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A Generalized Regular Form for Multivariable Sliding Mode Control

W. PERRUQUETTI*, J. P. RICHARD and P. BORNE

Laboratory of Automatic Control (LAIL UPRESA CNRS 8021), Ecole Centrale de LILLE, BP 48, 59 651 Villeneuve d'Ascq Cedex – France

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The paper shows how to compute a diffeomorphic state space transformation in order to put the initial mutivariable nonlinear model into an appropriate *regular form*. This form is an extension of the one proposed by Lukyanov and Utkin [9], and constitutes a guidance for a "natural" choice of the sliding surface. Then stabilization is achieved *via* a sliding mode strategy. In order to overcome the chattering phenomenon, a new non linear gain is introduced.

Keywords: Sliding mode control; Multivariable systems; Non linear systems; Stabilization

I. INTRODUCTION

Sliding mode control is based on variable structure systems theory: the control commutates in order to force the systems motions to behave on a desired surface (called the sliding surface). Sliding regimes are unaffected by perturbations satisfying the well-known matching conditions (see [1, 2, 12]). The choice of the surface is mostly related to some stabilization problem: the shape of the surface is selected *a priori*, leading to a set of parameters that are to be computed (adjusted) in order to obtain the desired dynamics ([2, 3, 10, 15]). For

*Corresponding author. e-mail: wilfrid.perruquetti@ec-lille.fr

LTI systems, a natural shape for the sliding surface is a linear one (hyperplan), but for nonlinear ones, one can tranform the initial system into:

1. a linear one and then, select an hyperplan (see [4, 12–14]);
2. transform the initial system into an appropriate form which provides a guide for the choice of the surface.

The last approch is investigated in this paper. For this, we consider MIMO systems modeled by:

$$\dot{x} = f(x) + G(x)u, \quad (M) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control vector (m inputs), $f: \mathbb{R}^n \to \mathbb{R}^n$, a smooth drift vector field, $G(x) = (g_1(x), \ldots, g_m(x))$ is an $(n \times m)$-matrix and the $g_i: \mathbb{R}^n \to \mathbb{R}^n$ are smooth vector fields, $g_{ij}(x)$ is the control gain of the jth input acting on the ith state space variable.

Section 2 gives first a static feedback in order to obtain (1) with an input gain matrix of full rank and then, solves the question of obtaining (through a regular change of coordinates) an equivalent "regular form" to (1), defined as follows:

$$(RF) \equiv \begin{cases} \dot{z}_1 = f_1^R(z_1, z_2) + G^R(z_1, z_2)u, \\ \dot{z}_2 = f_2^R(z_1, z_2), \\ z_1 \in \mathbb{R}^d, z_2 \in \mathbb{R}^{(n-d)} \end{cases} \tag{2}$$

Here, the number d may be greater or smaller than the number of input m (see Theorem 2), whereas in [9] $d = m$: thus our result is more general. On the basis of such a regular form, results of Section 3 are devoted to the stabilizing controller design. Moreover, in order to overcome the "chattering" phenomenon, that is an important drawback of sliding mode control, we introduce a new nonlinear gain for the signum function such that the gain decreases as the motions converge towards the origin.

The following notations are used throughout the paper:

- for smooth n-vector fields, $f(x)$, $g(x)$, [8]: $[f, g](x) \triangleq (\partial g(x)/\partial x) f(x) - (\partial f(x)/\partial x) g(x)$, (Lie bracket). The Ad operator is defined by $Ad_f^0 g(x) \triangleq g(x)$, $Ad_f^1(g)(x) \triangleq [f, g](x), \ldots, Ad_f^k g(x) \triangleq [f, Ad_f^{k-1} g](x)$.
- for a smooth real-valued function $\lambda(x)$, [8]: $d\lambda(x) \triangleq ((\partial\lambda/\partial x_1), \ldots, (\partial\lambda/\partial x_i), \ldots, (\partial\lambda/\partial x_n))$, (the "gradient" of λ).

