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A Hybrid Domain Analysis for Systems with Delays in State and Control

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The solution of time-delay systems is obtained by using a hybrid function. The properties of the hybrid functions consisting of block-pulse functions and Chebyshev polynomials are presented. The method is based upon expanding various time functions in the system as their truncated hybrid functions. The operational matrix of delay is introduced. The operational matrices of integration and delay are utilized to reduce the solution of time-delay systems to the solution of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique.

Keywords: Delay systems; Analysis; Hybrid; State; Control

1. INTRODUCTION

There are three classes of sets of orthogonal functions which are widely used. The first includes sets of piecewise constant basis functions (PCBF's) (*e.g.*, Walsh, block-pulse, *etc.*). The second consists of sets of orthogonal polynomials (*e.g.*, Laguerre, Legendre, Chebyshev, *etc.*). The third is the widely used sets of sine-cosine functions in Fourier series. While orthogonal polynomials and sine-cosine functions together form a class of continuous basis functions, PCBF's have inherent discontinuities or jumps. References [1] and [2] have demonstrated the advantages of piecewise constant basis spectral methods over Fourier spectral techniques. If a continuous function is

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approximated by PCBF's, the resulting approximation is piecewise constant. On the other hand if a discontinuous function is approximated by continuous basis functions, the discontinuities are not properly modeled. Signals frequently have mixed features of continuity and jumps. These signals are continuous over certain segments of time, with discontinuities or jump occuring at the transitions of the segments. In such situations, neither the continuous basis functions nor PCBF's taken alone would form an efficient basis for the representation of such signals.

Delays occur frequently in biological, chemical, electronic and transportation systems [3]. Time-delay systems are therefore a very important class of systems whose control and optimization have been of interest to many investigators. Much progress has been made towards the solution of delay systems using the orthogonal functions (OF's). Special attention has been given to applications of Walsh functions [4], block-pulse functions [5], Laguerre polynomials [6], Legendre polynomials [7], Chebyshev polynomials [8] and Fourier series [9]. In general, the computed response of the delay systems *via* OF's is not in good agreement with the exact response of the system [10].

In the present paper we introduce a new direct computational method to solve delay systems. This method consists of reducing the delay problem to a set of algebraic equations by first expanding the candidate function as a hybrid function with unknown coefficients. These hybrid functions, which consists of block-pulse functions plus Chebychev polynomials are given. The operational matrix of delay is introduced. This matrix together with the operational matrix of integration are then used to evaluate the coefficients of the hybrid function for the solution of delay systems. Here we will demonstrate the results by considering three illustrative examples.

2. PROPERTIES OF HYBRID FUNCTIONS

2.1. Hybrid Functions of Block-pulse and Chebyshev Polynomials

Hybrid functions $b(n,m,t)$,' $n=1,2,\ldots,N$, $m=0,1,\ldots,M-1$, have three arguments: n is the order of block-pulse functions, m is the order of Chebyshev polynomials, and t is the normalized time. They are defined on the interval $[0, t_f)$ as

$$b(n,m,t) = \begin{cases} T_m\big((2N/t_f)t - 2n + 1\big), & t \in \big[\big((n - 1/N)\big)t_f, (n/N)t_f\big) \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Here $T_m(t)$ are the well-known Chebyshev polynomials of order m which are orthogonal with respect to the weight function $w(t) = 1/\sqrt{1-t^2}$ on the interval $[-1, 1]$ and satisfy the following formulae:

$$T_{\circ}(t) = 1, \quad T_1(t) = t, \tag{2}$$

$$T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t), \quad m = 1, 2, 3, \ldots . \tag{3}$$

