# Dissipativity Theory and Stability of Feedback Interconnections for Hybrid Dynamical Systems*

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In this paper we develop a unified dynamical systems framework for a general class of systems possessing left-continuous flows; that is, left-continuous dynamical systems. These systems are shown to generalize virtually all existing notions of dynamical systems and include hybrid, impulsive, and switching dynamical systems as special cases. Furthermore, we generalize dissipativity, passivity, and nonexpansivity theory to left-continuous dynamical systems. Specifically, the classical concepts of system storage functions and supply rates are extended to left-continuous dynamical systems providing a generalized hybrid system energy interpretation in terms of stored energy, dissipated energy over the continuous-time dynamics, and dissipated energy over the resetting events. Finally, the generalized dissipativity notions are used to develop general stability criteria for feedback interconnections of left-continuous dynamical systems. These results generalize the positivity and small gain theorems to the case of left-continuous, hybrid, and impulsive dynamical systems.

*Keywords*: Left-continuous systems; Hybrid systems; Impulsive systems; Stability; Dissipativity; Passivity; Nonexpansivity; Hybrid feedback systems; Feedback interconnections

## 1. INTRODUCTION

In the light of the increasingly complex nature of dynamical systems requiring controls, the predominant considerations in control law design for modern engineering systems have focused on general multiechelon hierarchical nonlinear switching control architectures that minimize control law complexity subject to the achievement of control law robustness. Multiechelon systems are classified as hybrid systems (see [1, 2] and the numerous references therein) and typically possess a hierarchical structure characterized by continuous-time dynamics at the lower-level units and logical decision-making units at the higher-level of the hierarchy. The logical decision making units serve to coordinate and reconcile the (sometimes competing) actions of the lower-level units. Due to their multiechelon hierarchical structure, hybrid dynamical systems are capable of simultaneously exhibiting continuous-time dynamics, discrete-time dynamics, logic commands, discrete-events, and resetting events. Such systems include dynamical switching systems [3 – 5], nonsmooth impact mechanical systems [6, 7], biological systems [8], sampled-data systems [9], discrete-event systems [10], intelligent vehicle/highway systems [11], constrained mechanical systems [6], and flight control systems [12], to cite but a few examples. The mathematical descriptions of some of these systems can be characterized by impulsive differential equations [13 – 16]. Impulsive dynamical systems can be viewed as a subclass of hybrid systems and consist of three elements; namely, a continuous-time differential equation, which governs the motion of the dynamical system between impulsive or resetting events; a difference equation, which governs the way the system states are instantaneously changed when a resetting event occurs; and a criterion for determining when the states of the system are to be reset.

Even though numerous results focusing on specific forms of hybrid systems have been developed in the literature (see for example [17] and the numerous references therein), the development of a general model for hybrid dynamical systems has received little attention in the literature. Notable exceptions include [18 – 20]. In particular, the authors in [18, 19] introduce a general (undisturbed) hybrid dynamical system model whose flow is defined on an arbitrary metric space evolving over a notion of generalized abstract time. For this class of

hybrid dynamical systems the authors in [18, 19] provide a thorough treatment of Lyapunov stability and Lagrange stability results. Alternatively, the authors in [20] give a unified framework for an autonomous hybrid control model and develop an optimal control framework for synthesizing hybrid controllers.

In this paper we develop a unified dynamical systems framework for a general class of systems possessing left-continuous flows; that is, left-continuous dynamical systems.[1] A left-continuous dynamical system is a precise mathematical object and is defined on the semi-infinite interval as a mapping between vector spaces satisfying an appropriate set of axioms and includes hybrid inputs and hybrid outputs that take their values in appropriate vector spaces. The notion of a left-continuous dynamical system introduced in the paper generalizes virtually all existing notions of dynamical systems and includes hybrid, impulsive, and switching dynamical systems as special cases. Furthermore, using generalized left-continuous storage functions and hybrid supply rates we extend the notions of classical dissipativity theory [21] and exponential dissipativity theory [22] to left-continuous dynamical systems. The overall approach provides an interpretation of a generalized hybrid energy balance of a left-continuous dynamical system in terms of the stored or, accumulated generalized energy, dissipated energy over the continuous-time dynamics, and dissipated energy at the resetting events. Furthermore, as in the continuous-time dynamical systems case possessing continuous flows [21], we show that the set of all possible storage functions of a left-continuous dynamical system forms a convex set and is bounded from below by the system's available stored generalized energy which can be recovered from the system, and bounded from above by the system's required generalized energy supply needed to transfer the system from an initial state of minimum generalized energy to a given state.

Next, using the concepts of dissipativity and exponential dissipativity for left-continuous systems, we develop feedback interconnection stability results for left-continuous dynamical systems. Specifically, general stability criteria are given for Lyapunov, asymptotic, and exponential stability of feedback left-continuous systems. In the case of

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[1] Right-continuous dynamical systems; that is, systems possessing right-continuous flows, can also be analogously considered.

quadratic hybrid supply rates involving net system power and input-output energy, these results generalize the positivity and small gain theorems to the case of left-continuous dynamical systems and hence hybrid and impulsive dynamical systems. In particular, we show that if the left-continuous dynamical systems $\mathcal{G}$ and $\mathcal{G}_c$ are dissipative (respectively, exponentially dissipative) with respect to quadratic hybrid supply rates corresponding to net system power, or weighted input and output energy, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is Lyapunov (respectively, asymptotically) stable.

The contents of the paper are as follows. In Section 2 we introduce the notion of a left-continuous dynamical system as a precise mathematical object satisfying a set of axioms. In Section 3 we show that the notion of a left-continuous dynamical system introduced in Section 2 generalizes virtually all existing notions of dynamical systems and includes hybrid systems, impulsive systems, and switching systems as special cases. In Section 4 we present stability results for left-continuous dynamical systems. Furthermore, for a special class of left-continuous dynamical systems; namely, strong left-continuous dynamical systems, we present an invariant set stability theorem for systems with left-continuous flows. Then, in Section 5, we extend the notion of dissipative dynamical systems to develop the concept of dissipativity theory for left-continuous dynamical systems using left-continuous storage functions and hybrid supply rates. In Section 6 we use the results of Sections 4 and 5 to state and prove feedback interconnection stability results for dissipative left-continuous dynamical systems. In the special case of quadratic hybrid supply rates involving net system power and input-output energy, these results generalize the positivity and small gain theorems to the case of hybrid systems. Finally, we draw conclusions in Section 7.

## 2. LEFT-CONTINUOUS DYNAMICAL SYSTEMS

In this section we establish definitions, notation, and introduce the notion of a left-continuous dynamical system. The following definition is concerned with left-continuous dynamical systems or, systems with left-continuous flows. For this definition $\mathcal{U} \triangleq \mathcal{U}_c \times \mathcal{U}_d$ is an input space and consists of bounded continuous $U$-valued functions on the

semi-infinite interval $[0, \infty)$.[2] The set $U \triangleq U_c \times U_d$, where $U_c \subseteq \mathbb{R}^{m_c}$ and $U_d \subseteq \mathbb{R}^{m_d}$, contains the set of input values; that is, for every $u = (u_c, u_d) \in \mathcal{U}$ and $t \in [0,\infty)$, $u(t) \in U$, $u_c(t) \in U_c$, and $u_d(t) \in U_d$. Furthermore, $\mathcal{Y} \triangleq \mathcal{Y}_c \times \mathcal{Y}_d$ is an output space and consists of bounded continuous $Y$-valued functions on the semi-infinite interval $[0, \infty)$. The set $Y \triangleq Y_c \times Y_d$, where $Y_c \subseteq \mathbb{R}^{l_c}$ and $Y_d \subseteq \mathbb{R}^{l_d}$, contains the set of output values; that is, for every $y = (y_c, y_d) \in \mathcal{Y}$ and $t \in [0,\infty)$, $y(t) \in Y$, $y_c(t) \in Y_c$, and $y_d(t) \in Y_d$. Finally, $\mathcal{D} \subseteq \mathbb{R}^n$ and $\|\cdot\|$ denotes the Euclidean norm. The notions of openness, convergence, continuity, and compactness that we use in the paper refer to the topology generated on $\mathbb{R}^n$ by the norm $\|\cdot\|$.

DEFINITION 2.1  A *left-continuous dynamical system* on $\mathcal{D}$ is the octuple $(\mathcal{D}, \mathcal{U}, U, \mathcal{Y}, Y, s, h_c, h_d)$, where  $s : [0,\infty) \times [0,\infty) \times \mathcal{D} \times \mathcal{U} \to \mathcal{D}$,  $h_c : \mathcal{D} \times U_c \to Y_c$, and $h_d : \mathcal{D} \times U_d \to Y_d$ are such that the following axioms hold:

  (i) (Left-continuity): For every $t_0 \in [0, \infty)$, $x_0 \in \mathcal{D}$, and $u \in \mathcal{U}$, $s(t_0, \cdot, x_0, u)$ is left-continuous; that is, $\lim_{\tau \to t^-} s(\tau, t_0, x_0, u) = s(t, t_0, x_0, u)$ for all $t \in (t_0, \infty)$.

  (ii) (Consistency): For every $x_0 \in \mathcal{D}$, $u \in \mathcal{U}$, and $t_0 \in [0, \infty)$, $s(t_0, t_0, x_0, u) = x_0$.

  (iii) (Determinism): For every $t_0 \in [0, \infty)$ and $x_0 \in \mathcal{D}$, $s(t, t_0, x_0, u_1) = s(t, t_0, x_0, u_2)$ for all $t \in [t_0, \infty)$ and $u_1, u_2 \in \mathcal{U}$ satisfying $u_1(\tau) = u_2(\tau)$, $\tau \in [t_0, t]$.

  (iv) (Semi-group property): $s(t_2, t_0, x_0, u) = s(t_2, t_1, s(t_1, t_0, x_0, u), u)$ for all $t_0, t_1, t_2 \in [0, \infty)$, $t_0 \le t_1 \le t_2$, $x_0 \in \mathcal{D}$, and $u \in \mathcal{U}$.

  (v) (Read-out map):[3] There exists $y \in \mathcal{Y}$ such that $y(t) = (h_c(s(t, t_0, x_0, u), u_c(t)), h_d(s(t, t_0, x_0, u), u_d(t)))$ for all $x_0 \in \mathcal{D}$, $u \in \mathcal{U}$, $t_0 \in [0, \infty)$, and $t \in [t_0, \infty)$.

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[2] Instead of the semi-infinite interval, a completely ordered time set $\mathbb{T}$ having a hybrid topological structure involving isolated points and closed sets homeomorphic to intervals on the real line can be considered [18]. For example, continuous-time and discrete-events can be defined on $\mathbb{T} \subset [0,\infty) \times \mathcal{T}$, where $\mathcal{T}$ is a countable subset of $[0, \infty)$. Alternatively, in our formulation continuous-time and discrete-events are defined on a single semi-infinite interval $[0, \infty)$, where the graph of the semi-infinite interval corresponding to the discrete-events is piecewise constant and left-continuous.
[3] More generally, the read-out maps $h_c$ and $h_d$ can be explicit functions of time; that is, $h_c : [0,\infty) \times \mathcal{D} \times U_c \to Y_c$ and $h_d : [0,\infty) \times \mathcal{D} \times U_d \to Y_d$.

Henceforth, we denote the left-continuous dynamical system $(\mathcal{D}, \mathcal{U}, U, \mathcal{Y}, Y, s, h_{\mathrm{c}}, h_{\mathrm{d}})$ by $\mathcal{G}$. For the remainder of the paper we refer to the left-continuous dynamical system $\mathcal{G}$ as the dynamical system $\mathcal{G}$. Furthermore, we refer to $s(t, t_0, x_0, u)$, $t \geq t_0$, as the *trajectory* of $\mathcal{G}$ corresponding to $x_0 \in \mathcal{D}$, $t_0 \in [0, \infty)$, and $u \in \mathcal{U}$; and for a given trajectory $s(t, t_0, x_0, u)$, $t \geq t_0$, we refer to $t_0 \in [0, \infty)$ as an *initial time* of $\mathcal{G}$, $x_0 \in \mathcal{D}$ as an *initial condition* of $\mathcal{G}$, and $u \in \mathcal{U}$ as an *input* to $\mathcal{G}$. The trajectory $s(t, t_0, x_0, u)$, $t \geq t_0$, of $\mathcal{G}$ is *bounded* if there exists $\gamma > 0$ such that $\|s(t, t_0, x_0, u)\| \leq \gamma$, $t \geq t_0$,. The dynamical system $\mathcal{G}$ is *isolated* if the input space consists of one constant element only; that is, $u(t) \equiv u^*$, and the dynamical system $\mathcal{G}$ is *undisturbed* if $u^* = 0$. Furthermore, an *equilibrium point* of the undisturbed dynamical system $\mathcal{G}$ is a point $x \in \mathcal{D}$ satisfying $s(t, t_0, x, 0) = x$, $t \geq t_0$. An equilibrium point $x \in \mathcal{D}$ of the undisturbed dynamical system $\mathcal{G}$ is *Lyapunov stable* if, for all $\varepsilon > 0$ and $t_0 \in [0, \infty)$, there exists $\delta(\varepsilon) > 0$ such that if $\|x - x_0\| < \delta$, then $\|x - s(t, t_0, x_0, 0)\| < \varepsilon$, $t \geq t_0$. An equilibrium point $x \in \mathcal{D}$ of the undisturbed dynamical system $\mathcal{G}$ is *asymptotically stable* if $x$ is Lyapunov stable and there exists $\delta > 0$ such that if $\|x - x_0\| < \delta$, then $\lim_{t \to \infty} s(t, t_0, x_0, 0) = x$. An equilibrium point $x \in \mathcal{D}$ of the undisturbed dynamical system $\mathcal{G}$ is *exponentially stable* if there exist positive constants $\alpha$, $\beta$, and $\delta$ such that if $\|x - x_0\| < \delta$, then $\|x - s(t, t_0, x_0, 0)\| \leq \alpha \|x - x_0\| e^{-\beta t}$, $t \geq t_0$. Finally, an equilibrium point $x \in \mathcal{D}$ of the undisturbed dynamical system $\mathcal{G}$ is *globally asymptotically* (resp., *exponentially*) *stable* if asymptotic (resp., exponential) stability holds for any $x_0 \in \mathbb{R}^n$ and $t_0 \in [0, \infty)$. The next definition provides a specialization of Definition 2.1 to the case of stationary left-continuous dynamical systems wherein $\mathcal{U}$ is closed under the shift operator.