- $\mathrm{Sgn}(\zeta)$ and $\mathrm{Sgn}(z)$ are respectively scalar and vector signum functions defined as follows:

$$\mathrm{Sgn}(\zeta) = \begin{cases} -1 \text{ if } \zeta < 0, \\ \ \ 1 \text{ if } \zeta > 0, \end{cases} \tag{3}$$

$$\mathrm{SGN}(z) = (\mathrm{Sgn}(z_1), \ldots, \mathrm{Sgn}(z_l)), z \in \mathbb{R}^l. \tag{4}$$

- $k(\cdot)$ stands for a function k of the state variables that may be constant.

II. OBTAINING THE REGULAR FORM

The problem is to find a diffeomorphic state space transformation $z = \phi(x)$ changing (M) (1) into (RF) (2). The proposed procedure is: firstly, transform (M) (1) into a model with full rank gain matrix; lastly, define conditions on the existence of a regular form.

II.A. On the Rank of the Input Gain Matrix

In [9], the hypothesis $\mathrm{rank}(G(x)) = m$ was combined with some additional integrability conditions to obtain (RF) (2) with $d = m$. The next theorem shows how to recover this hypothesis from the general case through a static feedback:

THEOREM 1 *Let $x_0 \in \mathbb{R}$. If $rank(G(x_0)) = r$, then there is a static feedback $u = W(x)(v^T, 0, \ldots, 0)^T$, $v \in \mathbb{R}^r$, with W nonsingular in a neighbourhood $\mathcal{N}(x_0)$ of x_0, such that*:

$$G(x)W(x) = (G'(x)|0_{n \times (m-r)}), \quad \forall x \in \mathcal{N}(x_0). \tag{5}$$

combined with (M) (1) leads to:

$$\dot{x} = f(x) + G'(x)v, \quad (M') \tag{6}$$

where v is the new control vector, $G'(x)$ is a $(n \times r)$-matrix of full rank r.

Proof This comes out directly by performing a Gauss reduction on columns (using elementary columns transformations: interchange, sum, scalar multiplication by $a(x)$). Recall that in order to compute

(5), one can perform on $G(x)$ and on the $(m \times m)$-identity matrix the same operation which leads respectively to the right-hand side of Eq. (5) and to $W(x)$. Note that $\det(W(x_0)) = 1$. ■

In the following, we consider that $\mathrm{rank}(G(x_0)) = m$, otherwise ($\mathrm{rank}(G(x_0)) = r < m$), the previous theorem allows us to consider the system (M) (1) with static feedback (5) which leads to (M') (6).

II.B. On the Existence of a Regular Form

The given results are local, but when asumption H1 (see theorem statements) holds everywhere in the state space, then the diffeomorphism is global and so are the results. The following statement can be regarded as an extension of previous results for nonlinear systems using differential geometric approach (see [8]):

THEOREM 2 *Let Δ be a distribution such that*:

(H1) Δ *is nonsingular at* x_0 (*i.e., of constant dimension* $\dim\Delta = d_\Delta \leq n$),
(H2) Δ *is involutive*: $\forall \tau_1 \in \Delta, \forall \tau_2 \in \Delta : [\tau_1, \tau_2] \in \Delta$,
(H3) *span* $\{g_1(x), \ldots, g_m(x)\} \subset \Delta$,

then there exist a neighbourhood $\mathcal{N}(x_0)$ *of* x_0 *and a local diffeomorphism* $z = \phi(x)$ *defined on* $\mathcal{N}(x_0)$, *such that* (M) (1) *is transformed into* (RF) (2) *with* $d = d_\Delta \leq n$.

The involutive closure[1] of span $\{g_1(x), \ldots, g_m(x)\}$ denoted by $\mathbf{\Delta_G}$, satisfies the assumptions of Theorem 2.

