partial f_3}{\partial \omega_x}\right)_{x^o} \delta \omega_x + \left(\frac{\partial f_3}{\partial \omega_y}\right)_{x^o} \delta \omega_y,\end{aligned} \tag{9.42}$$

$$\begin{aligned}\delta\dot{x}_4 = {} & \left(\frac{\partial f_4}{\partial x_1}\right)_{x^o} \delta x_1 + \left(\frac{\partial f_4}{\partial x_2}\right)_{x^o} \delta x_2 + \left(\frac{\partial f_4}{\partial x_3}\right)_{x^o} \delta x_3 + \left(\frac{\partial f_4}{\partial x_4}\right)_{x^o} \delta x_4 \\ & + \left(\frac{\partial f_4}{\partial x_5}\right)_{x^o} \delta x_5 + \left(\frac{\partial f_4}{\partial x_6}\right)_{x^o} \delta x_6 + \left(\frac{\partial f_4}{\partial x_7}\right)_{x^o} \delta x_7 + \left(\frac{\partial f_4}{\partial x_8}\right)_{x^o} \delta x_8 \\ & + \left(\frac{\partial f_4}{\partial v_x}\right)_{x^o} \delta v_x + \left(\frac{\partial f_4}{\partial v_y}\right)_{x^o} \delta v_y + \left(\frac{\partial f_4}{\partial \omega_x}\right)_{x^o} \delta \omega_x + \left(\frac{\partial f_4}{\partial \omega_y}\right)_{x^o} \delta \omega_y,\end{aligned} \tag{9.43}$$

$$\delta\dot{x}_5 = \left(\frac{\partial f_5}{\partial x_7}\right)_{x^o} \delta x_7, \tag{9.44}$$

$$\delta\dot{x}_6 = \left(\frac{\partial f_6}{\partial x_8}\right)_{x^o} \delta x_8, \tag{9.45}$$

$$\delta\dot{x}_7 = \left(\frac{\partial f_7}{\partial x_5}\right)_{x^o} \delta x_5 + \left(\frac{\partial f_7}{\partial x_6}\right)_{x^o} \delta x_6, \tag{9.46}$$

$$\delta\dot{x}_8 = \left(\frac{\partial f_8}{\partial x_5}\right)_{x^o} \delta x_5 + \left(\frac{\partial f_8}{\partial x_6}\right)_{x^o} \delta x_6. \tag{9.47}$$

Hence we have the complete expression of the matrices $A(t)$ and $B(t)$,

$$A(t) =$$
$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{31}(t) & A_{32}(t) & A_{33}(t) & A_{34}(t) & A_{35}(t) & A_{36}(t) & A_{37}(t) & A_{38}(t) & 0 & 0 \\ A_{41}(t) & A_{42}(t) & A_{43}(t) & A_{44}(t) & A_{45}(t) & A_{46}(t) & A_{47}(t) & A_{48}(t) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{75}(t) & A_{76}(t) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{85}(t) & A_{86}(t) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}, \tag{9.48}$$

$$B(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ B_{31}(t) & B_{32}(t) & B_{33}(t) & B_{34}(t) \\ B_{41}(t) & B_{42}(t) & B_{43}(t) & B_{44}(t) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{9.49}$$

where the non-zero entries are obvious from Eqs. (9.40) to (9.47).

Now assume that

(1) The initial covariance matrix of the states is zero,

$$P(t_o) = 0. \tag{9.50}$$

(2) The noise process $\mathbf{u}$ is a zero-mean stationary Gaussian white process,

$$E[\mathbf{u}] = 0, \tag{9.51}$$

$$E[\mathbf{u}(\mathbf{t})\mathbf{u}^{\mathrm{T}}(\sigma)] = R_u\delta(t-\sigma), \tag{9.52}$$

where $\delta(t-\sigma)$ is the Dirac delta function.