DEFINITION 2.2   A *stationary left-continuous dynamical system* on $\mathcal{D}$ is the octuple $(\mathcal{D}, \mathcal{U}, U, \mathcal{Y}, Y, s, h_{\mathrm{c}}, h_{\mathrm{d}})$, where $s: [0, \infty) \times [0, \infty) \times \mathcal{D} \times \mathcal{U} \to \mathcal{D}$, $h_{\mathrm{c}} : \mathcal{D} \times U_{\mathrm{c}} \to Y_{\mathrm{c}}$, and $h_{\mathrm{d}} : \mathcal{D} \times U_{\mathrm{d}} \to Y_{\mathrm{d}}$ are such that Axioms (i) − (v) hold and:

(vi) (Stationarity): For every $t_0, t \in [0, \infty)$, $t \geq t_0$, $\tau \in \mathbb{R}$, $x_0 \in \mathcal{D}$, $u$, $u_T \in \mathcal{U}$, such that $u_T(t) = u(t + T)$, $t \in [0, \infty)$, $s(t + \tau, t_0 + \tau, x_0, u_T) = s(t, t_0, x_0, u)$.

Note that without loss of generality, for a given stationary left-continuous dynamical system $\mathcal{G}$, we can set $t_0 = 0$ by redefining

$u(t) \triangleq u(t - t_0)$. Hence, we will denote the trajectory $s(t, t_0, x_0, u)$, $t \geq t_0$, of a stationary left-continuous dynamical system $\mathcal{G}$ as $s(t, 0, x_0, u)$, $t \geq 0$. In the sequel, for every $t_0 \in [0, \infty)$, $x_0 \in \mathcal{D}$, and $u \in \mathcal{U}$, let $\mathcal{T}_{t_0, x_0, u} \subseteq [t_0, \infty)$ denote a dense subset of the semi-infinite interval $[t_0, \infty)$ such that $\mathcal{T}_{t_0, x_0, u}^{\mathrm{c}} \triangleq [t_0, \infty) \backslash \mathcal{T}_{t_0, x_0, u}$ is (finitely or infinitely) countable. For notational convenience we write $\mathcal{T}$ and $\mathcal{T}^{\mathrm{c}}$ for $\mathcal{T}_{t_0, x_0, u}$ and $\mathcal{T}_{t_0, x_0, u}^{\mathrm{c}}$, respectively. Furthermore, we refer to $\mathcal{T}^{\mathrm{c}}$ as the set of *resetting times* or *resetting events*.

DEFINITION 2.3   A *strong left-continuous dynamical system* on $\mathcal{D}$ is the octuple $(\mathcal{D}, \mathcal{U}, U, \mathcal{Y}, Y, s, h_{\mathrm{c}}, h_{\mathrm{d}})$, where $s: [0, \infty) \times [0, \infty) \times \mathcal{D} \times \mathcal{U} \rightarrow \mathcal{D}, h_{\mathrm{c}}: \mathcal{D} \times U_{\mathrm{c}} \rightarrow Y_{\mathrm{c}}$, and $h_{\mathrm{d}}: \mathcal{D} \times U_{\mathrm{d}} \rightarrow Y_{\mathrm{d}}$ are such that Axioms (i)−(vi) hold and:

(vii) (Quasi-continuous dependence): For every $t_0 \in [0, \infty)$, $x_0 \in \mathcal{D}$, and $u \in \mathcal{U}$, there exists $\mathcal{T} \subseteq [t_0, \infty)$ such that $[t_0, \infty) \backslash \mathcal{T}$ is countable and for every $\varepsilon > 0$ and $t \in \mathcal{T}$, there exists $\delta(\varepsilon, x_0, t) > 0$, such that if $\|x_0 - x\| < \delta(\varepsilon, x_0, t)$, $x \in \mathcal{D}$, then $\|s(t, t_0, x_0, u) - s(t, t_0, x, u)\| < \varepsilon$, for all $u \in \mathcal{U}$.

The quasi-continuous dependence property (vii) is a generalization of the standard continuous dependence property for dynamical systems with continuous flows to dynamical systems with discontinuous flows [23]. Specifically, by letting $\mathcal{T} = [t_0, \infty)$ the quasi-continuous dependence property specializes to the classical continuous dependence of solutions of a given dynamical system with respect to the system's initial conditions $x_0 \in \mathcal{D}$ [24]. If, in addition, $x_0 = 0$, $u \equiv 0$, $s(t, 0, 0, 0) = 0$, $t \geq 0$, and $\delta(\varepsilon, 0, t)$ can be chosen independent of $t$, then continuous dependence implies the classical Lyapunov stability of the zero trajectory $s(t, 0, 0, 0) = 0$, $t \geq 0$ Hence, Lyapunov stability of motion can be interpreted as continuous dependence of solutions uniformly in $t$ for all $t \geq 0$. Conversely, continuous dependence of solutions can be interpreted as Lyapunov stability of motion for every fixed time $t$ [24]. Analogously, Lyapunov stability of left-continuous dynamical systems as defined in [23] can be interpreted as quasi-continuous dependence of solutions uniformly in $t$ for all $t \in \mathcal{T}$. In applying Definition 2.1 it may be convenient to replace Axiom (vii) with a stronger condition which may be easier to verify in practice. The following proposition provides sufficient conditions for $\mathcal{G}$ to be a strong left-continuous dynamical system.

PROPOSITION 2.1 [23]  *Let $\mathcal{G}$ be a stationary left-continuous dynamical system such that the following condition holds:*

(vii)′  *For every $t_0 \in [0,\infty)$, $x_0 \in \mathcal{D}$, $\varepsilon$, $\eta > 0$, and $T \in \mathcal{T}$, there exists $\delta(\varepsilon, x_0, T) > 0$ such that if $\|x_0 - x\| < \delta(\varepsilon, x_0, T)$, $x \in \mathcal{D}$, then, for every $t \in \mathcal{T} \cap [0,T]$, such that $|t - \tau| > \eta$, for all $\tau \in \mathcal{T}^c \cap [0,T]$, and for all $u \in \mathcal{U}$, $\|s(t, t_0, x, u) - s(t, t_0, x, u)\| < \varepsilon$. Furthermore, if $t \in \mathcal{T}$ is an accumulation point of $\mathcal{T}^c$, then $s(t, t_0, \cdot, u)$ is continuous for all $u \in \mathcal{U}$.*

*Then $\mathcal{G}$ is a strong left-continuous dynamical system.*

## 3. SPECIALIZATION TO HYBRID AND IMPULSIVE DYNAMICAL SYSTEMS

In this section we show that hybrid dynamical systems [18, 20] and impulsive dynamical systems [8, 13−16] are a specialization of left-continuous dynamical systems. We start our presentation by considering a definition of a controlled hybrid dynamical system that includes the definition given in [20] as a special case. For this definition let $\mathcal{Q} \subseteq \mathcal{N}$, where $\mathcal{N}$ denotes the set of nonnegative integers.

DEFINITION 3.1  A *hybrid dynamical system* $\mathcal{G}_H$ is the 13-tuple $(\mathcal{D}, \mathcal{Q}, \mathcal{U}, U, \mathcal{Y}, Y, q, x, s_c, f_d, \mathcal{S}, h_c, h_d)$, where  $q : [0,\infty) \times [0,\infty) \times \mathcal{D} \times \mathcal{Q} \times \mathcal{U} \to \mathcal{Q}$,  $x : [0,\infty) \times [0,\infty) \times \mathcal{D} \times \mathcal{Q} \times \mathcal{U} \to \mathcal{D}$,  $s_c = \{s_{cq}\}_{q \in \mathcal{Q}}$,  $s_{cq} : [0,\infty) \times [0,\infty) \times \mathcal{D} \times \mathcal{U} \to \mathcal{D}$,  $\mathcal{S} = \{\mathcal{S}_q\}_{q \in \mathcal{Q}}$,  $\mathcal{S}_q \subset [0,\infty) \times \mathcal{D} \times \mathcal{U}$,  $f_d = \{f_{dq}\}_{q \in \mathcal{Q}}$,  $f_{dq} : \mathcal{S}_q \to \mathcal{D} \times \mathcal{Q}$,  $h_c : \mathcal{D} \times \mathcal{U}_c \to Y_c$,  and  $h_d : \mathcal{D} \times \mathcal{U}_d \to Y_d$  are such that the following axioms hold:

(i)  For every $q \in \mathcal{Q}$, $t_0 \in [0,\infty)$, and $u \in \mathcal{U}$, $s_{cq}(\cdot, t_0, \cdot, u)$ is jointly continuous on $[t_0, \infty) \times \mathcal{D}$.

(ii)  For every $q_0 \in \mathcal{Q}$, $t_0 \in [0,\infty)$, $x_0 \in \mathcal{D}$, and $u \in \mathcal{U}$, $s_{cq}(t_0, t_0, x_0, u) = x_0$.

(iii)  For every $q \in \mathcal{Q}$, $t_0 \in [0,\infty)$, and $x_0 \in \mathcal{D}$, $s_{cq}(t, t_0, x_0, u) = s_{cq}(t, t_0, x_0, u_2)$ for all $t \in [t_0, \infty)$ and $u_1$, $u_2 \in \mathcal{U}$ satisfying $u_1(\tau) = u_2(\tau)$, $\tau \in [t_0, t]$.

(iv)  For every $q \in \mathcal{Q}$, $t_0, t_1, t_2 \in [0,\infty)$, $t_0 \leq t_1 \leq t_2$, $x_0 \in \mathcal{D}$, and $u \in \mathcal{U}$, $s_{cq}(t_2, t_0, x_0, u) = s_{cq}(t_2, t_1, s_{cq}(t_1, t_0, x_0, u), u)$.

(v)  For every $q \in \mathcal{Q}$, $t_0 \in [0,\infty)$, $x_0 \in \mathcal{D}$, and $u \in \mathcal{U}$, $q(\cdot)$ and $x(\cdot)$ are such that $q(t, t_0, x_0, q_0, u) = q_0$ and $x(t, t_0, x_0, q_0, u) = s_{cq_0}(t, t_0, x_0, u)$,

for all $t_0 \le t \le t_1$, where $t_1 \triangleq \min\{t \ge t_0 : (t, s_{cq0}(t, t_0, x_0, u), u(t)) \notin S_{q0}\}$. Furthermore, $[x^T(t_1^+, t_0, x_0, q_0, u), q^T(t_1^+, t_0, x_0, q_0, u)]^T = f_{dq0}$ $(t_1, x(t_1), u(t_1)) + [x^T(t_1, t_0, x_0, q_0, u), q^T(t_1, t_0, x_0, q_0, u)]^T$ and for $(x_1, q_1) \triangleq (x(t_1^+, t_0, x_0, q_0, u), q(t_1^+, t_0, x_0, q_0, u))$, $q(\cdot)$ and $x(\cdot)$ are such that $q(t, t_0, x_0, q_0, u) = q_1$ and $x(t, t_0, x_0, q_0, u) = s_{cq1}(t, t_1, x_1, u)$, for all $t_1 < t \le t_2$, where $t_2 \triangleq \min\{t > t_1 : (t, s_{cq1}(t, t_1, x_1, u), u(t)) \notin S_{q1}\}$, and so on.

(vi) There exists $y \in \mathcal{Y}$ such that $y(t) = (h_c(x(t, t_0, x_0, q_0, u), u_c(t)), h_d(x(t, t_0, x_0, q_0, u), u_d(t)))$ for all $x_0 \in \mathcal{D}$, $u \in \mathcal{U}$, $t_0 \in [0, \infty)$, and $t \in [t_0, \infty)$.

It follows from Definition 3.1 that hybrid dynamical systems involve switching between a countable collection of continuous dynamical systems. To ensure that the switching or resetting times are well defined and distinct we make the following additional assumptions:

A1. If $(t, x(t, t_0, x_0, q_0, u), u(t)) \in \bar{S}_q \backslash S_q$, where $\bar{S}_q$ denotes the closure of the set $S_q$, then there exists $\varepsilon > 0$ such that, for all $0 < \delta < \varepsilon$,

$$s_{cq}(t + \delta, t, x(t, t_0, x_0, q_0, u), u(t + \delta)) \notin S_q.$$

A2. If $(t_k, x(t_k, t_0, x_0, q_0, u), u(t_k)) \in \partial S_q \cap S_q$, where $\partial S_q$ denotes the boundary of the set $S_q$, then there exists $\varepsilon > 0$ such that, for all $0 < \delta < \varepsilon$,

$$s_{c\hat{q}}(t_k + \delta, t_k, x(t_k^+, t_0, x_0, q_0, u), u(t_k + \delta)) \notin S_{\hat{q}}, \quad \hat{q} \in \mathcal{Q}.$$

Assumption A1 ensures that if a trajectory reaches the closure of $S_q$ at a point that does not belong to $S_q$, then the trajectory must be directed away from $S_q$; that is, a trajectory cannot enter $S_q$ through a point that belongs to the closure of $S_q$ but not to $S_q$. Equivalently, A1 implies that a trajectory can only reach $S_q$ through a point belonging to both $S_q$ and its boundary. Furthermore, A2 ensures that when a trajectory intersects (the boundary of) a resetting set $S_q$, it instantaneously exits $S_q$ and the continuous-time dynamics becomes the active element of the hybrid dynamical system. Since a continuous trajectory starting outside $S_q$ and intersecting the interior of $S_q$ must first intersect the boundary of $S_q$, it follows that no trajectory can reach the interior of $S_q$.