Proof Under assumptions (H1)–(H3), one can find $(n - d_\Delta)$ real valued functions λ_i such that the anihilator $\Delta^\perp = \{\omega^* \in (\mathbb{R}^n)^* : \langle \omega^*, v\rangle = 0,\ v \in \Delta\rangle$ of Δ is spanned by the covectors $d\lambda_i$ (Frobenius Theorem). Selecting $\phi_{d_\Delta + i} \triangleq \lambda_i$, for $i \in \{1 \cdots n - d_\Delta\}$ and completing the basis with real valued functions ϕ_i, for $i \in \{1 \cdots d_\Delta\}$, such that $\mathrm{rank}(d\phi_i : i \in \{1 \cdots n\}) = n$, leads to the result. ■

Remark 1 Note that, for $m = 1$ with $g_1(x_0) \neq 0$, the distribution $\Delta = \mathrm{span}\{g_1(x)\}$ is involutive and (M) (1) can be transformed into (RF) (2) with $d = \dim\Delta = 1$. Now, if $\Delta = \mathrm{span}\{g_1(x), \ldots, g_m(x)\}$ satisfy

[1] see [8] for its construction.

(H2) and (H3) of Theorem 2, then one obtains the classical result of [9] ($d = d_\Delta = m$). So, our result is an extension which, as we will see in the following sections, provides a guidance for the design of sliding mode controller in the general case: d may be greater or smaller[2] than m.

III. STABILIZING SLIDING MODE CONTROLLER

From (RF) (2), it is clear that, in order to make the origin of (M) (1) stable, one can design a sliding mode control in the following way:

1. let $s = z_1 - p(z_2) \in \mathbb{R}^d$, where the d-vector valued function p is to be defined,
2. design a sliding mode control such that a sliding regime occurs on the manifold $s = 0$ of dimension $(n - d)$,
3. define p such that the origin $z_2 = 0$ is locally asymptotically stable for $\dot{z}_2 = f_2^R(p(z_2), z_2)$ (dynamics in sliding regime).

In this procedure like in the center manifold theory (see [5]), there are two time scales: a fast one (the hitting phase is in finite time) and a slow one (asymptotic stabilization in sliding regime). Thus the sliding manifold plays the same role as the center manifold does in the theory of the same name.

This idea was first used in [9] under three assumptions: first $d = m$, secondly $\operatorname{rank}(G(x)) = m$ and under an integrability condition on $G(x)$ (which is always fulfilled when $m = 1$). In the following we give a generalization of these results. According to the previous section, we can distinguish the two following cases[3]:

1. $r = \operatorname{rank}(G(x_0)) = \dim\Delta_G$;
2. $r = \operatorname{rank}(G(x_0)) < \dim\Delta_G$.

Here, we shall consider the first Case 1; the second case will be treated through Example 2. Note that this first case is also considered in [9], but in the paper, all trajectories (this means for all initial

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[2] if we do not use Theorem 1.

[3] After eventually applying static feedback of Theorem 1 (see Section II.A).

conditions) belonging to the surface must converge asymptotically. In our case, only local asymptotic stability is required. The sliding gain is calculated in such a way that the motions reach the sliding surface in its stable part.

THEOREM 3 *Let us suppose that*:

(H1) Δ_G *exists with* $dim\Delta_G = r = rank(G(x_0))$, *one obtains* (RF) (2) *with* $d = r$,

(H2) f_2^R *is at least* $C^1(\mathbb{R}^n;\mathbb{R}^{(n-r)})$,

(H3) *the origin* $0 \in \mathbb{R}^{(n-r)}$ *of system*:

$$\dot{z}_2 = f_2^R(p(z_2), z_2), \tag{7}$$

is locally asymptotically stable, with $p(z_2) \in C^1(\mathbb{R}^{(n-r)};\mathbb{R}^r)$, $p(0) = 0 \in \mathbb{R}^r$,

Then, defining the sliding surface as $s = z_1 - p(z_2)$, *we have*:

(C1) *there exists a gain* $k(\cdot)$ *providing a local asymptotic stabilization of the origin w.r.t.* (RF) (2), *by means of the control*:

$$u = (G^R(z))^{-1}(-f_1^R(z) - k(\cdot)SGN(s) + \frac{\partial p(z_2)}{\partial z_2} f_2^R(z)), \tag{8}$$

(C2) *if in addition* $0 \in \mathbb{R}^{(n-r)}$, *is globally asymptotically stable for* (7), *then the origin of* (RF) (2) *is globally asymptotically stable under the control* (8) *defined with any non zero constant gain* $k(\cdot)$.