Consider the model of Eq. (9.39) under the assumptions of Eqs. (9.50)–(9.52). The covariance matrix then $P(t)$ satisfies the Lyapunov equation

$$\dot{P}(t) = A(t)P(t) + P(t)A^{\mathrm{T}}(t) + B(t)R_u B^{\mathrm{T}}(t), \tag{9.53}$$

hence

$$P(t) = \int_{t_o}^{t}\int_{t_o}^{t} \Phi(t,\sigma)B(\sigma)R_u\delta(\sigma - s)B^{\mathrm{T}}(s)\Phi^{\mathrm{T}}(t,s)\,\mathrm{d}s\,\mathrm{d}\sigma \tag{9.54}$$

$$= \int_{t_o}^{t} \Phi(t,\sigma)B(\sigma)R_u B^{\mathrm{T}}(\sigma)\Phi^{\mathrm{T}}(t,\sigma)\,\mathrm{d}\sigma, \tag{9.55}$$

where the state transition matrix of the linearized dynamic system, $\Phi(t,\sigma)$, satisfies

$$\frac{\partial\Phi(t,\sigma)}{\partial t} = A(t)\Phi(t,\sigma), \tag{9.56}$$

$$\Phi(\sigma,\sigma) = I_{8\times 8}. \tag{9.57}$$

Recall that the miss distance vector is defined as the relative position at nominal predicted PCA time t_f^{o}. Then the small perturbation of the nominal predicted PCA miss vector due to sensor noise becomes

$$\delta\mathbf{M} = \begin{bmatrix} \delta x_{\mathrm{m}} - \delta x_{\mathrm{T}} \\ \delta y_{\mathrm{m}} - \delta y_{\mathrm{T}} \end{bmatrix}(t_f^{\mathrm{o}}). \tag{9.58}$$

The magnitude of the miss at t_f^{o} is

$$\|\delta\mathbf{M}\|_2^2 = (\delta x_{\mathrm{m}} - \delta x_{\mathrm{T}})^2(t_f^{\mathrm{o}}) + (\delta y_{\mathrm{m}} - \delta y_{\mathrm{T}})^2(t_f^{\mathrm{o}}). \tag{9.59}$$

The mean-square miss distance is given by

$$E[\|\delta\mathbf{M}\|_2^2] = E[\delta x_{\mathrm{m}}^2] + E[\delta x_{\mathrm{T}}^2] + E[\delta y_{\mathrm{m}}^2] + E[\delta y_{\mathrm{T}}^2] - 2(E[\delta x_{\mathrm{m}}\delta x_{\mathrm{T}}] + E[\delta y_{\mathrm{m}}\delta y_{\mathrm{T}}]). \tag{9.60}$$

Hence the root mean square of the predicted PCA miss becomes

$$\sqrt{E[\|\delta\mathbf{M}\|_2^2]} = \sqrt{P_{11}(t_f^{\mathrm{o}}) + P_{22}(t_f^{\mathrm{o}}) + P_{55}(t_f^{\mathrm{o}}) + P_{66}(t_f^{\mathrm{o}}) - 2(P_{15}(t_f^{\mathrm{o}}) + P_{26}(t_f^{\mathrm{o}}))}. \tag{9.61}$$

Suppose that the interceptor and target are above the sensible atmosphere, and the earth is a perfectly non-rotating homogeneous sphere. Also suppose that there is no interaction with other celestial bodies. Assume that the interceptor, on a hyperbolic orbit, has small perturbations in cut-off conditions which cause a miss and the interceptor activates the guidance law around the nominal intercept point to reduce the predicted PCA miss. In this numerical example, we investigate the effect of navigation errors on the performance of the guidance law in terms of rms miss. The performance of the guidance law depends primarily on the navigation measurements, which are the relative position and velocity of the missile and target. Assume that the interceptor has the initial conditions $x_{\mathrm{m}}(0) = 6566.69\,\mathrm{km}$, $y_{\mathrm{m}}(0) = 5679.22\,\mathrm{km}$, $v_{\mathrm{m}x}(0) = -2.08\,\mathrm{km/s}$, $v_{\mathrm{m}y} = 16.55\,\mathrm{km/s}$, and the target has the initial conditions $x_{\mathrm{T}}(0) = 6613.32\,\mathrm{km}$, $y_{\mathrm{T}}(0) = 5827.58\,\mathrm{km}$, $v_{\mathrm{T}x}(0) = -2.94\,\mathrm{km/s}$, $v_{\mathrm{T}y} = 10.42\,\mathrm{km/s}$. The radius of the earth is 6378.135 km. The constant k is $3.98601 \times 10^5\,\mathrm{km^3/s}$. Assume that, in Eq. (9.52), $R_u = \mathrm{diag}[0.01\,\mathrm{km}^2, 0.01\,\mathrm{km}^2, 0.01\,(\mathrm{km/s})^2, 0.01\,(\mathrm{km/s})^2]$.