2.2. Function Approximation

A function $f(t)$ defined over the interval 0 to t_f may be expanded as

$$f(t) \simeq \sum_{n=1}^{N} \sum_{m=0}^{M-1} c(n,m)b(n,m,t) = C^T B(t), \tag{4}$$

where

$$C = [c(1,0), \ldots, c(1, M-1) | c(2,0), \ldots, c(2, M-1) | \cdots \\ | c(N,0), \ldots, c(N, M-1)]^T, \tag{5}$$

$$B(t) = [b(1,0,t), \ldots, b(1, M-1, t) | b(2,0,t), \ldots, b(2, M-1, t) | \cdots \\ | b(N,0,t), \ldots, b(N, M-1, t)]^T. \tag{6}$$

The integration of the vector $B(t)$ defined in Eq. (6) can be approximated by

$$\int_0^t B(t')dt' \simeq PB(t), \tag{7}$$

where P is the $NM \times NM$ operational matrix for integration and is given in [11].

2.3. The Delay Operational Matrix of the Hybrid of Block-pulse and Chebyshev Polynomials

The delay function $B(t-\tau)$ is the shift of the function $B(t)$ defined in Eq. (6) along the time axis by τ. The general expression is given by

$$B(t-\tau) = DB(t), \quad t > \tau \tag{8}$$

where D is the delay operational matrix of hybrid functions. To find D, we first choose N in the following manner:

$$N = \begin{cases} (1/\tau) & \text{if } (1/\tau) \text{ is an integer number} \\ [(1/\tau)] + 1 & \text{otherwise,} \end{cases} \tag{9}$$

where [ ] denotes greatest integer value.

It is noted that for the case $\tau < t < 2\tau$, the only terms with nonzero values are $b(1, m, t-\tau)$ for $m = 0, 1, 2, \ldots, M-1$. If we expand $b(1, m, t-\tau)$ in terms of $b(2, m, t)$, the coefficient is an $M \times M$ identity matrix, since $b(1, m, t-\tau) = b(2, m, t)$. In a similar manner, for $2\tau < t < 3\tau$, only $b(2, m, t-\tau)$ for $m = 0, 1, 2, \ldots, M-1$ has nonzero values. If we expand $b(2, m, t-\tau)$ in terms of $b(3, m, t)$, then the coefficient is an $M \times M$ identity matrix. Thus, if we expand $B(t-\tau)$ in terms of $B(t)$ we find the $NM \times NM$ matrix D as

$$D = \begin{pmatrix} 0 & I & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & I & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & I & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{10}$$

3. TIME-VARYING LINEAR DELAY SYSTEMS

Consider a linear time-varying system with delay in both the state and control described by

$$\begin{aligned} \dot{X}(t) = E(t)X(t) + F(t)X(t-\tau) + G(t)U(t) \\ + H(t)U(t-\tau), \quad 0 \le t \le 1, \end{aligned} \tag{11}$$

$$X(0) = X_0, \tag{11a}$$

$$X(t) = \phi(t), \quad -\tau \leq t < 0, \tag{11b}$$

$$U(t) = \psi(t), \quad -\tau \leq t < 0, \tag{11c}$$

where $X(t) \in R^l$, $U(t) \in R^q$, $E(t)$, $F(t)$, $G(t)$ and $H(t)$ are matrices of appropriate dimensions, X_0 is a constant specified vector, and $\phi(t)$ and $\psi(t)$ are arbitrary known functions. Let

$$E(t) = [e_{ij}(t)] \quad i = 1,2,\ldots,l, \; j = 1,2,\ldots,l, \tag{12}$$

$$F(t) = [f_{ij}(t)] \quad i = 1,2,\ldots,l, \; j = 1,2,\ldots,l, \tag{13}$$

$$G(t) = [g_{ik}(t)] \quad i = 1,2,\ldots,l, \; k = 1,2,\ldots,q, \tag{14}$$

$$H(t) = [h_{ik}(t)] \quad i = 1,2,\ldots,l, \; k = 1,2,\ldots,q, \tag{15}$$

$$X(t) = [x_1(t), x_2(t), \ldots, x_l(t)]^T, \tag{16}$$

$$U(t) = [u_1(t), u_2(t), \ldots, u_q(t)]^T. \tag{17}$$

Assume that each $e_{ij}(t)$, $f_{ij}(t)$ and $x_i(t)$, $i = 1,2,\ldots,l$, $j = 1,2,\ldots,l$, can be written in terms of hybrid functions as