To show that $\mathcal{G}_H$ is a left-continuous dynamical system, let $s$ : $[0,\infty) \times [0,\infty) \times (\mathcal{D} \times \mathcal{Q}) \times \mathcal{U} \to \mathcal{D} \times \mathcal{Q}$ be such that $s(t_0, t_0, (x_0, q_0), u) = (x_0, q_0)$ and for every $k = 1, 2, \ldots$,

$$s(t, t_0, (x_0, q_0), u) = (s_{cq_{k-1}}(t, t_{k-1}, x_{k-1}, u), q_{k-1}), \quad t_{k-1} < t \le t_k, \quad (1)$$

$$s(t_k^+, t_0, (x_0, q_0), u) = f_{dq_{k-1}}(t_k, x_k, u(t_k)) + [x_k^T, q_k^T]^T. \quad (2)$$

Note that $s$ satisfies Axioms (i)–(v) of Definition 2.1 so that the controlled hybrid dynamical system $\mathcal{G}_H$ generates a left-continuous dynamical system on $\mathcal{D} \times \mathcal{Q}$ given by the octuple $(\mathcal{D} \times \mathcal{Q}, \mathcal{U}, U, \mathcal{Y}, Y, s, h_c, h_d)$. Since the resetting events $\mathcal{T}^c = \{t_1, t_2, \ldots\}$ can be a function of time $t$, the system state $x(t, t_0, x_0, q_0, u)$, and the system input $u$, hybrid dynamical systems can involve system jumps at variable times and hence in general are time-varying left-continuous dynamical systems. In the case where the resetting events are defined by a prescribed sequence of times which are independent of the system trajectories and system inputs; that is, $\mathcal{S}_q = \mathcal{T}_q \times \mathcal{D} \times \mathcal{U}$, where $\mathcal{T}_q \subset [0,\infty)$ and $q \in \mathcal{Q}$ is a closed discrete set, we refer to $\mathcal{G}_H$ as a *time-dependent hybrid dynamical system*. Alternatively, in the case where the resetting events are defined by the manifold $\mathcal{S}_q = [0,\infty) \times \mathcal{S}_{xq} \times \mathcal{U}$, where $\mathcal{S}_{xq} \subset \mathcal{D}$, $q \in \mathcal{Q}$; that is, $\mathcal{S}_q$ is independent of time and the inputs, we refer to $\mathcal{G}_H$ as a *state-dependent hybrid dynamical system*. More generally, if the resetting events are defined by the manifold $\mathcal{S}_q = [0,\infty) \times \mathcal{S}_{xq} \times \mathcal{S}_{uq}$, where $\mathcal{S}_{uq} \subset \mathcal{U}$, $q \in \mathcal{Q}$, we refer to $\mathcal{G}_H$ as an *input/state-dependent hybrid dynamical system*. Note that if $\{s_{cq}\}_{q \in \mathcal{Q}}$ are continuous trajectories such that Axiom (vi) in Definition 2.2 holds, then state- and input/state-dependent hybrid dynamical systems are stationary left-continuous dynamical systems.

It is important to note that the analysis of hybrid dynamical systems can be quite involved. In particular, such systems can exhibit Zenoness, beating, as well as confluence wherein solutions exhibit infinitely many resettings in a finite-time, encounter the same resetting surface a finite or infinite number of times in zero time, and coincide after a given point of time. Even though A1 and A2 allow for the possibility of confluence and Zeno solutions, A2 precludes the possibility of beating. In the case of stationary state-dependent hybrid dynamical systems several interesting observations can be made

regarding quasi-continuous dependence and Zeno solutions. Specifically, if the first resetting time is continuous with respect to all initial conditions and all system solutions are non-Zeno, then the hybrid dynamical system can be shown to satisfy Axiom (vii). Furthermore, if the second resetting time is continuous with respect to all initial conditions on the resetting surfaces $\mathcal{S}_q$, $q \in \mathcal{Q}$, and all solutions starting from $\mathcal{D} \backslash \cup_{q \in \mathcal{Q}} \bar{\mathcal{S}}_q$ are Zeno, then all the trajectories approach the set $\cup_{q \in \mathcal{Q}} \bar{\mathcal{S}}_q \backslash \mathcal{S}_q$ as $t \to \infty$. For a precise statement and proof of these facts as applied to autonomous impulsive dynamical systems the interested reader is referred to [23].

The notion of a controlled hybrid dynamical system $\mathcal{G}_H$ given by Definition 3.1 generalizes all the existing notions of dynamical systems wherein the state space has a fixed dimension. For example, if $\mathcal{Q} = \{q\}$ and $\mathcal{S} = \varnothing$, then $\mathcal{G}_H$ denotes a continuous-time dynamical system with a continuous flow [21]. Alternatively, if $\mathcal{Q} = \{q\}$, $\mathcal{S} = \mathcal{S}_q$, and $s_{cq}$ denotes the solution to an ordinary differential equation

$$\dot{x}_q(t) = f_{cq}(t, x_q(t), u_c(t)), \quad x_q(t_0) = x_0, \quad t \geq t_0, \tag{3}$$

where $x_q(t) \in \mathcal{D}$, $t \geq t_0$, and $f_{cq} : [0, \infty) \times \mathcal{D} \times \mathcal{U}_c \to \mathbb{R}^n$, then the hybrid dynamical system $\mathcal{G}_H$ is characterized by the impulsive differential equation

$$\dot{x}(t) = f_{cq}(t, x(t), u_c(t)), \quad x(t_0) = x_0, \quad (t, x(t), u_c(t)) \notin \mathcal{S}_q, \tag{4}$$

$$\Delta x(t) = f_{dq}(t, x(t), u_d(t)), \quad (t, x(t), u_c(t)) \in \mathcal{S}_q. \tag{5}$$

More generally, if $\mathcal{Q}$ is a (finitely or infinitely) countable set and $\{s_{cq}\}_{q \in \mathcal{Q}}$ denote the solutions to a set of ordinary differential equations then $\mathcal{G}_H$ can be represented by a set of coupled ordinary differential equations and difference equations or, equivalently, a set of impulsive differential equations with discontinuous vector fields. Specifically, for every $q \in \mathcal{Q}$, let $s_{cq}$ denote the solution to the ordinary differential equation

$$\dot{x}_g(t) = f_{cq}(t, x_q(t), u_c(t)), \quad x_q(t_0) = x_0, \quad t \geq t_0, \tag{6}$$

where $x_q(t) \in \mathcal{D}$, $t \geq t_0$, and $f_{cq} : [0, \infty) \times \mathcal{D} \times \mathcal{U}_c \to \mathbb{R}^n$. In this case, the hybrid dynamical system $\mathcal{G}_H$ is characterized by the impulsive

differential equation

$$
\begin{bmatrix} \dot{x}(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} f_{cq(t)}(t, x(t), u_c(t)) \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x(t_0) \\ q(t_0) \end{bmatrix} = \begin{bmatrix} x_0 \\ q_0 \end{bmatrix}, \quad (t, x(t), u_c(t)) \notin \mathcal{S}_{q(t)},
$$

(7)

$$
\begin{bmatrix} \Delta x(t) \\ \Delta q(t) \end{bmatrix} = f_{dq(t)}(t, x(t), u_d(t)), \quad (t, x(t), u_c(t)) \in \mathcal{S}_{q(t)}.
$$

(8)

Finally, note that if $\Delta x(t) = 0$ in (8), then (7) specializes to the case of switched hybrid systems involving continuous flows but discontinuous vector fields [3 – 5]; that is, a Filippov dynamical system.

We close this section by noting that several of the classical hybrid dynamical system models developed in the literature [25 – 30] are a special case of the impulsive dynamical system (7), (8). Specifically, the Witsenhausen model [25], the Tavernini model [26], the Nerode-Kohn model [27], and the Antsaklis-Stiver-Lemmon model [28] are a special case of an autonomous version of (7), (8) with $\Delta x(t) \equiv 0, u_c(t) \equiv 0$, and $u_d(t) \equiv 0$. Hence, these models belong to the class of switched hybrid system models with continuous flows and discontinuous vector fields. Alternatively, the Back-Guckenheimer-Myers model [29] is a special case of an autonomous version of (7), (8) with $u_c(t) \equiv 0$ and $u_d(t) \equiv 0$. Finally, the Brockett models [30] are a special case of an autonomous version of (7, 8) with $\Delta x(t) \equiv 0$. For a further discussion of these models the interested reader is referred to [20].

## 4. STABILITY ANALYSIS OF LEFT-CONTINUOUS DYNAMICAL SYSTEMS

In this section we present Lyapunov, asymptotic, and exponential stability results for left-continuous dynamical systems. Furthermore, for strong left-continuous dynamical systems we present an invariant set stability theorem that generalizes the Barbashin-Krasovskii-LaSalle principle [31 – 35] to systems with left-continuous flows. For

the statement of the following result we define

$$\dot{V}(t, s(t, t_0, x_0, u)) \triangleq \lim_{\tau \to t^-} \frac{1}{t - \tau} [V(t, s(t, t_0, x_0, u)) - V(\tau, s(\tau, t_0, x_0, u))],$$
(9)

for a given continuous function $V: [t_0, \infty) \times \mathcal{D} \to [0, \infty)$ whenever the limit on the right hand side exists. Note that $V(t, s(t, t_0, x_0, u))$ is left-continuous on $[t_0, \infty)$, and is continuous everywhere on $[t_0, \infty)$ except on the discrete set $\mathcal{T}^c$. Furthermore, we assume that the origin is an equilibrium point of the undisturbed left-continuous dynamical system $\mathcal{G}$.

THEOREM 4.1  *Suppose there exists a continuous function $V: [t_0, \infty) \times \mathcal{D} \to [0, \infty)$ and class $\mathcal{K}$ functions $\alpha(\cdot), \beta(\cdot)$ satisfying*

$$\alpha(\|x\|) \le V(t, x) \le \beta(\|x\|), \quad x \in \mathcal{D}, \quad t \in [t_0, \infty),$$
(10)

$$V(t, s(t, t_0, x_0, 0)) \le V(t, s(\tau, t_0, x_0, 0)), \quad t \ge \tau \ge t_0.$$
(11)

*Then the equilibrium point $x = 0$ of the undisturbed left-continuous dynamical system $\mathcal{G}$ is Lyapunov stable. If, in addition, for every $x_0 \in \mathcal{D}$, $V(\cdot)$ is such that $\dot{V}(t, s(t, t_0, x_0, 0)), t \in \mathcal{T}$, exists and*

$$\dot{V}(t, s(t, t_0, x_0, 0)) \le -\gamma(\|s(t, t_0, x_0, 0)\|), \quad t \in \mathcal{T},$$
(12)

*where $\gamma: [0, \infty) \to [0, \infty)$ is a class $\mathcal{K}$ function, then the equilibrium point $x = 0$ of the undisturbed left-continuous dynamical system $\mathcal{G}$ is asymptotically stable. Furthermore, if there exist scalars $\varepsilon, \hat{\alpha}, \hat{\beta} > 0$ and $p \ge 1$ such that*

$$\hat{\alpha}\|x\|^p \le V(t, x) \le \hat{\beta}\|x\|^p, \quad x \in \mathcal{D}, \quad t \in [t_0, \infty),$$
(13)

$$\dot{V}(t, s(t, t_0, x_0, 0)) \le -\varepsilon V(t, s(t_0, x_0, 0)), \quad t \in \mathcal{T},$$
(14)

*then the equilibrium point $x = 0$ of the undisturbed left-continuous dynamical system $\mathcal{G}$ is exponentially stable. Finally, if $\mathcal{D} = \mathbb{R}^n$ and $\alpha(\cdot)$ is a class $\mathcal{K}_\infty$ function, then (12) implies (resp., (13), (14) imply) that the equilibrium point $x = 0$ of the undisturbed left-continuous dynamical system $\mathcal{G}$ is globally asymptotically (resp., exponentially) stable.*

*Proof* The proof follows from standard arguments. For a similar proof see [18, 36].                                                    ∎

The following definitions are necessary for the main results of this section. For the next definition let the map $s_t : \mathcal{D} \to \mathcal{D}$ be defined by $s_t(x) \triangleq s(t, 0, x, 0)$, $x \in \mathcal{D}$, for a given $t \geq 0$.

DEFINITION 4.1    A set $\mathcal{M} \subset \mathcal{D}$ is a *positively invariant set* for the undisturbed stationary left-continuous dynamical system $\mathcal{G}$ if $s_t(\mathcal{M}) \subseteq \mathcal{M}$, for all $t \geq 0$, where $s_t(\mathcal{M}) \triangleq \{s_t(x) : x \in \mathcal{M}\}$. A set $\mathcal{M} \subset \mathcal{D}$ is an *invariant set* for the undisturbed stationary left-continuous dynamical system $\mathcal{G}$ if $s_t(\mathcal{M}) = \mathcal{M}$ for all $t \geq 0$.

Next, we present a generalization of the Barbashin-Krasovskii-LaSalle invariance principle [31−35] to strong left-continuous dynamical systems. This result is predicated on a generalized positive limit set theorem for systems with left-continuous flows satisfying the quasi-continuous dependence property given in Axiom (vii). For details of this result see [23]. For the remainder of the results of this paper define the notation

$$V^{-1}(\gamma) \triangleq \{x \in \mathcal{Q} : V(x) = \gamma\},$$

where $\gamma \in \mathbb{R}$, $\mathcal{Q} \subseteq \mathcal{D}$, and $V : \mathcal{Q} \to \mathbb{R}$ is a $C^0$ function, and let $\mathcal{M}_\gamma$ denote the largest invariant set (with respect to the strong left-continuous dynamical system $\mathcal{G}$) contained in $V^{-1}(\gamma)$.

THEOREM 4.2 [23]    *Let* $s(t, 0, x_0, 0)$, $t \geq 0$, *denote a trajectory of the undisturbed strong left-continuous dynamical system* $\mathcal{G}$ *and let* $\mathcal{D}_c \subset \mathcal{D}$ *be a compact positively invariant set with respect to* $\mathcal{G}$. *Assume there exists a* $C^0$ *function* $V : \mathcal{D}_c \to \mathbb{R}$ *such that* $V(s(t, 0, x_0, 0)) \leq V(s(\tau, 0, x_0, 0))$, $0 \leq \tau \leq t$, *for all* $x_0 \in \mathcal{D}_c$. *If* $x_0 \in \overset{\circ}{\mathcal{D}}_c$, *then* $s(t, 0, x_0, 0) \to \mathcal{M} \triangleq \cup_{\gamma \in \mathbb{R}} \mathcal{M}_\gamma$ *as* $t \to \infty$. *If, in addition,* $0 \in \mathcal{D}_c$, $V(0) = 0$, $V(x) > 0$, $x \in \mathcal{D}_c$, $x \neq 0$, *and there exists an unbounded infinite sequence* $\{\tau_n\}_{n=1}^\infty$ *such that* $V(s(\tau_{n+1}, 0, x_0, 0)) < V(s(\tau_n, 0, x_0, 0))$, $n = 1, 2, \ldots$, *then the origin is an asymptotically stable equilibrium point of the undisturbed strong left-continuous dynamical system* $\mathcal{G}$.

*Remark 4.1*    If in Theorem 4.2 $\mathcal{M}$ contains no invariant set other than the set $\{0\}$, then the origin is attractive and $\mathcal{D}_c$ is a subset of the domain of attraction. If, in addition, $V(0) = 0$, $V(x) > 0$, $x \in \mathcal{D}$, $x \neq 0$,

and $0 \in \overset{\circ}{\mathcal{D}}_{\mathrm{c}}$, then the origin is an asymptotically stable equilibrium point of the undisturbed strong left-continuous dynamical system $\mathcal{G}$.

