Mathematical Problems in Engineering,
Volume 7, pp. 393–412
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Robust Stability and $\mathcal{H}_\infty$-Estimation for Uncertain Discrete Systems with State-Delay

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(Received 1 December 1999; In final form 30 January 2001)

In this paper, we investigate the problems of robust stability and $\mathcal{H}_\infty$-estimation for a class of linear discrete-time systems with time-varying norm-bounded parameter uncertainty and unknown state-delay. We provide complete results for robust stability with prescribed performance measure and establish a version of the discrete Bounded Real Lemma. Then, we design a linear estimator such that the estimation error dynamics is robustly stable with a guaranteed $\mathcal{H}_\infty$-performance irrespective of the parameteric uncertainties and unknown state delays. A numerical example is worked out to illustrate the developed theory.

Keywords: Robust stability; Linear estimation; Discrete-time systems; State-delay; $\mathcal{H}_\infty$-performance

1 INTRODUCTION

State estimation (filtering) is perhaps one of the oldest problems studied in systems theory [1]. In recent years, robust state estimation arose out of the desire to determine estimates of unmeasurable state variables for dynamical systems with uncertain parameters. From this prespective, robust state estimation can be viewed as an extension of the celebrated Kalman filter [1] to uncertain dynamical systems. The past decade has witnessed major developments in robust state estimation problem using various approaches [2–11]. Of particular interest to our work is the $\mathcal{H}_\infty$

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filtering in which the design is based on minimizing the $\mathcal{H}_\infty$-norm of the system. This design reflects a worst-case gain of the transfer function from the disturbance inputs to the estimation error output. In addition, $\mathcal{H}_\infty$ filtering is superior to standard $\mathcal{H}_\infty$ filtering since no statistical assumption is made on the input signals.

On another front of research, the class of dynamical systems with time-delay has attracted the attention of numerous investigators in the last two decades. Design of robust state estimators to different classes of continuous-time systems with parametric uncertainties and state-delay have been pursued in [12–15]. Despite the significant role of time-delays in discrete-time systems, a little attention has been paid to the class of uncertain discrete-time systems with delays. A preliminary result to bridge this gap is reported in [16] by developing a robust Kalman filter for a class of discrete uncertain systems with state-delay. A comprehensive coverages of the available results on time-delay systems can be found in [17].

This paper contributes to the further developement of robust state estimation techniques of classes of uncertain time-delay systems. The objective is to build upon the results of [12–15] for continuous-time systems. Specifically, the work reported here extends [12–15] to another dimension by considering the $\mathcal{H}_\infty$-estimation of a class of discrete-time systems with real time-varying norm-bounded parameteric uncertainties and unknown state-delay. On the other hand, our approach in this paper casts the results of [2,3,6] about $\mathcal{H}_\infty$ filtering for delay-free systems into the context of linear uncertian discrete-time systems with unknown-but-bounded state-delay. In addition, it generalizes the techniques of [4,9] by including another source of uncertainties due to bounded state-delays. Towards our objective in this paper, we address the important problem of robust stability of the class of linear uncertian discrete-time systems with unknown-but-bounded state-delay and construct appropriate stability measures. Basically, we provide a version of the discrete Bounded Real Lemma which generalizes the available results in [8]. Then, we design a linear filter which provides both robust stability and a guaranteed $\mathcal{H}_\infty$-performance for the estimation error irrespective of the parameteric uncertainties and unknown delays. Our results come in line with most robust results of time-delay systems that yield only sufficient conditions [17] due to the presence of the delay term as an additional uncertainties.

The main results of this paper are summarized by: Theorem 1 on necessary and sufficient conditions for robust stability and Lemma 1 on sufficient conditions for robust stability with prescribed performance level, Lemma 2 as a version of the discrete Bounded Real Lemma, Theorem 2 which establishes the solvability conditions for a robust $\mathcal{H}_\infty$-estimator in the form of algebraic matrix inequalities (AMIs) and finally Theorem 3 which provides expressions for the filter gain matrices. Several corollaries are given to link our results with those published before.