$$e_{ij}(t) = E_{ij}^T B(t), \tag{18}$$

$$f_{ij}(t) = F_{ij}^T B(t), \tag{19}$$

$$x_i(t) = B^T(t) X_i, \tag{20}$$

where E_{ij}, F_{ij} and X_i can be obtained similarly to Eq. (5). Using Eqs. (12) and (18) we get

$$E(t) = \begin{pmatrix} E_{11}^T B(t) & E_{12}^T B(t) & \cdots & E_{1l}^T B(t) \\ E_{21}^T B(t) & E_{22}^T B(t) & \cdots & E_{2l}^T B(t) \\ \vdots & \vdots & & \vdots \\ E_{l1}^T B(t) & E_{l2}^T B(t) & \cdots & E_{ll}^T B(t) \end{pmatrix}$$
$$= \begin{pmatrix} E_{11}^T & E_{12}^T & \cdots & E_{1l}^T \\ E_{21}^T & E_{22}^T & \cdots & E_{2l}^T \\ \vdots & \vdots & & \vdots \\ E_{l1}^T & E_{l2}^T & \cdots & E_{ll}^T \end{pmatrix} \begin{pmatrix} B(t) & 0 & \cdots & 0 \\ 0 & B(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B(t) \end{pmatrix}. \tag{21}$$

Further by using Eqs. (16) and (20) we have

$$X(t)=\begin{pmatrix} B^T(t)X_1 \\ B^T(t)X_2 \\ \vdots \\ B^T(t)X_l \end{pmatrix}=\begin{pmatrix} B^T(t) & 0 & \cdots & 0 \\ 0 & B^T(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B^T(t) \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_l \end{pmatrix}. \tag{22}$$

In a similar manner we get

$$X(0)=\begin{pmatrix} B^T(t)d_1 \\ B^T(t)d_2 \\ \vdots \\ B^T(t)d_l \end{pmatrix}, \quad \phi(t-\tau)=\begin{pmatrix} B^T(t)r_1 \\ B^T(t)r_2 \\ \vdots \\ B^T(t)r_n \end{pmatrix},$$

where

$$d_i=[x_i(0),0,\ldots,0|x_i(0),0,\ldots,0|\cdots|x_i(0),0,\ldots,0]^T, \quad i=1,2,\ldots,l. \tag{23}$$

Let

$$\hat{B}(t)=I_l\otimes B^T(t),$$

where I_l is the l-dimensional identity matrix and $\otimes$ denotes Kronecker product [12].

Thus we have

$$X(t)=\hat{B}(t)X, \tag{24}$$

$$X(0)=\hat{B}(t)d, \tag{25}$$

$$\phi(t-\tau)=\hat{B}(t)R, \tag{26}$$

where X, d, and R are vectors of order $lMN\times 1$ given by

$$X=[X_1,X_2,\ldots,X_l]^T,$$
$$d=[d_1,d_2,\ldots,d_l]^T,$$
$$R=[r_1,r_2,\ldots,r_l]^T.$$

Using Eqs. (21) and (22), $E(t)X(t)$ can be written as

$$E(t)X(t) = \begin{pmatrix} E_{11}^T B(t)B^T(t) & E_{12}^T B(t)B^T(t) & \cdots & E_{1l}^T B(t)B^T(t) \\ E_{21}^T B(t)B^T(t) & E_{22}^T B(t)B^T(t) & \cdots & E_{2l}^T B(t)B^T(t) \\ \vdots & \vdots & & \vdots \\ E_{l1}^T B(t)B^T(t) & E_{l2}^T B(t)B^T(t) & \cdots & E_{ll}^T B(t)B^T(t) \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_l \end{pmatrix} \quad (27)$$