Figure 9 describes the nominal PPCA miss as a function of the guidance law gain in the absence of navigation error. The plot

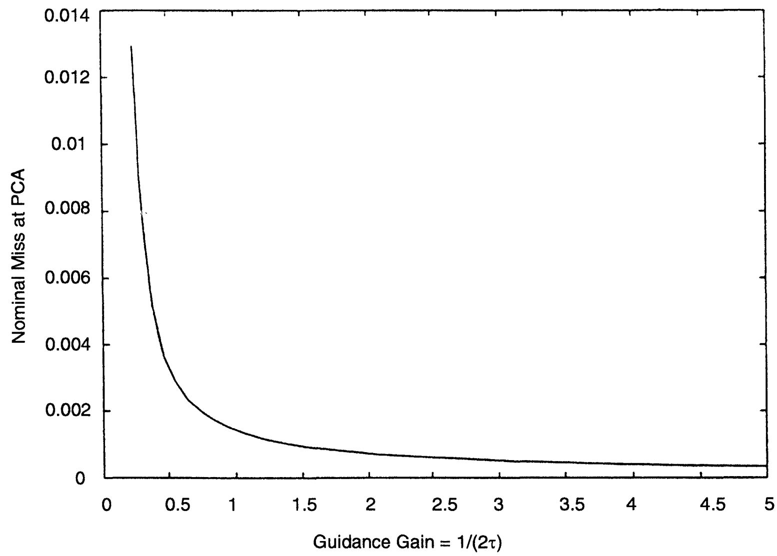

FIGURE 9 Nominal PCA miss as a function of guidance law gain.

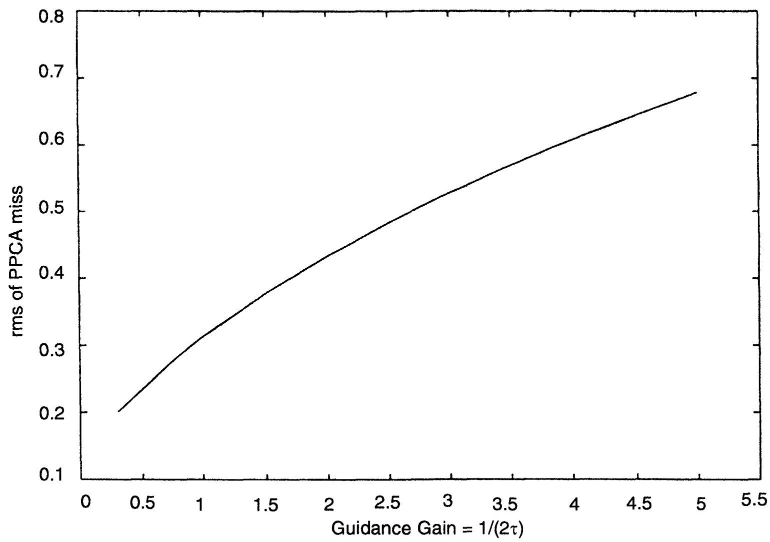

FIGURE 10 Rms PCA miss as a function of guidance law gain.

shows that using large guidance gains for the interception reduces the nominal PCA miss.

In the presence of sensor error, however, it is no longer true that a larger gain yields a better performance of the guidance law in the sense of the predicted PCA miss. Figure 10 illustrates the rms of the PPCA miss as a function of guidance law gain when the measurements of the relative position and velocity of the missile and target are contaminated by noise. Contrary to the case of nominal miss, increasing the guidance law gain increases the rms of the PPCA miss, which results in a degradation of the interception performance.

10 CONCLUSION

For an interceptor that follows a Keplerian trajectory and nominally intercepts a moving target, we have obtained a closed form linear expression for the miss distance in terms of the perturbations of the booster cut-off conditions, where the miss distance reflects the predicted miss at the point of closest approach between the interceptor and the target. This has been accomplished by first determining the position and velocity of the interceptor and target at a common time, then computing the miss distance under the assumption of rectilinear uniform motion. This common time was itself chosen as the time at which the interceptor, flying along the perturbed trajectory, is closest to the point of nominal intercept. The results of this miss distance analysis can in principle be applied in a variety of situations, from comet interception missions to anti-ballistic missiles. They can also be used to compute the expected value of the miss distance, given the statistical properties of the cut-off errors.