NOTATIONS AND FACTS *In the sequel, we denote by W^t and W^{-1} the transpose and the inverse of any square matrix W. We use $W > 0$ ($W < 0$) to denote a positive- (negative-) definite matrix W; and I is used to denote the identity matrix of appropriate order. Sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise.*

Fact 1 (Schur Complement) Given constant matrices $\Omega_1, \Omega_2, \Omega_3$ where $\Omega_1 = \Omega_1^t$ and $0 < \Omega_2 = \Omega_2^t$ then $\Omega_1 + \Omega_3^t \Omega_2^{-1} \Omega_3 < 0$ if and only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^t \\ \Omega_3 & -\Omega_2 \end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix} -\Omega_2 & \Omega_3 \\ \Omega_3^t & \Omega_1 \end{bmatrix} < 0$$

Fact 2 (Matrix Inversion Lemma) For any real nonsingular matrices Σ_1, Σ_3 and real matrices Σ_2, Σ_4 with appropriate dimensions, it follows that

$$(\Sigma_1 + \Sigma_2\Sigma_3\Sigma_4)^{-1} = \Sigma_1^{-1} - \Sigma_1^{-1}\Sigma_2\left[\Sigma_3^{-1} + \Sigma_4\Sigma_1^{-1}\Sigma_2\right]^{-1}\Sigma_4\Sigma_1^{-1}$$

Fact 3 Let $\Sigma_1, \Sigma_2, \Sigma_3$ be real constant matrices of compatible dimensions and $H(t)$ be a real matrix function satisfying $H^t(t)H(t) \leq I$. Then $\forall\ \rho > 0$ and any matrix $0 < R = R^t$ such that $\rho\Sigma_2^t\Sigma_2 < R$

$$(\Sigma_3 + \Sigma_1 H(t)\Sigma_2)R^{-1}(\Sigma_3^t + \Sigma_2^t H^t(t)\Sigma_1^t) \leq \rho^{-1}\Sigma_1\Sigma_1^t + \Sigma_3\left[R - \rho\Sigma_2^t\Sigma_2\right]^{-1}\Sigma_3^t$$

2 PROBLEM DESCRIPTION

We consider a class of uncertain time-delay systems represented by:

$$\begin{aligned} (\Sigma_\Delta): \quad x(k+1) &= [A_o + \Delta A(k)]x(k) + B_o w(k) + E_o x(k-\tau) \\ &= A_\Delta(k)x(k) + B_o w(k) + E_o x(k-\tau) \qquad (1) \end{aligned}$$

$$y(k) = [C_o + \Delta C(k)]x(k) + D_o w(k) = C_\Delta(k)x(k) + D_o w(k) \quad (2)$$

$$z(k) = H_o x(k) \quad (3)$$

where $x(k) \in \Re^n$ is the stable, $w(k) \in \Re^m$ is the input noise which belongs to ℓ_2 $[0, \infty)$, $y(k) \in \Re^p$ is the measured output, $z(k) \in \Re^r$, is a linear combination of the state variables to be estimated and the matrices $A_o \in \Re^{n\times n}$, $C_o \in \Re^{p\times n}$, $D_o \in \Re^{p\times m}$, $E_o \in \Re^{n\times n}$ and $H_o \in \Re^{r\times n}$ are real constant matrices representing the nominal plant. Here, τ is an unknown constant scalar representing the amount of delay in the state. For system (Σ_Δ), we have the following assumption:

ASSUMPTION 1 *The matrices $\Delta A(k)$ and $\Delta C(k)$ are represented by:*

$$\begin{bmatrix} \Delta A(k) \\ \Delta C(k) \end{bmatrix} = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \Delta(k) M, \quad \Delta'(k)\Delta(k) \leq I, \ \forall k \geq 0 \quad (4)$$

where $L_1 \in \Re^{n\times\alpha}$, $L_2 \in \Re^{p\times\alpha}$ and $M \in \Re^{\beta\times n}$ are known constant matrices and $\Delta(k) \in \Re^{\alpha\times\beta}$ is unknown matrices. The initial condition is specified as $\alpha_o(.) = \langle x_o, \phi(s)\rangle$, where $\phi(.) \in \ell_2[-\tau, 0]$. In this paper, we examine two problems:

(1) Internal stability of system (Σ_Δ) for all admissible uncertainties satisfying (4),
(2) Estimating the variable $z(k)$ given the measurements $\{y(\beta) : 0 \leq \beta \leq k\}$. Since problem (1) is a prerequisite to problem (2) and stability analysis is a key issue in the subsequent development, we focus our attention initially on a relevant stability measure for system (Σ_Δ) for all admissible uncertainties.

3 ROBUST STABILITY RESULTS

In this section, we study the problem of internal stability of system (Σ_Δ) using a quadratic Lyapunov-Krasovkii function and develop appropriate measures accordingly. In the sequel, we refer to the following systems:

$$(\Sigma_{\Delta o}):\ x(k+1) = A_\Delta(k)x(k) + E_o x(k-\tau) \tag{5}$$

$$\begin{aligned}(\Sigma_{\Delta w}):\ x(k+1) &= A_\Delta(k)x(k) + B_o w(k) + E_o x(k-\tau)\\ z(k) &= H_o x(k)\end{aligned} \tag{6}$$

and develop complete results for internal stability of discrete uncertain systems with unknown state-delay.

Define the scalar valued function $V : \Re^n \to \Re$ as

$$V(x,k) = x^t(k)Px(k) + \sum_{\alpha=k-\tau}^{k-1} x^t(\alpha)Qx(\alpha) \tag{7}$$

where $0 < P = P^t \in \Re^{n\times n}$ and $0 < Q = Q^t \in \Re^{n\times n}$. Observe that $V(x,k) = 0$ for $x \equiv 0$ and $V(x,k) > 0$ when $x \neq 0$. Along the trajectories of system (5), the first-order difference $\Delta V(x,k) := V(x,k+1) - V(x,k)$ is given by:

$$\begin{aligned}\Delta V(x,k) = {} & x^t\{A_\Delta^t(k)PA_\Delta(k) - P + Q\}x\\ & - x^t(k-\tau)[Q - E_o^t PE_o]x(k-\tau)\\ & + x^t(k-\tau)E_o^t PA_\Delta(k)x(k) + x^t A_\Delta^t(k)PE_o x(k-\tau)\end{aligned}$$

By a standard 'completion of the square' argument, we get:

$$\begin{aligned}\Delta V(x,k) < {} & x^t\{A_\Delta^t(k)PA_\Delta(k) - P + Q + A_\Delta^t(k)PE_o[Q - E_o^t PE_o]^{-1}\\ & \times E_o^t PA_\Delta(k)\}x - \{x(k-\tau) - [Q - E_o^t PE_o]^{-1}E_o^t A_\Delta(k)x\}^t\\ & \times [Q - E_o^t PE_o]\{x(k-\tau) - [Q - E_o^t PE_o]^{-1}E_o^t A_\Delta(k)x\}\end{aligned} \tag{8}$$

Inequality (8) can be overbounded in the form:

$$\begin{aligned}\Delta V(x,k) < {} & x^t\{A_\Delta^t(k)PA_\Delta(k) - P + Q\\ & + A_\Delta^t(k)PE_o[Q - E_o^t PE_o]^{-1}E_o^t PA_\Delta(k)\}x\end{aligned} \tag{9}$$

Based on (9), we provide the following definition:

Definition 1 System $(\Sigma_{\Delta o})$ *is robustly stable (RS)* for variable delay τ if given a matrix $0 < Q = Q^t \in \Re^{n\times n}$, there exists a matrix

$0 < P = P^t \in \Re^{n\times n}$ satisfying the algebraic Riccati inequality (ARI):

$$A_\Delta^t(k)PA_\Delta(k) - P + Q + A_\Delta^t(k)PE_o[Q - E_o^tPE_o]^{-1}E_o^tPA_\Delta(k) < 0 \tag{10}$$

for all admissible uncertainties satisfying (4).