The following property of the product of two Chebyshev polynomial vectors will also be used. Let

$$T(t) = [T_0(t), T_1(t), \ldots, T_{M-1}(t)]^T,$$
$$A = [a_0, a_1, \ldots, a_{M-1}]^T.$$

Then we have

$$T(t)T^T(t)A = \tilde{A}T^T(t), \quad (28)$$

where $\tilde{A}$ is an $M \times M$ matrix given in [13] by

$$\tilde{A} = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{M-1} \\ (1/2)a_1 & a_0 + (1/2)a_2 & (1/2)(a_1 + a_3) & \cdots & (1/2)a_{M-2} \\ (1/2)a_2 & (1/2)(a_1 + a_3) & a_0 + (1/2)a_4 & \cdots & (1/2)a_{M-3} \\ \vdots & \vdots & \vdots & & \vdots \\ (1/2)a_{M-1} & (1/2)a_{M-2} & (1/2)a_{M-3} & \cdots & a_0 \end{pmatrix}.$$

Let

$$B_n(t) = [b(n,0,t), b(n,1,t), \ldots, b(n,M-1,t)]^T, \quad n = 1, 2, \ldots, N,$$
$$\bar{C}_n = [c(n,0), c(n,1), \ldots, c(n,M-1)]^T, \quad n = 1, 2, \ldots, N.$$

Then using Eqs. (5) and (6) we get

$$B(t) = [B_1(t), B_2(t), \ldots, B_N(t)]^T, \quad (29)$$

$$C = [\bar{C}_1, \bar{C}_2, \ldots, \bar{C}_N]^T. \quad (30)$$

By using Eqs. (29) and (30) we obtain

$$B(t)B^T(t)C = \begin{pmatrix} B_1(t)B_1^T(t)\bar{C}_1 & 0 & \cdots & 0 \\ 0 & B_2(t)B_2^T(t)\bar{C}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_N(t)B_N^T(t)\bar{C}_N \end{pmatrix}. \tag{31}$$

Similarly to Eq. (28) we have

$$B_n(t)B_n^T(t)\bar{C}_n = \tilde{\bar{C}}_n B_n(t), \quad n = 1, 2, \ldots, N. \tag{32}$$

Using Eqs. (31) and (32), we get

$$B(t)B^T(t)C = \tilde{C}B(t), \tag{33}$$

where $\tilde{C}$ is an $NM \times NM$ diagonal matrix given by

$$\tilde{C} = \begin{pmatrix} \tilde{\bar{C}}_1 & 0 & \cdots & 0 \\ 0 & \tilde{\bar{C}}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{\bar{C}}_N \end{pmatrix}.$$

From Eq. (33) we get

$$E_{ij}^T B(t)B^T(t) = B^T(t)\tilde{E}_{ij}^T.$$

Using Eq. (27) we obtain

$$E(t)X(t) = \hat{B}(t)E^*X, \tag{34}$$

where

$$E^* = \begin{pmatrix} \tilde{E}_{11}^T & \tilde{E}_{12}^T & \cdots & \tilde{E}_{1l}^T \\ \tilde{E}_{21}^T & \tilde{E}_{22}^T & \cdots & \tilde{E}_{2l}^T \\ \vdots & \vdots & & \vdots \\ \tilde{E}_{l1}^T & \tilde{E}_{l2}^T & \cdots & \tilde{E}_{ll}^T \end{pmatrix}.$$

We can also approximate $X(t-\tau)$ in terms of hybrid functions as

$$X(t-\tau) = \begin{cases} \hat{B}(t)R, & 0 < t < \tau \\ \hat{B}(t)\hat{D}X, & \tau < t < 1, \end{cases} \tag{35}$$

where

$$\hat{D} = I_l \otimes D^T,$$

and D is delay operational matrix given in Eq. (10).