Next, we have proposed a new guidance law to ensure that, in the absence of gravity, the magnitude of the PCA miss decays exponentially and that the magnitude of the relative velocity remains constant as a function of time. The same guidance law has been applied to interceptors flying in a gravity field under the assumption that the gravitational acceleration is much smaller than a characteristic acceleration. The time histories of the magnitude of the miss distance were illustrated with various guidance law parameters. The results of numerical simulations show that the predicted PCA miss decays quasi-exponentially with respect to time, and state that a large guidance law gain reduces the predicted PCA miss rapidly. In addition an interception with a large gain yields a tighter engagement, but requires approximately constant velocity of the interceptor as a function of time.

Finally, we studied the performance degradation of the new guidance law in terms of the rms of the predicted PCA miss, due to random navigation errors in implementing the guidance system. The effect of the guidance law parameter on the accuracy on the interception was investigated, when the engagement was perturbed by sensor noise. The results of numerical examples suggest that increasing the guidance law gain increases the rms of the predicted PCA miss, which results in a degradation of the interception performance, despite the fact that it reduces the nominal predicted PCA miss. Hence a trade-off is required for the optimal value of the guidance gain.

References

[1] R.H. Battin, *An Introduction to the Mathematics and Methods of Astrodynamics*, AIAA, New York, 1987.

[2] S. Herrick, *Astrodynamics Vol. 1*, Van Nostrand Reinhold Company, London, 1971, pp. 77–89.

[3] J.E. Prussing and B.A. Conway, *Orbital Mechanics*, Oxford, New York, 1993, pp. 62–79.
[4] J.M.A. Danby, *Fundamentals of Celestial Mechanics*, The MacMillan Co., New York, 1962.
[5] R.R. Bate, D.D. Mueller and J.E. White, *Fundamentals of Astrodynamics*, Dover Publications, Inc., New York, 1971, Chap. 5.
[6] R.H. Battin, *Astronautical Guidance*, McGraw-Hill, Inc., New York, 1964, pp. 70–75.
[7] E.V.B. Stearns, *Navigation and Guidance in Space*, Prentice-Hall, Inc., 1963, pp. 245–247.
[8] H.H. Koelle, *Handbook of Astronautical Engineering*, McGraw-Hill, New York, 1961.
[9] C.T. Leondes, *Guidance and Control of Aerospace Vehicles*, McGraw-Hill, New York, 1963.
[10] R.H. Battin and R.M. Vaughan, An elegant Lambert algorithm, *Journal of Guidance and Control*, **7**(6), 1984, 662–670.
[11] R.H. Battin, Lambert's problem revisited, *AIAA Journal*, **15**, May 1977, pp. 707–713.
[12] R.H. Battin, T.J. Fill and S.W. Shepperd, A new transformation invariant in the orbital boundary-value problem, *Journal of Guidance and Control*, **1**(1), January–February 1978, 50–55.
[13] S.L. Nelson and P. Zarchan, Alternative approach to the solution of Lambert's problem, *Journal of Guidance, Control, and Dynamics*, **15**, July–August 1992, 1003–1009.
[14] T.J. Brand, An new approach to Lambert guidance, Charles Stark Draper Lab., MA, Rept. R-694, June 1971.
[15] R.H. Gooding, A procedure for the solution of Lambert's orbital boundary-value problem, *Celestial Mechanics*, **48**, 1990, 145–165.
[16] J.E. Prussing, A geometrical interpretation of the angles α and β in Lambert's problem, *Journal of Guidance and Control*, **2**(5), September–October 1979, 442–443.
[17] F.T. Sun, N.X. Vinh and T.J. Chern, Analytical study of the solution families of the extended godal's time equation for Lambert's problem, *J. Astr. Sci.*, **35**, 1987, 213–234.
[18] J.E. White, A Lambert targeting procedure for rocket systems that lack velocity control, *AIAA Paper*, 89-3442-CP, 1989.
[19] E.R. Lancaster and R.C. Blanchard, A unified form of Lambert's theorem, *NASA Technical Note*, D-5368, 1969.
[20] E.R. Lancaster, R.C. Blanchard and R.A. Devaney, A note on Lambert's theorem, *Journal of Spacecraft and Rockets*, **3**, 1966, 1436–1438.