Remark 1 It is readily seen using Fact 1 that (10) is equivalent to the algebraic matrix inequality (AMI):

$$\begin{bmatrix} -P+Q & 0 & A_\Delta^t(k) \\ 0 & -Q & E_o^t \\ A_\Delta(k) & E_o & -P^{-1} \end{bmatrix} < 0, \quad \forall\,\Delta{:}||\Delta||^2 \leq 1 \tag{11}$$

Either (10) or (11) can be converted, via Fact 2, into the compact form:

$$A_\Delta^t(k)\{P^{-1} - E_oQ^{-1}E_o^t\}^{-1}A_\Delta(k) - P + Q < 0, \quad \forall\,\Delta{:}||\Delta||^2 \leq 1 \tag{12}$$

From now onwards, we are going to use (10), (11) or (12) interchangeably. A basic result is provided below.

THEOREM 1 *The uncertain discrete delay system* $(\Sigma_{\Delta o})$ *is* robustly *stable (RS) if and only given a matrix* $0 < Q = Q^t \in \Re^{n\times n}$ *and a scalar* $\mu > 0$ *there exist a matrix* $0 < P = P^t \in \Re^{n\times n}$ *satisfying the ARI:*

$$A_o^t\{P^{-1} - E_oQ^{-1}E_o^t - \mu L_1L_1^t\}^{-1}A_o - P + \mu^{-1}M^tM + Q < 0 \tag{13}$$

Proof ($\Rightarrow$:) Suppose that $0 < P = P^t$ satisfies (13) for some $\mu > 0$. For any Δ satisfying $\Delta^t\Delta \leq I$, we have

$$\begin{aligned} &A_\Delta^t(k)PA_\Delta(k) - P + Q + A_\Delta^t(k)PE_o[Q - E_o^tPE_o]^{-1}E_o^tPA_\Delta(k) \\ &= A_\Delta^t(k)\{P^{-1} - E_oQ^{-1}E_o^t\}^{-1}A_\Delta(k) - P + Q \\ &\leq A_o^t\{P^{-1} - E_oQ^{-1}E_o^t - \mu L_1L_1^t\}^{-1}A_o - P + \mu^{-1}M^tM + Q \\ &< 0 \end{aligned}$$

This is implied from (13) and Fact 3.

($\Leftarrow$:) Suppose that $0 < P = P^t$ such that system (Σ_Δ) is RS. It follows that (11) holds for all matrices $\Delta{:}||\Delta||^2 \leq 1$. That is

$$\begin{aligned} &x^t\{A_\Delta^t(k)PA_\Delta(k) - P + Q + A_\Delta^t(k)PE_o[Q - E_o^tPE_o]^{-1}E_o^tPA_\Delta(k)\}x \\ &\quad = x^t\{A_\Delta^t(k)\{P^{-1} - E_oQ^{-1}E_o^t\}^{-1}A_\Delta(k) - P + Q\}x \\ &\quad < 0 \end{aligned}$$

for all $x \neq 0$ and $||\Delta||^2 \leq 1$. By [17], this implies that there exists a scalar $\mu > 0$ such that

$$A_o^t\{P^{-1} - E_oQ^{-1}E_o^t - \mu L_1L_1^t\}^{-1}A_o - P + \mu^{-1}M^tM + Q < 0$$

which completes the proof. □

COROLLARY 1 *It follows from [8] that the existence of a matrix* $0 < P = P^t$ *satisfying (13) is equivalent to the existence of a stabilizing solution* $0 \leq \bar{P} = \bar{P}^t$ *to the algebraic Riccati equation (ARE)*

$$A_o^t\{\bar{P}^{-1} - E_o^tQ^{-1}E_o - \mu LL^t\}^{-1}A_o - \bar{P} + \mu^{-1}M^tM + Q = 0 \qquad (14)$$

Building on Lemma 1, we have the following result.