Moreover

$$\int_0^t \hat{B}(t')dt' = (I_l \otimes B^T(t))(I_l \otimes P^T) = \hat{B}(t)\hat{P}, \tag{36}$$

where P is operational matrix of integration given in Eq. (7).

Using Eqs. (13) and (35) we have

$$F(t)X(t-\tau) = \begin{cases} F(t)\hat{B}(t)R = \hat{B}(t)F^*R & 0 < t < \tau, \\ F(t)\hat{B}(t)\hat{D}X = \hat{B}(t)F^*\hat{D}X, & \tau < t < 1, \end{cases} \tag{37}$$

where

$$F^* = \begin{pmatrix} \tilde{F}_{11}^T & \tilde{F}_{12}^T & \cdots & \tilde{F}_{1l}^T \\ \tilde{F}_{21}^T & \tilde{F}_{22}^T & \cdots & \tilde{F}_{2l}^T \\ \vdots & \vdots & & \vdots \\ \tilde{F}_{l1}^T & \tilde{F}_{l2}^T & \cdots & \tilde{F}_{ll}^T \end{pmatrix}. \tag{38}$$

From Eq. (37) we get

$$\int_0^t F(t')X(t'-\tau)dt' = \begin{cases} \hat{B}(t)\hat{P}F^*R, & 0 < t < \tau, \\ VF^*R + \hat{B}(t)\hat{P}F^*\hat{D}X, & \tau < t < 1, \end{cases} \tag{39}$$

where

$$V = \int_0^\tau \hat{B}(t)dt = \hat{B}(t)Z. \tag{40}$$

By applying Eqs. (38)–(40) we have

$$\int_0^t F(t')X(t'-\tau)dt' = \hat{B}(t)\hat{P}F^*R + \hat{B}(t)ZF^*R + \hat{B}(t)\hat{P}F^*\hat{D}X. \tag{41}$$

Similarly by expanding each $g_{ik}(t)$, h_{ik} and $u_k(t)$, $i = 1, 2, \ldots, l$, $k = 1, 2, \ldots, q$, in terms of hybrid functions we get

$$g_{ik}(t) = G_{ik}^T B(t), \tag{43}$$

$$h_{ik}(t) = H_{ik}^T B(t), \tag{43a}$$

$$u_k(t) = B^T(t) U_k. \tag{43b}$$

Hence we have

$$G(t)U(t) = \hat{B}(t) G^* U, \tag{44}$$

where $U = [U_1, U_2, \ldots, U_q]^T$ and

$$G^* = \begin{pmatrix} \tilde{G}_{11}^T & \tilde{G}_{12}^T & \cdots & \tilde{G}_{1q}^T \\ \tilde{G}_{21}^T & \tilde{G}_{22}^T & \cdots & \tilde{G}_{2q}^T \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{G}_{l1}^T & \tilde{G}_{l2}^T & \cdots & \tilde{G}_{lq}^T \end{pmatrix}. \tag{45}$$

4. SOLUTION OF TIME-VARYING LINEAR DELAY SYSTEMS

By integrating Eq. (11) from 0 to t and using Eqs. (12)–(45) we have

$$\begin{aligned} \hat{B}(t)X - \hat{B}(t)d = \hat{B}(t)\hat{P}E^*X + \hat{B}(t)\hat{P}F^*R + \hat{B}(t)ZF^*R \\ + \hat{B}(t)\hat{P}F^*\hat{D}X + \hat{B}(t)\hat{P}G^*U + \hat{B}(t)\hat{P}H^*L \\ + \hat{B}(t)ZH^*L + \hat{B}(t)\hat{P}H^*\hat{D}_1 U, \end{aligned} \tag{46}$$

where

$$\psi(t-\tau) = \hat{B}_1(t)L, \quad \hat{D}_1 = I_q \otimes D^T, \quad \hat{B}_1 = I_q \otimes B^T$$

and H^* can be obtained in a similar manner to G^* in Eq. (45). From Eq. (46) we get

$$\begin{aligned} X = [I - \hat{P}E^* - \hat{P}F^*\hat{D}]^{-1}[d + \hat{P}F^*R + ZF^*R + \hat{P}G^*U + \\ \hat{P}H^*L + ZH^*L + \hat{P}H^*\hat{D}_1 U]. \end{aligned} \tag{47}$$