Proof (*constructive*) (H1) implies $G^R(z)$ is nonsingular and $(G^R(z))^{-1}$ exists (for all z in a neighbourhood of z_0).

Point (C1): $\|x\|_1 = \sum_{i=1}^n |x_i|$, $x \in \mathbb{R}^n$. From (H3) using a converse Lyapunov theorem (see [7]), there exist a Lyapunov function $V_2(z_2) \in C^1(\mathbb{R}^{(n-r)};\mathbb{R}_+)$ and $\rho_2 > 0$ such that for every motion of system (7) starting in $S_2(\rho_2) = \{z_2 \in \mathbb{R}^{(n-m)}: V_2(z_2) \leq \rho_2\}$, we have $\dot{V}_2|_{(7)} < 0$. So, let us consider $V(z) = (\alpha/2)s^\top s + V_2(z_2)$, with $\alpha > 0$, and restrict our attention to $S(\rho_2) = \{z \in \mathbb{R}^n: V(z) \leq \rho_2\}$:$\forall \rho_2 < \infty$, $S(\rho_2)$ is compact.

$$\dot{V}|_{(2)} = \alpha s^\top \dot{s} + \frac{\partial V_2^\top}{\partial z_2}[f_2^R(z_1, z_2) - f_2^R(p(z_2), z_2) + f_2^R(p(z_2), z_2)].$$

$$\left|\frac{\partial V_2^{\top}}{\partial z_2}[f_2^R(z_1,z_2)-f_2^R(p(z_2),z_2)]\right|$$
$$\leq \sum_{i=1}^{n}\left|\left(\frac{\partial V_2}{\partial z_2}\right)_i\right||(f_2^R(z_1,z_2)-f_2^R(p(z_2),z_2))_i|$$
$$\leq \left\|\frac{\partial V_2}{\partial z_2}\right\|_1 \|f_2^R(z_1,z_2)-f_2^R(p(z_2),z_2)\|_1$$

(H2) implies that f_2^R is uniformly Lipschitz on $S(\rho_2)$, and as $(z_1,z_2)\in S(\rho_2)\Rightarrow(p(z_2),z_2)\in S(\rho_2)$, thus $\exists L_{\rho_2}:\forall(z_1,z_2)\in S(\rho_2)$, $\|f_2^R(z_1,z_2)-f_2^R(p(z_2),z_2)\|_1\leq L_{\rho_2}\|z_1-p(z_2)\|_1=L_{\rho_2}s^{\top}$ SGN(s). With control (8), this leads to:

$$\dot{V}|_{(2)}\leq -\alpha k(\cdot)s^{\top}\mathrm{SGN}(s)+L_{\rho_2}\left\|\frac{\partial V_2}{\partial z_2}\right\|_1 s^{\top}\mathrm{SGN}(s)$$
$$+\frac{\partial V_2^{\top}}{\partial z_2}f_2^R(p(z_2),z_2),\forall z\in S(\rho_2). \tag{9}$$

We can now choose $k(\cdot)=1+(L_{\rho_2}/\alpha)\|(\partial V_2/\partial z_2)\|_1:\dot{V}|_{(2)}\leq -\alpha s^{\top}$ $\mathrm{Sgn}(s)+(\partial V_2/\partial z_2)^{\top}f_2^R(p(z_2),z_2)$. V_2 is a Lyapunov function for (7): if $z\in S(\rho_2)\subset\mathbb{R}^n$ then $z_2\in S_2(\rho_2)\subset\mathbb{R}^{(n-m)}$, so $\dot{V}_2|_{(7)}=(\partial V_2/\partial z_2 f_2^R(p(z_2),$ $z_2)$ is negative define $\forall z_2\in S_2(\rho_2)$. Therefore, $(\partial V_2/\partial z_2)^{\top}f_2^R(p(z_2),z_2)$ is negative $\forall_z\in S(\rho_2)\subset\mathbb{R}^n$, which ends the proof.