LEMMA 1 *System* $(\Sigma_{\Delta w})$ *is robustly stable with disturbance attenuation* γ *for variable delay* τ *if given a matrix* $0 < Q = Q^t \in \Re^{n\times n}$ *there exist a matrix* $0 < P = P^t \in \Re^{n\times n}$ *satisfying the ARI:*

$$\begin{aligned} &A_\Delta^t\{P^{-1} - E_oQ^{-1}E_o^t - \gamma^{-2}B_oB_o^t\}^{-1}A_\Delta - P + H_o^tH_o \\ &\quad + Q < 0, \quad \forall\, \Delta : ||\Delta||^2 \leq 1 \end{aligned} \qquad (15)$$

Proof By evaluating the first-order difference $\Delta V(x,k)$ of (7) along the trajectories of (6) and considering the Hamiltonian

$$H(x,w) = \Delta V(x,k) + \{z^t(k)z(k) - \gamma^2 w^t(k)w(k)\}$$

it yields:

$$\begin{aligned} H(x,w) = {} & x^t\{A_\Delta^tPA_\Delta - P + Q\}x + w^tB_o^tPB_ow + x^tH_o^tH_ox - \gamma^2w^tw \\ & + x^tA_\Delta^tPE_ox(k-\tau) + x^t(k-\tau)E_o^tPA_\Delta x \\ & + x^tA_\Delta^tPB_ow + w^tB_o^tPA_\Delta x + w^tB_o^tPE_ox(k-\tau) \\ & + x^t(k-\tau)E_o^tPB_ow - x^t(k-\tau)[Q - E_o^tPE_o]x(k-\tau) \end{aligned} \qquad (16)$$

In terms of $h(k) := [x^t(k)\ w^t(k)\ x^t(k-\tau)]^t$, we express the Hamiltonian in the form:

$$H(x,w) = h^t(k)\Omega h(k)$$

$$\Omega(\sigma,k) = \begin{bmatrix} \Omega_1 & \Omega_2 & \Omega_3 \\ \Omega_2^t & \Omega_4 & \Omega_5 \\ \Omega_3^t & \Omega_5^t & \Omega_6 \end{bmatrix} \tag{17}$$

where

$$\Omega_1 = A_\Delta^t P A_\Delta - P + Q + H_o^t H_o,\ \Omega_3 = A_\Delta^t P E_o,\ \Omega_5 = B_o^t P E_o \tag{18}$$

$$\Omega_2 = A_\Delta^t P B_o,\ \Omega_6 = -[Q - E_o^t P E_o],\ \Omega_4 = -[\gamma^2 I - B_o^t P B_o] \tag{19}$$

For system $(\Sigma_{\Delta w})$ with quadratic Lyapunov function (7), the stability condition with ℓ_2-gain contraint $H(x,w) < 0$ is implied by $\Omega < 0$. By Fact 1 and using (18)–(19), it follows that $\Omega(\sigma,k) < 0$ is equivalent to

$$\begin{bmatrix} -P + Q + H_o^t H_o & 0 & 0 \\ 0 & -\gamma^2 I & 0 \\ 0 & 0 & -Q \end{bmatrix} + \begin{bmatrix} A_\Delta^t \\ B_o^t \\ E_o^t \end{bmatrix} P[A_\Delta\ B_o\ E_o] < 0$$

The above inequality holds if and only if

$$\begin{bmatrix} -P + Q & 0 & 0 & H_o^t \\ 0 & -\gamma^2 I & 0 & 0 \\ 0 & 0 & -Q & 0 \\ H_o & 0 & 0 & -I \end{bmatrix} + \begin{bmatrix} A_\Delta^t \\ B_o^t \\ E_o^t \\ 0 \end{bmatrix} P[A_\Delta\ B_o\ E_o\ 0] < 0 \tag{20}$$

By repeated application Fact 1 to (20), we obtain the ARI (15). □

Remark 2 Applying Fact 1 again to the ARI (15) we obtain the AMI:

$$\begin{bmatrix} -P + Q & 0 & 0 & H_o^t & A_\Delta^t \\ 0 & -\gamma^2 I & 0 & 0 & B_o^t \\ 0 & 0 & -Q & 0 & E_o^t \\ H_o & 0 & 0 & -I & 0 \\ A_\Delta & B_o & E_o & 0 & -P^{-1} \end{bmatrix} < 0 \tag{21}$$