## 5. DISSIPATIVE LEFT-CONTINUOUS DYNAMICAL SYSTEMS: INPUT-OUTPUT AND STATE PROPERTIES

In this section we extend the notion of dissipative dynamical systems to develop the concept of dissipativity for left-continuous dynamical systems. In particular, using concepts of left-continuous system storage functions with appropriate hybrid supply rates, storage functions are developed as Lyapunov functions for left-continuous dynamical systems. Consider the left-continuous dynamical system $\mathcal{G}$ with input $u = (u_{\mathrm{c}}, u_{\mathrm{d}})$ and output $y = (y_{\mathrm{c}}, y_{\mathrm{d}})$. Then a function $(r_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}), r_{\mathrm{d}}(u_{\mathrm{d}}, y_{\mathrm{d}}))$, where $r_{\mathrm{c}} : U_{\mathrm{c}} \times Y_{\mathrm{c}} \to \mathbb{R}$ and $r_{\mathrm{d}} : U_{\mathrm{d}} \times Y_{\mathrm{d}} \to \mathbb{R}$ are such that $r_{\mathrm{c}}(0, 0) = 0$ and $r_{\mathrm{d}}(0, 0) = 0$, is called a *supply rate*[4] if $r_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}})$ is locally integrable; that is, for all input-output pairs $u_{\mathrm{c}}(t) \in U_{\mathrm{c}}$, $y_{\mathrm{c}}(t) \in Y_{\mathrm{c}}$, $r_{\mathrm{c}}(\cdot, \cdot)$ satisfies $\int_t^{\hat{t}} |r_{\mathrm{c}}(u_{\mathrm{c}}(s), y_{\mathrm{c}}(s))| \mathrm{d}s < \infty$, $t, \hat{t} \geq 0$. Note that since all input-output pairs $u_{\mathrm{d}}(t_k) \in U_{\mathrm{d}}$, $y_{\mathrm{d}}(t_k) \in Y_{\mathrm{d}}$, are defined for the resetting events $t_k \in T^{\mathrm{c}}$, $r_{\mathrm{d}}(\cdot, \cdot)$ satisfies $\sum_{k \in \mathcal{N}_{[t, \hat{t})}} |r_{\mathrm{d}}(u_{\mathrm{d}}(i), y_{\mathrm{d}}(i))| < \infty$, where $k \in \mathcal{N}_{[t, \hat{t})} \overset{\Delta}{=} \{k : t \leq t_k < \hat{t}\}$. For reminder of this paper, we use the notation $s(t, t_0, x_0, u)$, $t \geq t_0$, and $x(t)$, $t \geq t_0$, interchangeably to denote the trajectory of $\mathcal{G}$ with initial time $t_0$, initial condition $x_0$, and input $u$.

DEFINITION 5.1    A left-continuous dynamical system $\mathcal{G}$ is *dissipative with respect to the supply rate* $(r_{\mathrm{c}}, r_{\mathrm{d}})$ if the *dissipation inequality*

$$0 \leq \int_{t_0}^{T} r_{\mathrm{c}}(u_{\mathrm{c}}(t), y_{\mathrm{c}}(t)) \mathrm{d}t + \sum_{k \in \mathcal{N}_{[t_0, T)}} r_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y_{\mathrm{d}}(t_k)), \quad T \geq t_0, \quad (15)$$

is satisfied for all $T \geq t_0$, and $x_0 = 0$. A left-continuous dynamical system $\mathcal{G}$ is *exponentially dissipative with respect to the supply rate* $(r_{\mathrm{c}}, r_{\mathrm{d}})$ if there exists a constant $\varepsilon > 0$, such that the dissipation inequality (15) is satisfied, with $r_{\mathrm{c}}(u_{\mathrm{c}}(t), y_{\mathrm{c}}(t))$ replaced by $e^{\varepsilon t} r_{\mathrm{c}}(u_{\mathrm{c}}(t), y_{\mathrm{c}}(t))$ and $r_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y_{\mathrm{d}}(t_k))$ replaced by $e^{\varepsilon t_k} r_{\mathrm{d}}(u_{\mathrm{d}}(t_k), y_{\mathrm{d}}(t_k))$, for all $T \geq t_0$ and $x_0 = 0$. A left-continuous dynamical system is *lossless with respect to*

---

[4] More generally, the hybrid supply rate $(r_{\mathrm{c}}, r_{\mathrm{d}})$ can be an explicit function of time; that is, $r_{\mathrm{c}} : [0, \infty) \times U_{\mathrm{c}} \times Y_{\mathrm{c}} \to \mathbb{R}$ and $r_{\mathrm{d}} : [0, \infty) \times U_{\mathrm{d}} \times Y_{\mathrm{d}} \to \mathbb{R}$.

*the supply rate* $(r_c, r_d)$ if the dissipation inequality (15) is satisfied as an equality for all $T \geq t_0$ and $x_0 = s(T, t_0, 0, u) = 0$.

Next, define the *available storage* $V_a(t_0, x_0)$ of the left-continuous dynamical system $\mathcal{G}$ by

$$
V_a(t_0, x_0) \overset{\Delta}{=} - \inf_{(u_c(\cdot), u_d(\cdot)), T \geq t_0} \left[ \int_{t_0}^{T} r_c(u_c(t), y_c(t)) dt \right.
$$
$$
\left. + \sum_{k \in \mathcal{N}_{[t_0, T)}} r_d(u_d(t_k), y_d(t_k)) \right]. \quad (16)
$$

Note that $V_a(t_0, x_0) \geq 0$ for all $(t_0, x_0) \in \mathbb{R} \times \mathcal{D}$ since $V_a(t_0, x_0)$ is the supremum over a set of numbers containing the zero element $(T = t_0)$. It follows from (16) that the available storage of a left-continuous dynamical system $\mathcal{G}$ is the maximum amount of generalized stored energy which can be extracted from $\mathcal{G}$ at any time $T$. Furthermore, define the *available exponential storage* of the left-continuous dynamical system $\mathcal{G}$ by

$$
V_a(t_0, x_0) \overset{\Delta}{=} - \inf_{(u_c(\cdot), u_d(\cdot)), T \geq t_0} \left[ \int_{t_0}^{T} e^{\varepsilon t} r_c(u_c(t), y_c(t)) dt \right.
$$
$$
\left. + \sum_{k \in \mathcal{N}_{[t_0, T)}} e^{\varepsilon t_k} r_d(u_d(t_k), y_d(t_k)) \right].
$$
$$
(17)
$$

*Remark 5.1* Note that in the case of stationary left-continuous dynamical systems, the available storage is time-invariant; that is, $V_a(t_0, x_0) = V_a(x_0)$. Furthermore, the available exponential storage satisfies

$$
V_a(t_0, x_0) = - \inf_{(u_c(\cdot), u_d(\cdot)), T \geq t_0} \left[ \int_{t_0}^{T} e^{\varepsilon t} r_c(u_c(t), y_c(t)) dt \right.
$$
$$
\left. + \sum_{k \in \mathcal{N}_{[t_0, T)}} e^{\varepsilon t_k} r_d(u_d(t_k), y_d(t_k)) \right]
$$
$$
= -e^{\varepsilon t_0} \inf_{(u_c(\cdot), u_d(\cdot)), T \geq t_0} \left[ \int_{t_0}^{T} e^{\varepsilon t} r_c(u_c(t), y_c(t)) dt \right.
$$
$$
\left. + \sum_{k \in \mathcal{N}_{[t_0, T)}} e^{\varepsilon t_k} r_d(u_d(t_k), y_d(t_k)) \right]
$$
$$
= e^{\varepsilon t_0} V_a(x_0),
$$
$$
(18)
$$

where

$$
V_a(x_0) \overset{\Delta}{=} - \inf_{(u_c(\cdot), u_d(\cdot)), T \geq 0} \left[ \int_{t_0}^{T} e^{\varepsilon t} r_c(u_c(t), y_c(t)) dt \right.
$$

$$
\left. + \sum_{k \in \mathcal{N}_{[0,T)}} e^{\varepsilon t_k} r_d(u_d(t_k), y_d(t_k)) \right]. \tag{19}
$$

Next, we show that the available storage (resp., available exponential storage) is finite if and only if $\mathcal{G}$ is dissipative (resp., exponentially dissipative). In order to state this result we require two more definitions.

DEFINITION 5.2  Consider the left-continuous dynamical system $\mathcal{G}$ with input $u = (u_c, u_d)$ and output $y = (y_c, y_d)$. Assume $\mathcal{G}$ is dissipative with respect to the supply rate $(r_c, r_d)$. A $C^0$ nonnegative-definite function $V_s : \mathbb{R} \times \mathcal{D} \to \mathbb{R}$ satisfying

$$
V_s(T, x(T)) \leq V_s(t_0, x_0) + \int_{t_0}^{T} r_c(u_c(t), y_c(t)) dt
$$

$$
+ \sum_{k \in \mathcal{N}_{[t_0,T)}} r_d(u_d(t_k), y_d(t_k)), \tag{20}
$$

where $x(T) = s(T, t_0, x_0, u)$, $T \geq t_0$, is called a *storage function* for $\mathcal{G}$. A $C^0$ nonnegative-definite function $V : \mathbb{R} \times \mathcal{D} \to \mathbb{R}$ satisfying

$$
e^{\varepsilon T} V(T, x(T))) \leq e^{\varepsilon t_0} V(t_0, x_0) + \int_{t_0}^{T} e^{\varepsilon t} r_c(u_c(t), y_c(t)) dt
$$

$$
+ \sum_{k \in \mathcal{N}_{[t_0,T)}} e^{\varepsilon t_k} r_d(u_d(t_k), y_d(t_k)), \tag{21}
$$

is called an *exponential storage function* for $\mathcal{G}$.

Note that for every $t_0 \in [0, \infty)$, $x_0 \in \mathcal{D}$, and $u \in \mathcal{U}$, $V_s(t, s(t, t_0, x_0, u))$ is left-continuous on $[t_0, \infty)$, and is continuous everywhere on $[t_0, \infty)$ except on $\mathcal{T}^c$.

DEFINITION 5.3  A left-continuous dynamical system $\mathcal{G}$ with input $u = (u_c, u_d)$ and output $y = (y_c, y_d)$ is *zero-state observable* if $(u_c(t), u_d(t_k)) \equiv (0,0)$, $(y_c(t), y_d(t_k)) \equiv (0,0)$ implies $s(t, t_0, x_0, u) \equiv 0$. A left-continuous dynamical system $\mathcal{G}$ is *completely reachable* if for all $(t_0, x_i) \in \mathbb{R} \times \mathcal{D}$, there exist a finite time $t_i \leq t_0$, square integrable input

$u_c(t)$ defined on $[t_i, t_0]$, and input $u_d(t_k)$ defined on $k \in \mathcal{N}_{[t_i, t_0)}$, such that the state $s(t, t_i, x_i, u)$, can be driven from $x_i = 0$ to $s(t_0, t_i, x_i, u) = x_0$. Finally, a left-continuous dynamical system $\mathcal{G}$ is *minimal* if it is zero-state observable and completely reachable.

THEOREM 5.1 *Consider the left-continuous dynamical system $\mathcal{G}$ with input $u = (u_c, u_d)$ and output $y = (y_c, y_d)$, and assume that $\mathcal{G}$ is completely reachable. Then $\mathcal{G}$ is dissipative (resp., exponentially dissipative) with respect to the supply rate $(r_c, r_d)$ if and only if the available system storage $V_a(t_0, x_0)$ given by (16) (resp., the available exponential system storage $V_a(t_0, x_0)$ given by (17)) is finite for all $t_0 \in \mathbb{R}$ and $x_0 \in \mathcal{D}$. Moreover, if $V_a(t_0, x_0)$ is finite for all $t_0 \in \mathbb{R}$ and $x_0 \in \mathcal{D}$, then $V_a(t, x)$, $(t, x) \in \mathbb{R} \times \mathcal{D}$, is a storage function (resp., exponential storage function) for $\mathcal{G}$. Finally, all storage functions (resp., exponential storage functions) $V_s(t, x)$, $(t, x) \in \mathbb{R} \times \mathcal{D}$, for $\mathcal{G}$ satisfy*

$$0 \le V_a(t, x) \le V_s(t, x), \quad (t, x) \in \mathbb{R} \times \mathcal{D}. \tag{22}$$

*Proof*  Suppose $V_a(t, x)$, $(t, x) \in \mathbb{R} \times \mathcal{D}$, is finite. Now, it follows from (16) (with $T = t_0$) that $V_a(t, x) \ge 0$, $(t, x) \in \mathbb{R} \times \mathcal{D}$. Next, since $- V_a(t, x)$, $(t, x) \in \mathbb{R} \times \mathcal{D}$, is given by the infimum over all admissible inputs $(u_c(\cdot), u_d(\cdot))$ and $T \ge t_0$ in (16), it follows that for all admissible inputs $(u_c(\cdot), u_d(\cdot))$ and $t \in [t_0, T]$,

$$
\begin{aligned}
-V_a(t_0, x_0) &\le \int_{t_0}^{T} r_c(u_c(t), y_c(t))dt + \sum_{k \in \mathcal{N}_{[t_0, T)}} r_d(u_d(t_k), y_d(t_k)) \\
&= \int_{t_0}^{t} r_c(u_c(s), y_c(s))ds + \sum_{k \in \mathcal{N}_{[t_0, t)}} r_d(u_d(t_k), y_d(t_k)) \\
&\quad + \int_{t_0}^{T} r_c(u_c(s), y_c(s))ds + \sum_{k \in \mathcal{N}_{[t, T)}} r_d(u_d(t_k), y_d(t_k)),
\end{aligned}
$$

which implies

$$
\begin{aligned}
-V_a(t_0, x_0) &- \int_{t_0}^{t} r_c(u_c(t), y_c(t))dt - \sum_{k \in \mathcal{N}_{[t_0, t)}} r_d(u_d(t_k), y_d(t_k)) \\
&\le \int_{t_0}^{T} r_c(u_c(s), y_c(s))ds + \sum_{k \in \mathcal{N}_{[t, T)}} r_d(u_d(t_k), y_d(t_k)).
\end{aligned}
$$

Hence

$$V_a(t_0, x_0) + \int_{t_0}^{t} r_c(u_c(t), y_c(t))dt + \sum_{k \in \mathcal{N}_{[t_0, t)}} r_d(u_d(t_k), y_d(t_k))$$

$$\geq - \inf_{(u_c(\cdot), u_d(\cdot)), T \geq t} \left[ \int_{t}^{T} r_c(u_c(s), y_c(s))ds \right. \tag{23}$$

$$\left. + \sum_{k \in \mathcal{N}_{[t, T)}} r_d(u_d(t_k), y_d(t_k)) \right]$$

$$= V_a(t, s(t, t_0, x_0, u)),$$

which shows that $V_a(t, x)$, $(t, x) \in \mathbb{R} \times \mathcal{D}$, is a storage function for $\mathcal{G}$.