Point (C2): first show that solution tends to the sliding surface in finite time (see for example [10]), then the result follows obviously. ■

THEOREM 4 *If* (*H*1) *and* (*H*2) *of Theorem* 3 *hold and* (*H*3) *is replaced by*: (*H*3′) *there exist a Lyapunov function* $V_2(z_2)$ *and a constant* ρ_2 *such that* $S_2(\rho_2)=\{z_2\in\mathbb{R}^{(n-r)}:V_2(z_2)\leq\rho_2\}$ *is an estimate of the domain of asymptotic stability of the origin* $0_2\in\mathbb{R}^{(n-r)}$ *of* (7), *then, the control u defined by* (8) *with gain* $k(\cdot)=k'(\cdot)+(L_{\rho_2}/\alpha)\|(\partial V_2/\partial z_2)\|_1$, $k'(z)>0$, $\alpha>0$, *achieves asymptotic stability of the origin for system* (*RF*) (2) *with* $S(\rho_2)=\{z\in\mathbb{R}^n:(\alpha/2)s^{\top}s+V_2(z_2)\leq\rho_2\}$ *as an estimate of the domain of asymptotic stability.*

Proof direct extension of the proof of Theorem 3. ■

Remark 2 If, in addition, $\lim_{z_2\to 0}\|(\partial V_2/\partial z_2)\|=\lim_{z\to 0}k'(z)=0$, then "chattering" tends to zero as the motion approaches the origin. This condition is not very restrictive because, it is fulfilled if the

Lyapunov function V_2 is, locally, at least quadratic ($k'(\cdot)$ can be set to $(L_{\rho_2}/\alpha)(\partial V_2/\partial z_2)\|_1$ for example). Note that $k'(\cdot)$ can also be set to any other sigmoïd function zeroing at the origin: in [14], the sign function is replaced by a saturation function in order to smooth the discontinuity.

Example 1 Consider the following nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 + u_1, \\ \dot{x}_2 &= x_1 + u_1 + u_2, \\ \dot{x}_3 &= x_1 x_3 + u_1. \end{aligned} \tag{10}$$

Distribution $\Delta = \text{span}\ \{g_1, g_2\}$, $g_1 = (1,1,1)^\top$, $g_2 = (0,1,0)^\top$, is a two-dimensional involutive distribution. In order to find ϕ, one has to find a basis of $\Delta^\perp$:

$$\begin{cases} d\lambda g_1 = 0 \\ d\lambda g_2 = 0 \end{cases} \Longleftrightarrow \begin{cases} (\partial\lambda/\partial x_1) + (\partial\lambda/\partial x_2) + (\partial\lambda/\partial x_3) = 0 \\ (\partial\lambda/\partial x_2) = 0 \end{cases}, \tag{11}$$

which has solution $\lambda(x) = x_1 - x_3$. Thus $z = \phi(x) = (x_1, x_2, x_1 - x_3)^\top$ leads to (Theorem 2)

$$\begin{aligned} \dot{z}_1 &= z_2 + u_1, \\ \dot{z}_2 &= z_1 + u_1 + u_2, \\ \dot{z}_3 &= z_2 - z_1(z_1 - z_3). \end{aligned}$$

Here $\text{rank}(G) = 2$, so according to Theorem 4, let u be defined as $u = (-x_2 + v_1, x_2 - x_1 - v_1 + v_2)^\top$, and let s be defined as $s = (z_1 - z_3, z_2 + z_3)^\top = (x_3, x_1 + x_2 - x_3)^\top$. Let $v_1 = -\text{Sgn}(x_3) + x_2 - x_1x_3$, $v_2 = -\text{Sgn}(x_1 + x_2 - x_3) - x_2 + x_1x_3$. Then, in sliding regime $\dot{z}_3 = -z_3$, and global asymptotic stability of the origin is achieved (Fig. 1).