5. ILLUSTRATIVE EXAMPLES

In this section three examples are given to demonstrate the applicability, efficiency and accuracy of our proposed method. First by using Eq. (9) we determine N. Thus we have different intervals given by

$$[0,\tau),[\tau,2\tau),\ldots,[(N-1)\tau,1),$$

when $N=1/\tau$, to define $x(t)$ for t in the interval $[0,1/N)$, we map $[0,1/N)$ into $[-1,1)$ by mapping t into $2Nt-1$ and for t in the interval $[1/N,2/N)$ we map this interval into $[-1,1)$ by mapping t into $2Nt-3$ and similarly for the other intervals. When $N=[1/\tau]+1$, to define $x(t)$ for t in the interval $[0,\tau)$ we map $[0,\tau)$ into $[-1,1)$ by mapping t into $2/\tau t-1$, and for t in the interval $[\tau,2\tau)$ we map this interval into $(-1,1)$ by mapping t into $2/\tau t-3$, and similarly for the other intervals. When selecting M, we first choose an arbitrary number depending on the problem. If the exact solutions are polynomials, we can increase the value of M by 1 until two consecutive results are the same. When the exact solutions are not polynomials, we evaluate the results for two consecutive M for different t in $[0,1)$ until the results are similar up to a required number of decimal places.

Example 1 Consider the following delay system with delay in both control and state

$$\dot{x}(t)=-x(t)-2x\left(t-\frac{1}{4}\right)+2u\left(t-\frac{1}{4}\right), \tag{48}$$

$$x(t)=u(t)=0, \quad \text{for } -\frac{1}{4}\le t\le 0, \tag{49}$$

$$u(t)=1, \quad \text{for } t>0. \tag{50}$$

Although the above system is time invariant, the method described here can be used. The exact solution is [14]

$$x(t)=\begin{cases} 0, & 0\le t<(1/4)\\ 2-2\exp\left[-\left(t-(1/4)\right)\right], & (1/4)\le t<(1/2)\\ -2-2\exp\left[-\left(t-(1/4)\right)\right] & \\ \quad +(2+4t)\exp\left[-\left(t-(1/2)\right)\right], & (1/2)\le t<(3/4)\\ 6-2\exp\left[-\left(t-(1/4)\right)\right] & \\ \quad +(2+4t)\exp\left[-\left(t-(1/2)\right)\right] & \\ -((17/4)+2t+4t^2)\exp\left[-\left(t-(3/4)\right)\right], & (3/4)\le t<1. \end{cases}$$

Here, we solve this problem with hybrid functions by choosing $N=4$ and $M=7$. Let

$$x(t) = C^T B(t), \tag{50}$$

where

$$C = [c_{10}, \ldots, c_{16} | c_{20}, \ldots, c_{26} | c_{30}, \ldots, c_{36} | c_{40}, \ldots, c_{46}]^T, \tag{51}$$

and

$$B(t) = [b(1,0,t), \ldots, b(1,6,t) | b(2,0,t), \ldots, b(2,6,t) \\ | b(3,0,t), \ldots, b(3,6,t) | b(4,0,t), \ldots, b(4,6,t)]^T. \tag{52}$$

We also have

$$\int_0^t u\left(t' - \frac{1}{4}\right) dt' = \begin{cases} 0, & 0 \leq t \leq (1/4) \\ t - (1/4) & (1/4) \leq t \leq 1. \end{cases} \tag{53}$$