Conversely, suppose $\mathcal{G}$ is dissipative with respect to the supply rate $(r_c, r_d)$ and let $t_0 \in \mathbb{R}$ and $x_0 \in \mathcal{D}$. Since $\mathcal{G}$ is completely reachable it follows that there exists $\hat{t} < t_0$, $u_c(t)$, $t \geq \hat{t}$, and $u_d(t_k)$, $k \in \mathcal{N}_{[\hat{t}, \infty)}$, such that $x(\hat{t}) = 0$ and $s(t_0, \hat{t}, 0, u) = x_0$. Hence, since $\mathcal{G}$ is dissipative with respect to the supply rate $(r_c, r_d)$ it follows that, for all $T \geq t_0$,

$$0 \leq \int_{\hat{t}}^{T} r_c(u_c(t), y_c(t))dt + \sum_{k \in \mathcal{N}_{[\hat{t}, T)}} r_d(u_d(t_k), y_d(t_k))$$

$$= \int_{\hat{t}}^{t_0} r_c(u_c(t), y_c(t))dt + \sum_{k \in \mathcal{N}_{[\hat{t}, t_0)}} r_d(u_d(t_k), y_d(t_k))$$

$$+ \int_{t_0}^{T} r_c(u_c(t), y_c(t))dt + \sum_{k \in \mathcal{N}_{[t_0, T)}} r_d(u_d(t_k), y_d(t_k)),$$

and hence there exists $W : \mathbb{R} \times \mathcal{D} \rightarrow \mathbb{R}$ such that

$$-\infty < W(t_0, x_0) \leq \int_{t_0}^{T} r_c(u_c(t), y_c(t))dt + \sum_{k \in \mathcal{N}_{[t_0, T)}} r_d(u_d(t_k), y_d(t_k)).$$

$$\tag{24}$$

Now, it follows from (24) that, for all $(t, x) \in \mathbb{R} \times \mathcal{D}$,

$$V_a(t, x) = - \inf_{(u_c(\cdot), u_d(\cdot)), T \geq t_0} \left[ \int_{t_0}^{T} r_c(u_c(t), y_c(t))dt \right.$$

$$\left. + \sum_{k \in \mathcal{N}_{[t_0, T)}} r_d(u_d(t_k), y_d(t_k)) \right]$$

$$\leq - W(t, x), \tag{25}$$

and hence the available storage $V_a(t, x)$, $(t, x) \in \mathbb{R} \times \mathcal{D}$, is finite.

Next, if $V_s(t, x)$, $(t, x) \in \mathbb{R} \times \mathcal{D}$, is a storage function then it follows that, for all $T \geq t_0$ and $x_0 \in \mathcal{D}$,

$$
\begin{aligned}
V_s(t_0, x_0) &\geq V_s(T, s(T, t_0, x_0, u)) \\
&\quad - \int_{t_0}^{T} r_c(u_c(t), y_c(t))\mathrm{d}t - \sum_{k \in \mathcal{N}_{[t_0, T)}} r_d(u_d(t_k), y_d(t_k)) \\
&\geq - \left[ \int_{t_0}^{T} r_c(u_c(t), y_c(t))\mathrm{d}t + \sum_{k \in \mathcal{N}_{[t_0, T)}} r_d(u_d(t_k), y_d(t_k)) \right],
\end{aligned}
$$

which implies

$$
\begin{aligned}
V_s(t_0, x_0) &\geq - \inf_{(u_c(\cdot), u_d(\cdot)), T \geq t_0} \left[ \int_{t_0}^{T} r_c(u_c(t), y_c(t))\mathrm{d}t \right. \\
&\qquad\qquad\qquad\qquad \left. + \sum_{k \in \mathcal{N}_{[t_0, T)}} r_d(u_d(t_k), y_d(t_k)) \right] \\
&= V_a(t_0, x_0).
\end{aligned}
$$

Finally, the proof for the exponentially dissipative case follows a similar construction and hence is omitted. ∎

The following corollary is immediate from Theorem 5.1.

COROLLARY 5.1  *Consider the left-continuous dynamical system $\mathcal{G}$ and assume that $\mathcal{G}$ is completely reachable. Then $\mathcal{G}$ is dissipative (resp., exponentially dissipative) with respect to the supply rate $(r_c, r_d)$ if and only if there exists a $C^0$ storage function (resp., exponential storage function) $V_s(t, x)$, $(t, x) \in \mathbb{R} \times \mathcal{D}$, satisfying (20) (resp., (21)).*

The next result gives necessary and sufficient conditions for dissipativity, exponential dissipativity, and losslessness over an interval $t \in (t_k, t_{k+1}]$ involving the consecutive resetting times $t_k, t_{k+1} \in \mathcal{T}^c$.

THEOREM 5.2  *$\mathcal{G}$ is dissipative with respect to the supply rate $(r_c, r_d)$ if and only if there exists a $C^0$ nonnegative-definite function $V_s: \mathbb{R} \times \mathcal{D} \to \mathbb{R}$ such that, for all $k \in \mathcal{N}$,*

$$
\begin{aligned}
V_s(\hat{t}, s(\hat{t}, t_0, x_0, u)) &- V_s(t, s(t, t_0, x_0, u)) \\
&\leq \int_{t}^{\hat{t}} r_c(u_c(s), y_c(s))\mathrm{d}s, \quad t_k < t \leq \hat{t} \leq t_{k+1}, \quad (26)
\end{aligned}
$$

$$
V_s(t_k^+, s(t_k^+, t_0, x_0, u)) - V_s(t_k, s(t_k, t_0, x_0, u)) \leq r_d(u_d(t_k), y_d(t_k)). \quad (27)
$$

*Furthermore, $\mathcal{G}$ is exponentially dissipative with respect to the supply rate $(r_c, r_d)$ if and only if there exists a $C^0$ nonnegative-definite function $V_s: \mathbb{R} \times \mathcal{D} \to \mathbb{R}$ such that*

$$e^{\varepsilon \hat{t}} V_s(\hat{t}, s(\hat{t}, s(\hat{t}, t_0, x_0, u))) - e^{\varepsilon t} V_s(t, s(t, t_0, x_0, u))$$
$$\leq \int_t^{\hat{t}} e^{\varepsilon s} r_c(u_c(s), y_c(s)) ds, \qquad (28)$$
$$t_k < t \leq \hat{t} \leq t_{k+1},$$

$$V_s(t_k^+, s(t_k^+, t_0, x_0, u)) - V_s(t_k, s(t_k, t_0, x_0, u)) \leq r_d(u_d(t_k), y_d(t_k)). \quad (29)$$

*Finally, $\mathcal{G}$ is lossless with respect to the supply rate $(r_c, r_d)$ if and only if there exists a $C^0$ nonnegative-definite function $V_s: \mathbb{R} \times \mathcal{D} \to \mathbb{R}$ such that (26) and (27) are satisfied as equalities.*

*Proof* Let $k \in \mathcal{N}$ and suppose $\mathcal{G}$ is dissipative with respect to the supply rate $(r_c, r_d)$. Then, there exists a $C^0$ nonnegative-definite function $V_s: \mathbb{R} \times \mathcal{D} \to \mathbb{R}$ such that (20) holds. Now, since for $t_k < t \leq \hat{t} \leq t_{k+1}, \mathcal{N}_{[t,\hat{t})} = \varnothing$, (26) is immediate. Next, note that

$$V_s(t_k^+, s(t_k^+, t_0, x_0, u)) - V_s(t_k, s(t_k, t_0, x_0, u))$$
$$\leq \int_{t_k}^{t_k^+} r_c(u_c(s), y_c(s)) ds + r_d(u_d(t_k), y_d(t_k)),$$
$$(30)$$

which, since $\mathcal{N}_{[t_k, t_k^+)} = \{k\}$, implies (27).

Conversely, suppose (26) and (27) hold, let $\hat{t} \geq t \geq 0$, and let $\mathcal{N}_{[t,\hat{t})} = \{i, i+1, \ldots, j\}$. (Note that if $\mathcal{N}_{[t,\hat{t})} = \varnothing$ the converse is a direct consequence of (20)). In this case, it follows from (26) and (27) that

$$V_s(\hat{t}, x(\hat{t})) - V_s(t, x(t)) = V_s(\hat{t}, x(\hat{t})) - V_s(t_j^+, x(t_j^+))$$
$$+ V_s(t_j^+, x(t_j^+)) - V_s(t_{j-1}^+, x(t_{j-1}^+))$$
$$+ V_s(t_{j-1}^+, x(t_{j-1}^+)) - \cdots - V_s(t_i^+, x(t_i^+))$$
$$+ V_s(t_i^+, x(t_i^+)) - V_s(t, x(t))$$
$$\leq \int_t^{\hat{t}} r_c(u_c(s), y_c(s)) ds$$
$$+ \sum_{k \in \mathcal{N}_{[t,\hat{t})}} r_d(u_d(t_k), y_d(t_k)),$$

which implies that $\mathcal{G}$ is dissipative with respect to the supply rate $(r_c, r_d)$.

Finally, similar constructions show that $\mathcal{G}$ is exponentially dissipative with respect to the supply rate $(r_c, r_d)$ if and only if (28) and (29) are satisfied, and $\mathcal{G}$ is lossless with respect to the supply rate $(r_c, r_d)$ if and only if (26) and (27) are satisfied as equalities.    ∎

If in Theorem 5.2 $\dot{V}_s(\cdot, s(\cdot, t_0, x_0, u))$ exists a.e. on $[t_0, \infty)$ except the discrete set $\mathcal{T}^c$, then an equivalent statement for dissipativeness of the left-continuous dynamical system $\mathcal{G}$ with respect to the supply rate $(r_c, r_d)$ is

$$\dot{V}_s(t, s(t, t_0, x_0, u)) \leq r_c(u_c(t), y_c(t)), \quad t_k < t \leq t_{k+1}, \tag{31}$$

$$\Delta V_s(t_k, s(t_k, t_0, x_0, u)) \leq r_d(u_d(t_k), y_d(t_k)), \quad k \in \mathcal{N}, \tag{32}$$

where $\Delta V_s(t_k, s(t_k, t_0, x_0, u)) \triangleq V_s(t_k^+, s(t_k^+, t_0, x_0, u)) - V_s(t_k, s(t_k, t_0, x_0, u))$, $k \in \mathcal{N}$, denotes the difference of the storage function $V_s(t, x)$ at the times $t_k$, $k \in \mathcal{N}$, of the left-continuous dynamical system $\mathcal{G}$. Furthermore, an equivalent statement for exponential dissipativeness of the left-continuous dynamical system $\mathcal{G}$ with respect to the supply rate $(r_c, r_d)$ is given by

$$\dot{V}_s(t, s(t, t_0, x_0, u)) + \varepsilon V_s(t, s(t, t_0, x_0, u)) \leq r_c(u_c(t), y_c(t)),$$
$$t_k < t \leq t_{k+1}, \tag{33}$$

and (32).

The following theorem provides sufficient conditions for guaranteeing that all storage functions (resp., exponential storage functions) of a given dissipative (resp., exponentially dissipative) left-continuous dynamical system are positive definite.

THEOREM 5.3  *Consider the left-continuous dynamical system $\mathcal{G}$ and assume that $\mathcal{G}$ is completely reachable and zero-state observable. Furthermore, assume that $\mathcal{G}$ is· dissipative (resp., exponentially dissipative) with respect to the supply rate $(r_c, r_d)$ and there exist functions $\kappa_c : Y_c \to U_c$ and $\kappa_d : Y_d \to U_d$ such that $r_c(\kappa_c(y_c), y_c) < 0$, $y_c \neq 0$, and $r_d(\kappa_d(y_d), y_d) < 0$, $y_d \neq 0$. Then all the storage functions (resp., exponential storage functions) $V_s(t, x), (t, x) \in \mathbb{R} \times \mathcal{D},$ for $\mathcal{G}$ are positive definite, that is, $V_s(\cdot, 0) = 0$ and $V_s(t, x) > 0, (t, x) \in \mathbb{R} \times \mathcal{D}, x \neq 0.$*

*Proof* It follows from Theorem 5.1 that the available storage $V_a(t, x)$, $(t, x) \in \mathbb{R} \times \mathcal{D}$, is a storage function for $\mathcal{G}$. Next, suppose, *ad absurdum*, there exists $(t, x) \in \mathbb{R} \times \mathcal{D}$ such that $V_a(t, x) = 0$, or, equivalently,

$$\inf_{(u_c(\cdot), u_d(\cdot)), T \geq t_0} \left[ \int_{t_0}^{T} r_c(u_c(t), y_c(t)) dt \right.$$
$$\left. + \sum_{k \in \mathcal{N}_{[t_0, T)}} r_d(u_d(t_k), y_d(t_k)) \right] = 0. \tag{34}$$

Furthermore, suppose there exists $[t_s, t_f) \subset \mathbb{R}$ such that $y_c(t) \neq 0$, $t \in [t_s, t_f)$, or $y_d(t_k) \neq 0$, for some $k \in \mathcal{N}$. Now, since there exists $\kappa_c : Y_c \to U_c$ and $\kappa_d : Y_d \to U_d$ such that $r_c(\kappa_c(y_c), y_c) < 0$, $y_c \neq 0$, and $r_d(\kappa_d(y_d), y_d) < 0$, $y_d \neq 0$, the infimum in (34) occurs at a negative value which as a contradiction. Hence, $y_c(t) = 0$, a.e. $t \in \mathbb{R}$, and $y_d(t_k) = 0$ for all $k \in \mathcal{N}$. Next, since $\mathcal{G}$ is zero-state observable it follows that $x = 0$ and hence $V_a(t, x) = 0$ if and only if $x = 0$. The result now follows from (22). Finally, the proof for the exponentially dissipative case is similar and hence is omitted. ∎

Next, we introduce the concept of required supply of a left-continuous dynamical system $\mathcal{G}$. Specifically, define the *required supply* $V_r(t_0, x_0)$ of the left-continuous dynamical system $\mathcal{G}$ by

$$V_r(t_0, x_0) \triangleq \inf_{(u_c(\cdot), u_d(\cdot)), T \leq t_0} \left[ \int_{T}^{t_0} r_c(u_c(t), y_c(t)) dt \right.$$
$$\left. + \sum_{k \in \mathcal{N}_{[T, t_0)}} r_d(u_d(t_k), y_d(t_k)) \right], \tag{35}$$

where $T \leq t_0$ and $u \in \mathcal{U}$ are such that $s(t_0, T, 0, u) = x_0$. It follows from (35) that the required supply of a left-continuous dynamical system is the minimum amount of generalized energy which can be delivered to the left-continuous dynamical system in order to transfer it from a zero initial state to a given state $x_0$. Similarly, define the *required exponential supply* of the left-continuous dynamical system $\mathcal{G}$ by

$$V_r(t_0, x_0) \triangleq \inf_{(u_c(\cdot), u_d(\cdot)), T \leq t_0} \left[ \int_{T}^{t_0} e^{\varepsilon t} r_c(u_c(t), y_c(t)) dt \right.$$
$$\left. + \sum_{k \in \mathcal{N}_{[T, t_0)}} e^{\varepsilon t_k} r_d(u_d(t_k), y_d(t_k)) \right], \tag{36}$$

where $T \leq t_0$ and $u \in \mathcal{U}$ are such that $s(t_0, T, 0, u) = x_0$.