In order to reduce the chattering phenomenon one can use a nonlinear gain as proposed in Theorem 4 : $V_2(z_2) = (z_2^2/2)$ leads to choose the gains of $\text{Sgn}(x_3)$ as $|x_1 - x_3|$ and of $\text{Sgn}(x_1 + x_2 - x_3)$ as $2\,|x_1|\,|x_1 - x_3|$ (Fig. 2). Here, one can take two different gains, for this, rewrite $k(\cdot)s^\top\ \text{SGN}(s)$ as $\sum_{i=1}^{r} k_i(\cdot)|s_i|$ in the proof of Theorem 3.

Example 2 This second example illustrates the case $\text{rank}(G(x_0)) = r < d = \dim\Delta_G$. The obtained regular form in Section I provides informations: as $\text{rank}(G(x_0)) = r < d = \dim\Delta_G$, $G^R(z)$ the obtained input gain matrix in (RF) (2) can be splitted into $T(z)$ a low triangular $(r \times r)$-matrix and $R(z)$ a rectangular matrix. This is possible due to the

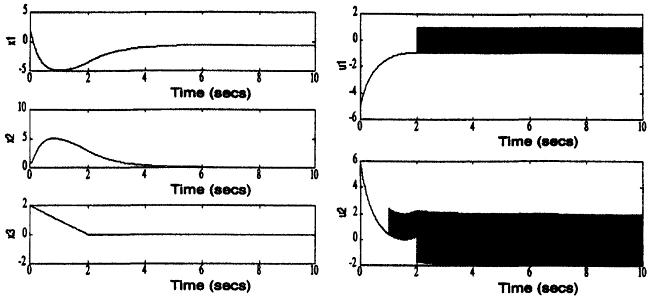

FIGURE 1 Stabilization of (10) using constant gains for sign functions.

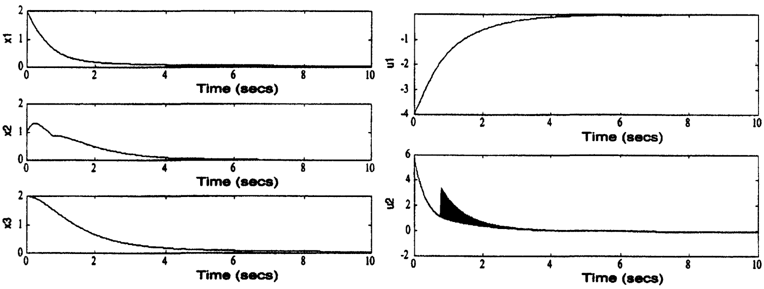

FIGURE 2 Stabilization of (10) using non linear gains obtained *via* Theorem 4.

use of the prestatic feedback $u = W(x)(v^T, 0, \ldots, 0)^T$ given by the Gauss reduction in the proof of Theorem 1 and using, if required, a permutation of the coordinates. Let us denote by $i \leq r$ the number of rows of T which are necessary to span the rows of R. Then using i integrators, one can obtained a dynamical extension for which Theorem 3 or 4 may apply. This procedure is going to be applied on a monocycle (see Fig. 3) with the pedaling rolling action (u_1) and the rotating action ($(d\theta/dt) = u_2$).

The model, which also express the dynamics of a two-wheel cart (see [6]), is:

$$\begin{aligned} \frac{dx_1}{dt} &= \sin\theta u_1, \\ \frac{dx_2}{dt} &= \cos\theta u_1, \\ \frac{d\theta}{dt} &= u_2. \end{aligned} \tag{12}$$

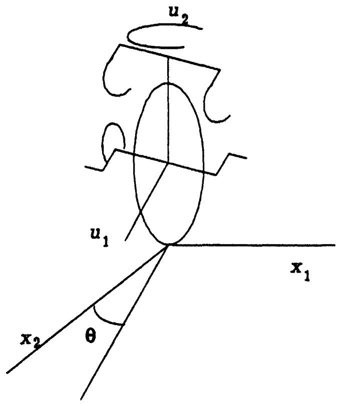

FIGURE 3 Monocycle.