From Eq. (53) we obtain

$$\int_0^t u\left(t' - \frac{1}{4}\right) dt' = \left[0,0,0,\ldots,0 \middle| \frac{1}{8}, \frac{1}{8}, 0, \ldots, 0 \middle| \frac{3}{8}, \frac{1}{8}, 0, \ldots, 0 \right. \\ \left. \middle| \frac{5}{8}, \frac{1}{8}, 0, \ldots, 0 \right] B(t) = U^T B(t). \tag{54}$$

Integrating Eq. (48) from 0 to t and using Eqs. (49)–(54) we get

$$C^T = -C^T P - 2C^T DP + 2U^T, \tag{55}$$

where P and D are operational matrices of integration and delay respectively. From Eq. (55) we obtain

$$C^T = 2U^T [I_{28} + P + 2DP]^{-1}.$$

Where I_{28} is the 28-dimensional identity matrix. In Table I a comparison is made between the exact solution and the approximate solution of $x(t)$ for $1/4 \leq t \leq 1$. The approximate value of $x(t)$ on $[0, 1/4]$ is equal to zero which is the same as the exact solution.

TABLE I Estimated and exact values of $x(t)$

t	*Hybrid*	*Exact*
0.25	0.00000000	0.00000000
0.30	0.09754115	0.09754115
0.35	0.19032516	0.19032516
0.40	0.27858404	0.27858404
0.45	0.36253849	0.36253849
0.50	0.44239843	0.44239843
0.55	0.51352714	0.51352714
0.60	0.57190846	0.57190846
0.65	0.61861659	0.61861659
0.70	0.65465130	0.65465130
0.75	0.68094260	0.68094260
0.80	0.69851567	0.69851567
0.85	0.70892964	0.70892964
0.90	0.71372360	0.71372360
0.95	0.71426052	0.71426052
1.00	0.71174280	0.71174280

Example 2 Consider a time-varying delay system described by

$$\dot{x}(t) = 16tx\left(t - \frac{1}{4}\right), \tag{56}$$

$$x(0) = 1,$$

$$x(t) = 0, \quad -\frac{1}{4} \le t < 0.$$

The exact solution is [15]

$$x(t) = \begin{cases} 1, & 0 \le t < (1/4) \\ 1 + 4(t-(1/4)) + 8(t-(1/4))^2, & (1/4) \le t < (1/2) \\ (5/2) + 8(t-(1/2)) + 24(t-(1/2))^2 \\ \quad +(128/3)(t-(1/2))^3 + 32(t-(1/2))^4, & (1/2) \le t < (3/4) \\ (163/24) + 20(t-(3/4)) + 68(t-(3/4))^2 \\ \quad +(416/3)(t-(3/4))^3 \\ +224(t-(3/4))^4 + (640/3)(t-(3/4))^5 \\ \quad +(256/3)(t-(3/4))^6, & (3/4) \le t < 1. \end{cases}$$

Here, we solve this problem by choosing $N=4$ and $M=7$. Let

$$x(t) = C^T B(t), \tag{57}$$

where C and $B(t)$ are given in Eqs. (51) and (52) respectively. By expanding t and $x(0)$ we get

$$t = \left[\frac{1}{8},\frac{1}{8},0,0\middle|\frac{3}{8},\frac{1}{8},0,0\middle|\frac{5}{8},\frac{1}{8},0,0\middle|\frac{7}{8},\frac{1}{8},0,0\right]^T B(t) = K^T B(t), \tag{58}$$

and

$$x(0) = [1,0,\dots,0|1,0,\dots,0|1,0,\dots,0|1,0,\dots,0]^T B(t) = e^T B(t). \tag{59}$$