Next, using the notion of required supply, we show that all storage functions are bounded from above by the required supply and bounded from below by the available storage. Hence, as in the case of systems with continuous flows [21], a dissipative left-continuous dynamical system can only deliver to its surroundings a fraction of its stored generalized energy and can only store a fraction of the generalized work done to it.

THEOREM 5.4   *Consider the left-continuous dynamical system $\mathcal{G}$ and assume that $\mathcal{G}$ is completely reachable. Then $\mathcal{G}$ is dissipative (resp., exponentially dissipative) with respect to the supply rate $(r_c, r_d)$ if and only if $0 \leq V_r(t, x) < \infty$, $t \in \mathbb{R}$, $x \in \mathcal{D}$. Moreover, if $V_r(t, x)$ is finite and nonnegative for all $(t_0, x_0) \in \mathbb{R} \times \mathcal{D}$, then $V_r(t, x)$, $(t, x) \in \mathbb{R} \times \mathcal{D}$, is a storage function (resp., exponential storage function) for $\mathcal{G}$. Finally, all storage functions (resp., exponential storage functions) $V_s(t, x)$, $(t, x) \in \mathbb{R} \times \mathcal{D}$, for $\mathcal{G}$ satisfy*

$$0 \leq V_a(t, x) \leq V_s(t, x) \leq V_r(t, x) < \infty, \quad (t, x) \in \mathbb{R} \times \mathcal{D}. \quad (37)$$

*Proof*   Suppose $0 \leq V_r(t, x) < \infty$, $(t, x) \in \mathbb{R} \times \mathcal{D}$. Next, since $V_r(t, x)$, $(t, x) \in \mathbb{R} \times \mathcal{D}$, is given by the infimum over all admissible inputs $(u_c(\cdot), u_d(\cdot))$ and $T \leq t_0$ in (35), it follows that for all admissible inputs $(u_c(\cdot), u_d(\cdot))$ and $T \leq t \leq t_0$,

$$
\begin{aligned}
V_r(t_0, x_0) &\leq \int_T^{t_0} r_c(u_c(t), y_c(t)) dt + \sum_{k \in \mathcal{N}_{[T, t_0)}} r_d(u_d(t_k), y_d(t_k)) \\
&= \int_T^t r_c(u_c(s), y_c(s)) ds + \sum_{k \in \mathcal{N}_{[T, t)}} r_d(u_d(t_k), y_d(t_k)) \\
&\quad + \int_t^{t_0} r_c(u_c(s), y_c(s)) ds + \sum_{k \in \mathcal{N}_{[t, t_0)}} r_d(u_d(t_k), y_d(t_k))
\end{aligned}
$$

and hence

$$
\begin{aligned}
V_r(t_0, x_0) \leq \inf_{(u_c(\cdot), u_d(\cdot)), T \leq t} &\left[ \int_T^t r_c(u_c(s), y_c(s)) ds \right. \\
&\left. + \sum_{k \in \mathcal{N}_{[T, t)}} r_d(u_d(t_k), y_d(t_k)) \right]
\end{aligned}
$$

$$+ \int_t^{t_0} r_c(u_c(s), y_c(s)) ds + \sum_{k \in \mathcal{N}_{[t,t_0)}} r_d(u_d(t_k), y_d(t_k))$$

$$= V_r(t, s(t, T, 0, u)) + \int_t^{t_0} r_c(u_c(s), y_c(s)) ds$$

$$+ \sum_{k \in \mathcal{N}_{[t,t_0)}} r_d(u_d(t_k), y_d(t_k)), \tag{38}$$

which shows that $V_r(t, x)$, $(t, x) \in \mathbb{R} \times \mathcal{D}$, is a storage function for $\mathcal{G}$ and hence $\mathcal{G}$ is dissipative.

Conversely, suppose $\mathcal{G}$ is dissipative with respect to the supply rate $(r_c, r_d)$ and let $t_0 \in \mathbb{R}$ and $x_0 \in \mathcal{D}$. Since $\mathcal{G}$ is completely reachable it follows that there exists $T < t_0$, and $u$ such that $s(t_0, T, 0, u) = x_0$. Hence, since $\mathcal{G}$ is dissipative with respect to the supply rate $(r_c, r_d)$ it follows that, for all $T \leq t_0$,

$$0 \leq \int_T^{t_0} r_c(u_c(t), y_c(t)) dt + \sum_{k \in \mathcal{N}_{[T,t_0)}} r_d(u_d(t_k), y_d(t_k)) \tag{39}$$

and hence

$$0 \leq \inf_{(u_c(\cdot), u_d(\cdot)), T \leq t_0} \left[ \int_T^{t_0} r_c(u_c(s), y_c(s)) ds \right.$$

$$\left. + \sum_{k \in \mathcal{N}_{[T,t_0)}} r_d(u_d(t_k), y_d(t_k)) \right], \tag{40}$$

which implies that

$$0 \leq V_r(t_0, x_0) < \infty, \quad (t_0, x_0) \in \mathbb{R} \times \mathcal{D}. \tag{41}$$

Next, if $V_s(\cdot, \cdot)$, is a storage function for $\mathcal{G}$ then it follows from Theorem 5.1 that

$$0 \leq V_a(t, x) \leq V_s(t, x), \quad (t, x) \in \mathbb{R} \times \mathcal{D}. \tag{42}$$

Furthermore, for all $T \in \mathbb{R}$ such that $x(T) = 0$ it follows that

$$V_s(t_0, x_0) \leq V_s(T, 0) + \int_T^{t_0} r_c(u_c(t), y_c(t)) dt$$

$$+ \sum_{k \in \mathcal{N}_{[T,t_0)}} r_d(u_d(t_k), y_d(t_k)) \tag{43}$$

and hence

$$V_s(t_0, x_0) \le \inf_{(u_c(\cdot), u_d(\cdot)), T \le t_0} \left[ \int_T^{t_0} r_c(u_c(t), y_c(t)) \mathrm{d}t \right.$$

$$\left. + \sum_{k \in \mathcal{N}_{[T, t_0)}} r_d(u_d(t_k), y_d(t_k)) \right]$$

$$= V_r(t_0, x_0) < \infty, \tag{44}$$

which implies (37). Finally, the proof for the exponentially dissipative case follows a similar construction and hence is omitted. ∎

Next, as a direct consequence of Theorems 5.1 and 5.4, we show that the set of all possible storage functions of a left-continuous dynamical system forms a convex set. An identical result holds for exponential storage functions.

PROPOSITION 5.1   *Consider the left-continuous dynamical system $\mathcal{G}$ with available storage $V_a(t, x)$, $(t, x) \in \mathbb{R} \times \mathcal{D}$, and required supply $V_r(t, x)$, $(t, x) \in \mathbb{R} \times \mathcal{D}$, and assume that $\mathcal{G}$ is completely reachable. Then*

$$V_s(t, x) \stackrel{\triangle}{=} \alpha V_a(t, x) + (1 - \alpha) V_r(t, x), \quad \alpha \in [0, 1], \tag{45}$$

*is a storage function for $\mathcal{G}$.*

*Proof*   The result is a direct consequence of the dissipation inequality (20) by noting that if $V_a(t, x)$ and $V_r(t, x)$ satisfy (20), then $V_s(t, x)$ satisfies (20). ∎

Finally, we provide two definitions of left-continuous dynamical systems which are dissipative (resp., exponentially dissipative) with respect to supply rates of a specific form.

DEFINITION 5.4   A left-continuous dynamical system $\mathcal{G}$ with input $u = (u_c, u_d)$, output $y = (y_c, y_d)$, $m_c = l_c$, and $m_d = l_d$ is *passive* (resp., *exponentially passive*) if $\mathcal{G}$ is dissipative (resp., exponentially dissipative) with respect to the supply rate $(r_c(u_c, y_c), r_d(u_d, y_d)) = (2u_c^T y_c, 2u_d^T y_d)$.

DEFINITION 5.5   A left-continuous dynamical system $\mathcal{G}$ with input $u = (u_c, u_d)$ and output $y = (y_c, y_d)$ is *nonexpansive* (resp., *exponentially*

*nonexpansive*) if $\mathcal{G}$ is dissipative (resp., exponentially dissipative) with respect to the supply rate $(r_c(u_c, y_c), r_d(u_d, y_d)) = (\gamma_c^2 u_c^T u_c - y_c^T y_c, \gamma_d^2 u_d^T u_d - y_d^T y_d)$, where $\gamma_c, \gamma_d > 0$ are given.

*Remark 5.2*   Note that a mixed passive-nonexpansive formulation of $\mathcal{G}$ can also be considered. Specifically, one can consider left-continuous dynamical systems $\mathcal{G}$ which are dissipative with respect to supply rates of the form $(r_c(u_c, y_c), r_d(u_d, y_d)) = (2u_c^T y_c, \gamma_d^2 u_d^T u_d - y_d^T y_d)$, where $\gamma_d > 0$, and *vice-versa*. Furthermore, supply rates for input passivity [37], output strict passivity [37], and input-output strict passivity [37] can also be considered. However, for simplicity of exposition we do not do so here.

In light of the above definitions, the following result is immediate.

PROPOSITION 5.2   *Consider the left-continuous dynamical system $\mathcal{G}$ with input $u = (u_c, u_d)$, output $y = (y_c, y_d)$, storage function $V_s(\cdot, \cdot)$, and supply rate $(r_c, r_d)$. Then the following statements hold*:

 (i) *If $\mathcal{G}$ is dissipative, $r_c(0, y_c) \leq 0$, $y_c \in Y_c$, $r_d(0, y_d) \leq 0$, $y_d \in Y_d$, and $V_s(\cdot, \cdot)$ satisfies (10), then the equilibrium point $x = 0$ of the undisturbed system $\mathcal{G}$ is Lyapunov stable.*

 (ii) *If $\mathcal{G}$ is exponentially dissipative, $r_c(0, y_c) \leq 0$, $y_c \in Y_c$, $r_d(0, y_d) \leq 0$, $y_d \in Y_d$, and $V_s(\cdot, \cdot)$ satisfies (10), then the equilibrium point $x = 0$ of the undisturbed system $\mathcal{G}$ is asymptotically stable. If, in addition, $V_s(\cdot, \cdot)$ satisfies (13), then the equilibrium point $x = 0$ of the undisturbed system $\mathcal{G}$ is exponentially stable.*

 (iii) *If $\mathcal{G}$ is passive (resp., nonexpansive) then the equilibrium point $x = 0$ of the undisturbed system $\mathcal{G}$ is Lyapunov stable.*

 (iv) *If $\mathcal{G}$ is exponentially passive (resp., exponentially nonexpansive) and $V_s(\cdot, \cdot)$ satisfies (10), then the equilibrium point $x = 0$ of the undisturbed system $\mathcal{G}$ is asymptotically stable. If, in addition, $V_s(\cdot, \cdot)$ satisfies (13), then the equilibrium point $x = 0$ of the undisturbed system $\mathcal{G}$ is exponentially stable.*

 (v) *If $\mathcal{G}$ is a strong left-continuous dynamical system, zero-state observable, and nonexpansive, then the equilibrium point $x = 0$ of the undisturbed system $\mathcal{G}$ is asymptotically stable.*

*Proof*   The result is a direct consequence of Theorems 5.2, 4.1 and 4.2 using standard arguments.                                           ∎

## 6. INTERCONNECTIONS OF DISSIPATIVE LEFT-CONTINUOUS DYNAMICAL SYSTEMS

In this section we consider interconnections of dissipative left-continuous dynamical systems. Specifically, consider a finite collection of left-continuous dynamical systems $\mathcal{G}_\alpha = (\mathcal{D}_\alpha, \mathcal{U}_\alpha, U_\alpha, \mathcal{Y}_\alpha, Y_\alpha, s_\alpha, h_{c\alpha}, h_{d\alpha})$, where $\alpha$ spans over a finite index set $\mathcal{A}$,[5] and consider the spaces $\tilde{\mathcal{U}}$, $\tilde{U}$, $\tilde{\mathcal{Y}}$. and $\tilde{Y}$. Here, the elements of $\mathcal{U}_\alpha$ and $\mathcal{Y}_\alpha$ are internal inputs and outputs, respectively, while the elements of $\tilde{\mathcal{U}}$ and $\tilde{\mathcal{Y}}$ are external inputs and outputs, respectively. Next, we introduce an interconnection function $\mathcal{I} : \tilde{U} \times \Pi_{\alpha \in \mathcal{A}} Y_\alpha \to \tilde{Y} \times \Pi_{\alpha \in \mathcal{A}} U_\alpha$, where $\Pi_{\alpha \in \mathcal{A}}$ denotes the Cartesian set product. Figure 1 illustrates the concept of a finite collection of left-continuous dynamical subsystems $\mathcal{G}_\alpha$ interconnected through the interconnection constraint $\mathcal{I}$ yielding an interconnected system $\tilde{\mathcal{G}} = \Pi_{\alpha \in \mathcal{A}} \mathcal{G}_\alpha / \mathcal{I}$. The following definition provides well posedness conditions for the interconnected system $\tilde{\mathcal{G}}$ to qualify as a left-continuous dynamical system.
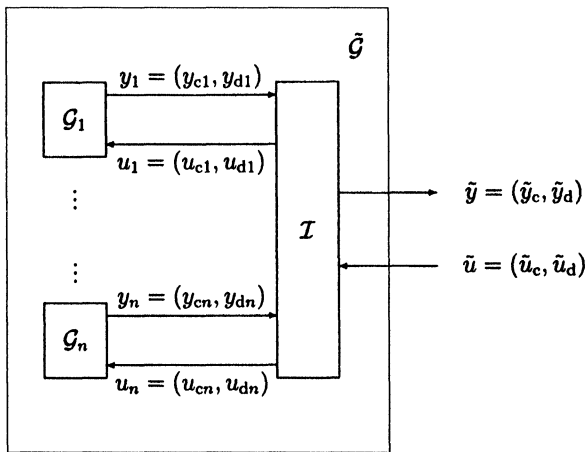


FIGURE 1   System interconnection $\tilde{\mathcal{G}} = \Pi_{\alpha \in \mathcal{A}} \mathcal{G}_\alpha \backslash \mathcal{I}$.