Let $g_1(x) = (\sin\theta,\ \cos\theta, 0)^{\top}$, $g_2 = (0,0,1)^{\top}$. As $g_3(x) = [g_1, g_2] = (-\cos\theta, \sin\theta, 0)^{\top} \notin \operatorname{span}\{g_1(x), g_2(x)\}$. Let $\Delta_G = \operatorname{span}\ \{g_1(x),\ g_2(x),\ g_3(x)\}$, this is a three-dimensional involutive distribution (compute the Lie brackets $[g_1, g_3]$ and $[g_2, g_3]$). Note that $\dim(\Delta_G) = 3$ implies that (12) is locally accessible. The proof of Theorem 2 implies that for this system one cannot obtain (RF) (2) with d less than 3 : thus (12) is in a regular form with $\operatorname{rank}(G(x_0)) = 2 < 3 = \dim\ \Delta_G$. From this regular form and the introduction of this example, permutting x_2 and θ, it comes out that the last and first rows of $G^R(x)$ are dependent thus adding an integrator on the first input will break this link and leads to:

$$\begin{aligned}
\frac{dx_1}{dt} &= \sin(\theta)\xi, \\
\frac{dx_2}{dt} &= \cos(\theta)\xi, \\
\frac{d\theta}{dt} &= u_2 = v_2, \\
\frac{d\xi}{dt} &= v_1, \quad u_1 = \xi.
\end{aligned} \tag{13}$$

Then, according to Theorem 4, let $s = (\xi, \theta)^{\top} - p(x_1, x_2)$, thus $\dot{s}_1 = v_1 - \sin(\theta)\xi = -k_1(\cdot)\mathrm{Sgn}(s_1)$, $\dot{s}_2 = v_2 - \cos(\theta)\xi = -k_2(\cdot)\mathrm{Sgn}(s_2)$.

In sliding regime,

$$\dot{x}_1 = \sin(p_2(x_1, x_2))p_1(x_1, x_2),$$
$$\dot{x}_2 = \cos(p_2(x_1, x_2))p_1(x_1, x_2),$$

which can be set respectively to the values $-\alpha x_1$ and $-\beta x_2$, for some $\alpha > 0$ and $\beta > 0$. For this, let $p_1(x_1, x_2) = -(\alpha x_1/\sin(\arctan(\alpha x_1/\beta x_2)))$, and $p_2(x_1, x_2) = \arctan(\alpha x_1/\beta x_2)$. Moreover, in order to stabilize the origin of (12), we need θ to tend to zero which is achieved if $\lim_{t\to+\infty}(\arctan(\alpha x_1(t)/\beta x_2(t))) = 0$. A sufficient condition is $(-\alpha+\beta) < 0$ (see [6]). Let us select $\alpha = 2$ and $\beta = 1$. Thus, global asymptotic stability of the origin of (12) is achieved using the control laws defined by:

$$\begin{aligned} u_1(t) &= \int_0^t \left(-\operatorname{Sgn}(s_1) - \frac{\xi(4\sin(\theta)x_1 + \cos(\theta)x_2)}{x_2\sqrt{(1+4(x_1^2/x_2^2))}} \right)(\tau)d\tau, \\ u_2 = v_2 &= -k_2(\cdot)\operatorname{Sgn}(s_2) + \frac{2\xi(\sin(\theta)x_2 - \cos(\theta)x_1)}{4x_1^2 + x_2^2}, \end{aligned} \tag{14}$$

$$\begin{aligned} s_1 &= \xi + \frac{2x_1}{\sin(\arctan(2x_1/x_2))}, \\ s_2 &= \theta - \arctan\left(\frac{2x_1}{x_2}\right) \end{aligned} \tag{15}$$

Note that Gulden and Utkin ([6]) used an other approach, imposing the cart (or monocycle) to approach the origin according to a “Lyapunov Navigation Function” (the tracked path is derived from this Lyapunov function).