Integrating Eq. (56) from 0 to t we obtain

$$C^T B(t) - e^T B(t) = 16\int_0^t C^T D B(t') B^T(t') K dt', \tag{60}$$

where D is the delay operational matrix. Also, from Eq. (33) we have

$$B(t)B^T(t)K = \tilde{K}B(t). \tag{61}$$

Using Eqs. (60) and (61) we get

$$C^T - e^T = 16 C^T D\tilde{K}P. \tag{62}$$

From Eq. (62) we obtain the vector C and using Eqs. (57) and (63) we get

$$x(t)=\begin{cases} T_0(8t-1), & 0\le t<(1/4)\\ (27/16)T_0(8t-3)+(3/4)T_1(8t-3)+(1/16)T_2(8t-3), & (1/4)\le t<(1/2)\\ (13225/3072)T_0(8t-5)+(271/128)T_1(8t-5) \\ \quad +(87/256)T_2(8t-5) \\ +(11/384)T_3(8t-5)+(1/1024)T_4(8t-5), & (1/2)\le t<(3/4)\\ (214757/16384)T_0(8t-7)+(30175/4096)T_1(8t-7) \\ \quad +(38757/32768)T_2(8t-7) \\ +(1171/8192)T_3(8t-7)+(569/49152)T_4(8t-7) \\ +(13/24576)T_5(8t-7)+(1/98304)T_6(8t-7), & (3/4)\le t<1. \end{cases}$$

Using Eqs. (2) and (3), for $T_0(t), \dots, T_6(t)$, the same value as the exact $x(t)$ would be obtained.

Example 3 Consider the delay system described by

$$\dot{x}(t) = x(t-0.3) + 2t, \quad 0 \le t \le 1, \tag{63}$$

$$x(0) = 1,$$
$$x(t) = 0, \quad -0.3 \le t < 0.$$

The exact solution is

$$x(t) = \begin{cases} 1 + t^2, & 0 \le t < 0.3 \\ (691/1000) + (109/100)t + (7/10)t^2 + (1/3)t^3, & 0.3 \le t < 0.6 \\ (4409/5000) + (209/500)t + (69/50)t^2 + (2/15)t^3 + (1/12)t^4, & 0.6 \le t < 0.9 \\ (1500917/2000000) + (35107/40000)t + (1617/2000)t^2 & \\ +(87/200)t^3 + (1/120)t^4 + (1/60)t^5, & 0.9 \le t \le 1. \end{cases}$$

Since $\tau = 0.3$ and $1/\tau$ is not an integer, using Eq. (9) we select $N = 4$ and also choose $M = 6$. Let

$$x(t) = C^T B(t), \tag{64}$$

where C and B can be obtained similarly to Eqs. (51) and (52). By expanding $x(0)$ and t^2 in terms of hybrid functions, we get

$$x(0) = f_1^T B(t), \tag{65}$$

and

$$t^2 = \left[\frac{27}{800}, \frac{9}{200}, \frac{9}{800}, 0, 0, 0 \middle| \frac{171}{800}, \frac{27}{200}, \frac{9}{800}, 0, 0, 0 \right.$$
$$\left. \middle| \frac{459}{800}, \frac{45}{200}, \frac{9}{800}, 0, 0, 0 \middle| \frac{891}{800}, \frac{63}{200}, \frac{9}{800}, 0, 0, 0 \right]^T = f_2^T B(t). \tag{66}$$

where f_1 can be calculated similarly to Eq. (59). Let

$$f = f_1 + f_2. \tag{67}$$

By integrating Eq. (63) from 0 to t and using Eqs. (65) and (68) we get

$$C^T = f^T[I_{24} - DP]^{-1}, \tag{68}$$

where D and P are operational matrices of delay and integration respectively and I_{24} is the 24-dimensional identity matrix. Solving Eq. (68) gives the exact value x(t).

6. CONCLUSION

The hybrid function operational matrix P together with the delay matrix D, are used to obtain the solution of a linear time-varying system with delay in both the state and control. The method is based upon reducing the system into a set of algebraic equations. It is also shown that the hybrid of block-pulse functions and Cheybyshev polynomials provides an exact solution for the cases when the exact solutions are polynomials. It is noted that exact solutions obtained in the examples (2) and (3) can not be obtained either with piecewise constant basis functions nor with continuous basis functions.

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