---

[5] More generally, countably infinite sets with an appropriate measure on $\mathcal{A}$ can also be considered.

DEFINITION   6.1   The   left-continuous   dynamical   system   $\tilde{\mathcal{G}} =$ $(\Pi_{\alpha \in \mathcal{A}}\mathcal{D}_\alpha, \tilde{\mathcal{U}}, \tilde{U}, \tilde{\mathcal{Y}}, \tilde{Y}, \Pi_{\alpha \in \mathcal{A}}s_\alpha, \tilde{h}_c, \tilde{h}_d)$ is an interconnection of the left-continuous dynamical systems $\mathcal{G}_\alpha = (\mathcal{D}_\alpha, \mathcal{U}_\alpha, U_\alpha, \mathcal{Y}_\alpha, Y_\alpha, s_\alpha, h_{c\alpha}, h_{d\alpha}$, $\alpha \in \mathcal{A}$, through the interconnection constraint $\mathcal{I}$ if for every $x_\alpha \in \mathcal{D}_\alpha$, $\tilde{u} \in \tilde{\mathcal{U}}$, and $t \in [t_0, \infty)$, there exist unique maps $\psi_\alpha : [0, \infty) \times \Pi_{\alpha \in \mathcal{A}}\mathcal{D}_\alpha \times$ $\tilde{U} \to U_\alpha$,   $(\tilde{h}_c, \tilde{h}_d): [0, \infty) \times \Pi_{\alpha \in \mathcal{A}}\mathcal{D}_\alpha \times \tilde{U} \to \tilde{Y}$,   and   $s_\alpha: [0, \infty) \times$ $[0, \infty) \times \Pi_{\alpha \in \mathcal{A}}\mathcal{D}_\alpha \times \tilde{U} \to \mathcal{D}_\alpha$, such that $u_\alpha(t) = \psi_\alpha(t, s_\alpha(t, t_0, x_\alpha, \tilde{u}), \tilde{u})$ and $\Pi_{\alpha \in \mathcal{A}}s_\alpha$ satisfies Axioms (i)–(iv).

A straight forward but key property of a left-continuous interconnected dynamical system is that if the component subsystems are dissipative and the interconnection constraint does not introduce any new supply or dissipation, then the interconnected system is dissipative. Hence, the following result is immediate. For this result let $\tilde{r}_c: \tilde{U}_c \times \tilde{Y}_c \to \mathbb{R}$ and $\tilde{r}_d: \tilde{U}_d \times \tilde{Y}_d \to \mathbb{R}$ be given.

PROPOSITION 6.1   *Let $\mathcal{G}_\alpha$, $\alpha \in \mathcal{A}$, be a finite collection of left-continuous dissipative dynamical systems with supply rates $r_\alpha = (r_{c\alpha}(u_{c\alpha}, y_{c\alpha})$, $r_{d\alpha}(u_{d\alpha}, y_{d\alpha}))$, where $r_{c\alpha}: U_{c\alpha} \times Y_{c\alpha} \to \mathbb{R}$ and $r_{d\alpha}: U_{d\alpha} \times Y_{d\alpha} \to \mathbb{R}$, and storage functions $V_{s\alpha}(\cdot, \cdot)$. Furthermore, let the interconnection constraint $\mathcal{I}: U \times \Pi_{\alpha \in \mathcal{A}}Y_\alpha \to Y \times \Pi_{\alpha \in \mathcal{A}}U_\alpha$ be such that $\tilde{r}_c = \sum_{\alpha \in \mathcal{A}}r_{c\alpha}$ and $\tilde{r}_d = \sum_{\alpha \in \mathcal{A}}r_{d\alpha}$. Then the interconnected system $\tilde{\mathcal{G}} = \Pi_{\alpha \in \mathcal{A}}\mathcal{G}_\alpha/\mathcal{I}$ is dissipative with supply rate $(\tilde{r}_c, \tilde{r}_d) = (\sum_{\alpha \in \mathcal{A}}r_{c\alpha}, \sum_{\alpha \in \mathcal{A}}r_{d\alpha})$ and storage function $V_s(\cdot, \cdot) = \sum_{\alpha \in \mathcal{A}}V_{s\alpha}(\cdot, \cdot)$.*

*Proof*   The result is a direct consequence of Theorem 5.2 by summing both sides of the inequalities

$$V_{s\alpha}(\hat{t}, x_\alpha(\hat{t})) - V_{s\alpha}(t, x_\alpha(t)) \leq \int_t^{\hat{t}} r_{c\alpha}(u_{c\alpha}(s), y_{c\alpha}(s))ds,$$

$$t_k < t \leq \hat{t} \leq t_{k+1},$$

$$V_{s\alpha}(t_k^+, x_\alpha(t_k^+)) - V_{s\alpha}(t_k, x_\alpha(t_k)) \leq r_{d\alpha}(u_d(t_k), y_d(t_k)),$$

for $k \in \mathcal{N}$, over $\alpha \in \mathcal{A}$ and using the assumptions $r_c = \sum_{\alpha \in \mathcal{A}}r_{c\alpha}$ and $r_d = \sum_{\alpha \in \mathcal{A}}r_{d\alpha}$. ∎

The following corollary is a direct consequence of Proposition 6.1.

COROLLARY 6.1   *Consider the left-continuous dynamical systems $\mathcal{G}_1$ and $\mathcal{G}_2$ with input-output pairs $(u_{c1}, u_{d1}; y_{c1}, y_{d1})$ and $(u_{c2}, u_{d2}; y_{c2}, y_{d2})$,*

*respectively. Then the following statements hold*:

(i) *If $\mathcal{G}_1$ and $\mathcal{G}_2$ are passive, then the parallel interconnection of $\mathcal{G}_1$ and $\mathcal{G}_2$ is passive.*

(ii) *If $\mathcal{G}_1$ and $\mathcal{G}_2$ are passive, then the negative feedback interconnection of $\mathcal{G}_1$ and $\mathcal{G}_2$ is passive.*

(iii) *If $\mathcal{G}_1$ and $\mathcal{G}_2$ are nonexpansive with gains $(\gamma_{c1}, \gamma_{d1})$ and $(\gamma_{c2}, \gamma_{d2})$, respectively, then the cascade interconnection of $\mathcal{G}_1$ and $\mathcal{G}_2$ is nonexpansive with gain $(\gamma_{c1} \gamma_{c2}, \gamma_{d1}\gamma_{d2})$.*

*Proof*   The result is a direct consequence of Proposition 6.1 by noting the interconnection constraints for cascade, parallel, and feedback interconnections are given by $(\tilde{u}_c, \tilde{u}_d) = (u_{c1}, u_{d2})$, $(u_{c2}, u_{d2}) = (y_{c1}, y_{d1})$, $(\tilde{y}_c, \tilde{y}_d) = (y_{c2}, y_{d2})$; $(\tilde{u}_c, \tilde{u}_d) = (u_{c1}, u_{d2}) = (u_{c2}, u_{d2})$, $(\tilde{y}_c, \tilde{y}_d) = (y_{c1} + y_{c2}, y_{d1} + y_{d2})$; and $(\tilde{u}_c, \tilde{u}_d) = (u_{c1} + y_{c2}, u_{d1} + y_{d2})$, $(\tilde{y}_c, \tilde{y}_d) = (y_{c1}, y_{d1}) = (u_{c2}, u_{d2})$, respectively. Now, the result is immediate by noting that the above interconnection constraints satisfy the required constraints on $\tilde{r}_c$ and $\tilde{r}_d$ in Proposition 6.1.                              ∎

Next, we consider stability of feedback interconnections of dissipative left-continuous dynamical systems. Specifically, using the notion of dissipative and exponentially dissipative left-continuous dynamical systems, with appropriate storage functions and supply rates, we construct Lyapunov functions for interconnected left-continuous dynamical systems by appropriately combining storage functions for each subsystem. Here, for simplicity of exposition, we restrict our attention to stationary left-continuous dynamical systems. Furthermore, we assume that for the dynamical system $\mathcal{G}$, $\mathcal{D} = \mathbb{R}^n$, $U_c = \mathbb{R}^{m_c}$, $U_d = \mathbb{R}^{m_d}$, $Y_c = \mathbb{R}^{l_c}$, and $Y_d = \mathbb{R}^{l_d}$. We begin by considering the negative feedback interconnection of the stationary left-continuous dynamical system $\mathcal{G}$ with the stationary left-continuous feedback system $\mathcal{G}_c$ given by the octuple $(\mathbb{R}^{n_c}, \hat{\mathcal{U}}, \mathbb{R}^{m_{cc}} \times \mathbb{R}^{m_{dc}}, \hat{\mathcal{Y}}, \mathbb{R}^{l_{cc}} \times \mathbb{R}^{l_{dc}}, s_c, h_{cc}, h_{dc})$, where $\hat{\mathcal{U}} \triangleq \mathcal{U}_{cc} \times \mathcal{U}_{dc}$, $\hat{\mathcal{Y}} \triangleq \mathcal{Y}_{cc} \times \mathcal{Y}_{dc}$, $s_c : [0, \infty) \times [0, \infty) \times \mathbb{R}^{n_c} \times \hat{\mathcal{U}} \rightarrow \mathbb{R}^{n_c}$, $h_{cc} : \mathbb{R}^{n_c} \times \mathbb{R}^{m_{cc}} \rightarrow \mathbb{R}^{l_{cc}}$, and $h_{dc} : \mathbb{R}^{n_c} \times \mathbb{R}^{m_{dc}} \rightarrow \mathbb{R}^{l_{dc}}$. We refer to $s_c(t, 0, x_{c0}, \hat{u})$, $t \geq 0$, as the trajectory of $\mathcal{G}_c$ corresponding to an initial condition $x_{c0} \in \mathbb{R}^{n_c}$ and input $\hat{u} = (u_{cc}, u_{dc}) \in \hat{\mathcal{U}}$, where $u_{cc} \in \mathcal{U}_{cc}$ and $u_{dc} \in \mathcal{U}_{dc}$. Furthermore, for $\mathcal{G}_c$, let $\mathcal{T}_c^c$ denote the set of resetting times and let $\mathcal{T}_c$ denote the complement of $\mathcal{T}_c^c$; that is, $[t_0, \infty) \backslash \mathcal{T}_c^c$. Note that with the feedback interconnection given by Figure 2,
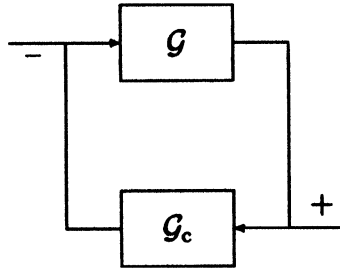
FIGURE 2   Feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$.

$(u_{cc}, u_{dc}) = (y_c, y_d)$ and $(y_{cc}, y_{dc}) = (-u_c, -u_d)$. For the ensuing results, we assume that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is well posed; that is, the feedback interconnection generates an undisturbed stationary left-continuous dynamical system on $\mathbb{R}^n \times \mathbb{R}^{n_c}$ with trajectory $\tilde{s}(t, 0, (x_0, x_{c0}), 0) \triangleq (s(t, 0, x_0, u), s_c(t, 0, x_{c0}, y))$ and initial condition $(x_0, x_{c0}) \in \mathbb{R}^n \times \mathbb{R}^{n_c}$.

THEOREM 6.1   *Consider the feedback system consisting of the stationary left-continuous dynamical systems $\mathcal{G}$ and $\mathcal{G}_c$ with input-output pairs $(u_c, u_d, y_c, y_d)$ and $(u_{cc}, u_{dc}; y_{cc}, y_{dc})$, respectively, and with $(u_{cc}, u_{dc}) = (y_c, y_d)$ and $(y_{cc}, y_{dc}) = (-u_c, -u_d)$. Assume $\mathcal{G}$ and $\mathcal{G}_c$ are zero-state observable, and dissipative with supply rates $(r_c(u_c, y_c),\ r_d(u_d,\ y_d))$ and $(r_{cc}(u_{cc}, y_{cc}),\ r_{dc}(u_{dc}, y_{dc}))$ and $C^0$ positive definite, radially unbounded storage functions $V_s(\cdot)$ and $V_{sc}(\cdot)$, respectively, such that $V_s(0) = 0$, $V_{sc}(0) = 0$. Furthermore, assume there exists a scalar $\sigma > 0$ such that $r_c(u_c, y_c) + \sigma r_{cc}(u_{cc}, y_{cc}) \leq 0$ and $r_d(u_d, y_d) + \sigma r_{dc}(u_{dc}, y_{dc}) \leq 0$. Then the following statements hold:*

(i) *The negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is Lyapunov stable.*

(ii) *If $\mathcal{G}$ and $\mathcal{G}_c$ are exponentially dissipative with respect to supply rates $(r_c(u_c, y_c),\ r_d(u_d, y_d))$ and $(r_{cc}(u_{cc}, y_{cc}),\ r_{dc}(u_{dc}, y_{dc}))$, respectively, and $V_s(\cdot)$ and $V_{sc}(\cdot)$ are such that there exist constants $\alpha$, $\alpha_c$, $\beta$, $\beta_c > 0$ such that*

$$\alpha\|x\|^2 \leq V_s(x) \leq \beta\|x\|^2, \quad x \in \mathbb{R}^n, \tag{46}$$

$$\alpha_c\|x_c\|^2 \leq V_{sc}(x_c) \leq \beta_c\|x_c\|^2, \quad x_c \in \mathbb{R}^{n_c}, \tag{47}$$

*then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is globally exponentially stable.*