Figure 4 illustrates the stabilization of the origin under the controls (15) with $k_{i=1,2}(\cdot) = 1$. The simulations were achieved from the following initial conditions: $x_1(0) = 1.2$, $x_2(0) = 2$, $\theta(0) = 0.2(\mathrm{rd})$, $\xi(0) = 0$ (no control at time zero). One can note that the first control u_1 is rather smooth (no chattering) : this is due to the presence of an integrator before the physical actuator. For the second control u_2, there is some chattering which can be smoothed using different technics (see for example [14] or sigmoïd functions). But, we can use a nonlinear gain as proposed in Theorem 4. Using the Lyapunov function $V_2(z_2) = x_1^2 + (x_2^2/2)$ leads to choose $k_2(\cdot) = 0.2\sqrt{4x_1^2 + x_2^2}$.

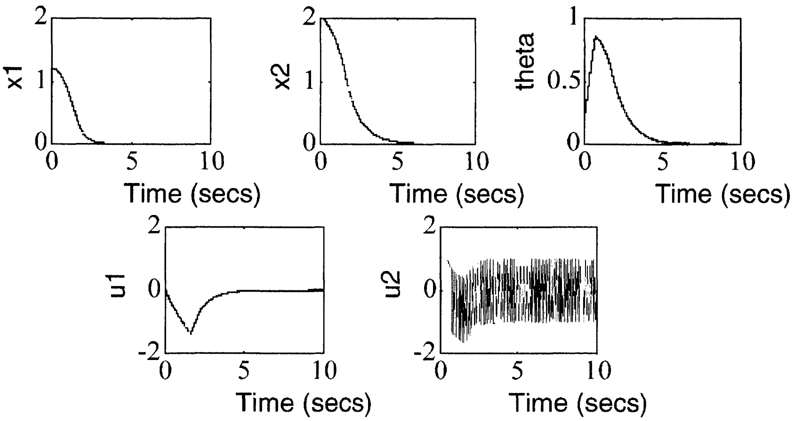

FIGURE 4 Stabilization of (12) using (14) with $k_{i=1,2}(\cdot) = 1$.

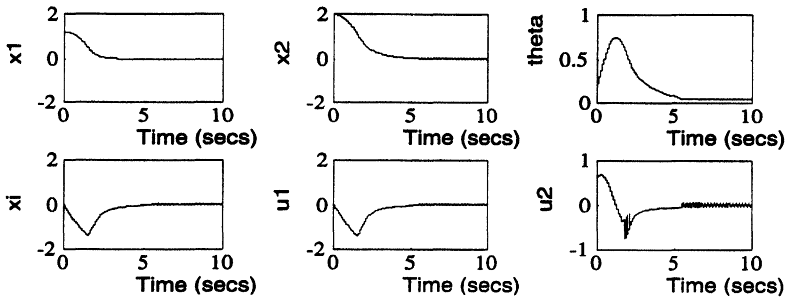

FIGURE 5 Stabilization of (12) using (14) with $k(\cdot) = 0.2\sqrt{4x_1^2 + x_2^2}$.

This gain replaces the gain "1" of the signum function $\mathrm{Sgn}(s_2)$ in (15) and yields the simulations of Figures 5.

IV. CONCLUSION

A first contribution of this paper is to complete the general problem of regular form initiated by Lukyanov and Utkin [9]:

1. The given results (Theorem 2 and consequences) are general (d may be greater or smaller than the number of inputs m) and do not depend on the rank of the input gain matrix (contrary to [9]).

2. The construction of Δ_G is easily obtained applying the algorithm for the involutive closure of span $\{g_1(x),\ldots,g_m(\mathrm{x})\}$.

In addition, Section III gives the design of a sliding mode controller achieving asymptotic stabilization of the origin. The proposed design procedures take into account the rank of the input gain matrix and the dimension of Δ_G which lead to the regular form. Lastly, a nonlinear gain is given in order to reduce the chattering phenomenon as the state converges asymptotically to the origin.

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