*Proof* Let $\tilde{\mathcal{T}}^c \triangleq \mathcal{T}^c \cup \mathcal{T}_c^c$, $\tilde{\mathcal{T}} \triangleq [t_0, \infty) \backslash \tilde{\mathcal{T}}^c$, and $t_k \in \tilde{\mathcal{T}}^c$, $k = 1, 2, \ldots$ i) Consider the Lyapunov function candidate $V(x, x_c) = V_s(x) + \sigma V_{sc}(x_c)$. Now, the corresponding Lyapunov left derivative of $V(x, x_c)$ along the state trajectories $(x(t), x_c(t)) = (s(t, 0, x_0, u), s_c(t, 0, x_{c0}, y))$, $t \in (t_k, t_{k+1}]$, is given by

$$
\begin{aligned}
\dot{V}(x(t), x_c(t)) &= \dot{V}_s(x(t)) + \sigma \dot{V}_{sc}(x_c(t)) \\
&\leq r_c(u_c(t), y_c(t)) + \sigma r_{cc}(u_{cc}(t), y_{cc}(t)) \\
&\leq 0, \quad t \in \tilde{\mathcal{T}},
\end{aligned}
\tag{48}
$$

and the Lyapunov difference of $V(x, x_c)$ at the resetting times $t_k \in \tilde{\mathcal{T}}^c$, $k \in \mathcal{N}$, is given by

$$
\begin{aligned}
\Delta V(x(t_k), x_c(t_k)) &= \Delta V_s(x(t_k)) + \sigma \Delta V_{sc}(x_c(t_k)) \\
&\leq r_d(u_d(t_k), y_d(t_k)) + \sigma r_{dc}(u_{dc}(t_k), y_{dc}(t_k)) \\
&\leq 0.
\end{aligned}
\tag{49}
$$

Now, Lyapunov stability of the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ follows as a direct consequence of Theorem 4.1. (ii) Next, if $\mathcal{G}$ and $\mathcal{G}_c$ are exponentially dissipative and (46), (47) hold, it follows that

$$
\begin{aligned}
\dot{V}(x(t), x_c(t)) &= \dot{V}_s(x(t)) + \sigma \dot{V}_{sc}(x_c(t)) \\
&\leq -\varepsilon_c V_s(x(t)) - \sigma \varepsilon_{cc} V_{sc}(x_c(t)) \\
&\quad + r_c(u_c(t), y_c(t)) + \sigma r_{cc}(u_{cc}(t), y_{cc}(t)) \\
&\leq -\min\{\varepsilon_c, \varepsilon_{cc}\} V(x(t), x_c(t)), \quad t_k < t \leq t_{k+1},
\end{aligned}
\tag{50}
$$

and $\Delta V(x(t_k), x_c(t_k))$, $t_k \in \tilde{\mathcal{T}}^c$, $k \in \mathcal{N}$, satisfies (49). Now, Theorem 4.1 implies that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is globally exponentially stable. ∎

The next result presents Lyapunov, asymptotic, and exponential stability of dissipative feedback systems with quadratic supply rates. For this result the notation $\mathbb{S}^n$ refers to the set of $n \times n$ symmetric matrices.

THEOREM 6.2  Let  $Q_c \in \mathbb{S}^{l_c}$,  $S_c \in \mathbb{R}^{l_c \times m_c}$,  $R_c \in \mathbb{S}^{m_c}$,  $Q_d \in \mathbb{S}^{l_d}$, $S_d \in \mathbb{R}^{l_d \times m_d}$, $R_d \in \mathbb{S}^{m_d}$, $Q_{cc} \in \mathbb{S}^{l_{cc}}$, $S_{cc} \in \mathbb{R}^{l_{cc} \times m_{cc}}$, $R_{cc} \in \mathbb{S}^{m_{cc}}$, $Q_{dc} \in \mathbb{S}^{l_{dc}}$,

$S_{dc} \in \mathbb{R}^{l_{dc} \times m_{dc}}$, and $R_{dc} \in \mathbb{S}^{m_{dc}}$. *Consider the closed-loop system consisting of the left-continuous dynamical systems $\mathcal{G}$ and $\mathcal{G}_c$ and assume $\mathcal{G}$ and $\mathcal{G}_c$ are zero-state observable. Furthermore, assume $\mathcal{G}$ is dissipative with respect to the quadratic supply rate $(r_c(u_c, y_c), r_d(u_d, y_d)) = (y_c^T Q_c y_c + 2 y_c^T S_c u_c + u_c^T R_c u_c, y_d^T Q_d y_d + 2 y_d^T S_d u_d + u_d^T R_d u_d)$ and has a radially unbounded storage function $V_s(\cdot)$, and $\mathcal{G}_c$ is dissipative with respect to the quadratic supply rate $(r_{cc}(u_{cc}, y_{cc}), r_{dc}(u_{dc}, y_{dc})) = (y_{cc}^T Q_{cc} y_{cc} + 2 y_{cc}^T S_{cc} u_{cc} + u_{cc}^T R_{cc} u_{cc}, y_{dc}^T Q_{dc} y_{dc} + 2 y_{dc}^T S_{dc} u_{dc} + u_{dc}^T R_{dc} u_{dc})$ and has a radially unbounded storage function $V_{sc}(\cdot)$. Finally, assume there exists a scalar $\sigma > 0$ such that*

$$
\begin{aligned}
\hat{Q}_c &\triangleq \begin{bmatrix} Q_c + \sigma R_{cc} & -S_c + \sigma S_{cc}^T \\ -S_c^T + \sigma S_{cc} & R_c + \sigma Q_{cc} \end{bmatrix} \leq 0, \\
\hat{Q}_d &\triangleq \begin{bmatrix} Q_d + \sigma R_{dc} & -S_d + \sigma S_{dc}^T \\ -S_d^T + \sigma S_{dc} & R_d + \sigma Q_{dc} \end{bmatrix} \leq 0.
\end{aligned}
\tag{51}
$$

*Then the following statements hold:*

(i) *The negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is Lyapunov stable.*

(ii) *If $\mathcal{G}$ and $\mathcal{G}_c$ are exponentially dissipative with respect to supply rates $(r_c(u_c, y_c), r_d(u_d, y_d))$ and $(r_{cc}(u_{cc}, y_{cc}), r_{dc}(u_{dc}, y_{dc}))$, respectively, and there exist constants $\alpha$, $\alpha_c$, $\beta$, $\beta_c > 0$ such that (46) and (47) hold, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is globally exponentially stable.*

(iii) *If $\hat{Q}_c < 0$, $\hat{Q}_d < 0$, and $\mathcal{G}$ and $\mathcal{G}_c$ are strong left-continuous dynamical systems, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is globally asymptotically stable.*

*Proof* (i) and (ii) are a direct consequence of Theorem 6.1 by noting

$$
r_c(u_c, y_c) + \sigma r_{cc}(u_{cc}, y_{cc}) = \begin{bmatrix} y_c \\ y_{cc} \end{bmatrix}^T \hat{Q}_c \begin{bmatrix} y_c \\ y_{cc} \end{bmatrix},
\tag{52}
$$

$$
r_d(u_d, y_d) + \sigma r_{dc}(u_{dc}, y_{dc}) = \begin{bmatrix} y_d \\ y_{dc} \end{bmatrix}^T \hat{Q}_d \begin{bmatrix} y_d \\ y_{dc} \end{bmatrix},
\tag{53}
$$

and hence $r_c(u_c, y_c) + \sigma r_{cc}(u_{cc}, y_{cc}) \leq 0$ and $r_d(u_d, y_d) + \sigma r_{dc}(u_{dc}, y_{dc}) < 0$. To show (iii) consider the Lyapunov function candidate

$V(x, x_c) = V_s(x) + \sigma V_{sc}(x)$. Noting that $u_{cc} = y_c$ and $y_{cc} = -u_c$ it follows that the corresponding Lyapunov left derivative of $V(x, x_c)$ along the trajectories $(x(t), x_c(t)) = (s(t, 0, x_0, u), s_c(t, 0, x_{c0}, y))$ satisfies

$$
\begin{aligned}
\dot{V}(x(t), x_c(t)) &= \dot{V}_s(x(t)) + \sigma \dot{V}_{sc}(x_c(t)) \\
&\leq r_c(u_c(t), y_c(t)) + \sigma r_{cc}(u_{cc}(t), y_{cc}(t)) \\
&= y_c^T(t) Q_c y_c(t) + 2 y_c^T(t) S_c u_c(t) + u_c^T(t) R_c u_c(t) \\
&\quad + \sigma[y_{cc}^T(t) Q_{cc} y_{cc}(t) + 2 y_{cc}^T(t) S_{cc} u_{cc}(t) + u_{cc}^T(t) R_{cc} u_{cc}(t)] \\
&= \begin{bmatrix} y_c(t) \\ y_{cc}(t) \end{bmatrix}^T \hat{Q}_c \begin{bmatrix} y_c(t) \\ y_{cc}(t) \end{bmatrix} \\
&\leq 0, \quad t_k < t \leq t_{k+1},
\end{aligned}
\tag{54}
$$

and, similarly, the Lyapunov difference satisfies

$$
\begin{aligned}
\Delta V(x(t_k), x_c(t_k)) &= \begin{bmatrix} y_d(t_k) \\ y_{dc}(t_k) \end{bmatrix}^T \hat{Q}_d \begin{bmatrix} y_d(t_k) \\ y_{dc}(t_k) \end{bmatrix} \\
&\leq 0, \quad k \in \mathcal{N},
\end{aligned}
\tag{55}
$$

which implies that $V(s(\cdot, 0, x_0, u), s_c(\cdot, 0, x_{c0}, y))$ is a nonincreasing function and hence it follows from Theorem 4.1 that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is Lyapunov stable.

Next, since $\mathcal{G}$ and $\mathcal{G}_c$ are zero-state observable it follows that $V(s(t, 0, x_0, 0), s_c(t, 0, x_{c0}, 0)) = V(s(0, 0, x_0, 0, s_c(0, 0, x_{c0}, 0)), t \geq 0$ if and only if $(s(t, 0, x_0, 0), s_c(t, 0, x_{c0}, 0)) = (0, 0)$, $t \geq 0$. Hence, the largest invariant set $\mathcal{M} \triangleq \cup_{\gamma \in \mathbb{R}} \mathcal{M}_\gamma$ contained in $\mathcal{R}$ is the set $\{(0, 0)\}$. Now, if follows from Theorem 4.2 and Remark 4.1 that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is globally asymptotically stable. ∎

The following result is a direct consequence of Theorem 6.2. For this result note that if a left-continuous dynamical system $\mathcal{G}$ is dissipative (resp., exponentially dissipative) with respect to a supply rate $(r_c(u_c, y_c), r_d(u_d, y_d)) = (2u_c^T y_c, 2u_d^T y_d)$, then, with $(\kappa_c(y_c), \kappa_d(y_d)) = (-k_c y_c, -k_d y_d)$, where $k_c, k_d > 0$, it follows that $(r_c(u_c, y_c), r_d(u_d, y_d)) = (-k_c y_c^T y_c - k_d y_d^T y_d) < (0, 0)$, $(y_c, y_d) \neq (0, 0)$. Alternatively, if a left-continuous dynamical system $\mathcal{G}$ is dissipative (resp., exponentially dissipative) with respect to a supply rate $(r_c(u_c, y_c), r_d(u_d, y_d)) = (\gamma_c^2 u_c^T u_c - y_c^T y_c, \gamma_d^2 u_d^T u_d - y_d^T y_d)$, where $\gamma_c$, $\gamma_d > 0$, then,

with $(\kappa_c(y_c), \kappa_d(y_d)) = (0,0)$, it follows that $(r_c(u_c, y_c), r_d(u_d, y_d)) = (-y_c^T y_c, -y_d^T y_d) < (0,0)$, $(y_c, y_d) \neq (0,0)$. Hence, if $\mathcal{G}$ is zero-state observable it follows from Theorem 5.3 that all storage functions of $\mathcal{G}$ are positive definite.

COROLLARY 6.2 *Consider the closed-loop system consisting of the stationary left-continuous dynamical systems $\mathcal{G}$ and $\mathcal{G}_c$ and assume $\mathcal{G}$ and $\mathcal{G}_c$ are zero-state observable. Then the following statements hold:*

(i) *If $\mathcal{G}$ and $\mathcal{G}_c$ are exponentially passive with storage functions $V_s(\cdot)$ and $V_{sc}(\cdot)$, respectively, such that (46) and (47) hold, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is exponentially stable.*

(ii) *If $\mathcal{G}$ and $\mathcal{G}_c$ are exponentially nonexpansive with storage functions $V_s(\cdot)$ and $V_{sc}(\cdot)$, respectively, such that (46) and (47) hold, and gains $\gamma_c$, $\gamma_d > 0$ and $\gamma_{cc}$, $\gamma_{dc} > 0$, respectively, such that $\gamma_c \gamma_{cc} \leq 1$ and $\gamma_d \gamma_{dc} \leq 1$, then the negative feedback interconnection of $\mathcal{G}$ and $G_c$ is exponentially stable.*

*Proof* The proof is a direct consequence of Theorem 6.2. Specifically, (i) follows from Theorem 6.2 with $Q_c = 0$, $Q_d = 0$, $Q_{cc} = 0$, $Q_{dc} = 0$, $S_c = I_{m_c}$, $S_d = I_{m_d}$, $S_{cc} = I_{m_{cc}}$, $S_{dc} = I_{m_{dc}}$, $R_c = 0$, $R_d = 0$, $R_{cc} = 0$, and $R_{dc} = 0$, while (ii) follows from Theorem 6.2 with $Q_c = -I_{l_c}$, $Q_d = -I_{l_d}$, $Q_{cc} = -I_{l_{cc}}$, $Q_{dc} = -I_{l_{dc}}$, $S_c = 0$, $S_d = 0$, $S_{cc} = 0$, $S_{dc} = 0$, and $R_c = \gamma_c^2 I_{m_c}$, $R_d = \gamma_d^2 I_{m_d}$, $R_{cc} = \gamma_{cc}^2 I_{m_{cc}}$, and $R_{dc} = \gamma_{dc}^2 I_{m_{dc}}$. ∎

*Remark 6.1* Global asymptotic stability of the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is also guaranteed if the strong left-continuous dynamical system $\mathcal{G}$ is input strict passive (see *Remark 5.2*) (resp., output strict passive) and the strong left-continuous dynamical compensator $\mathcal{G}_c$ is input strict passive (resp., output strict passive). Hence, the results of this section present a generalization of the results given in [22, 37] to left-continuous dynamical systems, hybrid systems, and impulsive systems.

## 7. CONCLUSION

In this paper we developed a unified framework for left-continuous dynamical systems. This general class of systems was shown to include

hybrid systems, impulsive systems, and switching systems as special cases. Furthermore, we extended the classical notions of dissipativity theory to left-continuous dynamical systems. Specifically, the concepts of storage functions and supply rates were extended to left-continuous dynamical systems providing a generalized hybrid system energy interpretation in terms of stored energy, dissipated energy over the continuous-time dynamics, and dissipated energy at the resetting events. Finally, general stability criteria were given for Lyapunov, asymptotic, and exponential stability of feedback interconnections of left-continuous dynamical systems.

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