Alternatively, by Fact 3, inequality (15) is equivalent to

$$A_o^t\{P^{-1} - E_o Q^{-1} E_o^t - \gamma^{-2} B_o B_o^t - \mu L_1 L_1^t\}^{-1} A_o \\ - P + H_o^t H_o + \mu^{-1} M^t M + Q < 0 \tag{22}$$

Remark 3 Observe that either (15) or (22) provides a sufficient robust stability condition with a disturbance attenuation γ. It comes in line with most robust stability results of time-delay systems that yield only sufficient conditions [17]. This is due to the presence of the delay term as an additional uncertainties. Therefore while it may be considered conservative, however the inclusion of a scalar parameter μ enables the designer to tune up the stability margin.

In view of Theorem 1 and Remark 2, we have the following result.

LEMMA 2 *For the uncertain time-delay system (1)–(3), the following statements are equivalent:*

(1) *System* $(\Sigma_{\sigma w})$ *is* robustly stable with disturbance attenuation γ
(2) *There exists a matrix* $0 < P = P^t$ *satisfying the ARI (15)*
(3) *There exists a matrix* $0 < P = P^t$ *satisfying the AMI (21)*
(4) *The following* $\mathcal{H}_\infty$ *norm bound is satisfied*

$$\left\| \begin{bmatrix} \mu^{-1/2} M \\ Q^{1/2} \end{bmatrix} [zI - A_o]^{-1} \; [\gamma^{-1} B \; \mu^{1/2} L_1 \; E_o Q^{-1/2}] \right\|_\infty < 1 \tag{23}$$

(5) *There exists a matrix* $0 \leq \bar{P} = \bar{P}^t$ *satisfying the ARE*

$$A_o^t\{\bar{P}^{-1} - E_o^t Q^{-1} E_o - \gamma^{-2} B_o B_o^t - \mu L L^t\}^{-1} A_o \\ - \bar{P} + \mu^{-1} M^t M + H_o^t H_o + Q = 0 \tag{24}$$

Proof (1)⇔(2) follows from Theorem 1. (2)⇔(3) follows by repeated applications of Fact 1 and using Fact 2. (3)⇔(4) follows from the results of [13]. (2)⇔(5) follows from [8] in line of Corollary 1. □

Remark 4 It is significant to observe that Lemma 2 establishes a version of the Bounded real Lemma [8] as applied to uncertain discrete-time systems with state-delay. Additionally, it provides alternative numerical techniques for testing the robust stability of the class of discrete systems under consideration.

4 $\mathcal{H}_\infty$-ESTIMATION RESULTS

The state-estimation problem we study in this paper is a robust $\mathcal{H}_\infty$-estimation problem and can be phrazed as follows:

For system (Σ_Δ), *design a linear estimator of* $z(k)$ *of the form*

$$\begin{aligned}(\Sigma_e)\colon\ \hat{x}(k+1) &= \hat{A}\hat{x}(k) + \hat{K}[y(k) - \hat{C}\hat{x}(k)] \\ &= [A_o + \delta A]\hat{x}(k) + \hat{K}[y(k) - (C_o + \delta C)\hat{x}(k)],\ \hat{x}(0) = 0 \\ \hat{z} &= H_o\hat{x} \end{aligned} \tag{25}$$

such that the estimation error $e(k) := z(k) - \hat{z}(k)$ *is quadratically stable* $\forall\, w(k) \in \ell_2[0,\infty]$ *and* $\|z - \hat{z}\|_2 \le \gamma\|w\|_2$.

In (24), $\delta A \in \Re^{n\times n}, \delta C \in \Re^{p\times n}, \hat{K} \in \Re^{n\times p}$ are the design matrices to be determined.

We now proceed to solve the robust $\mathcal{H}_\infty$-estimation problem. By defining $\tilde{x}(k) = x(k) - \hat{x}(k)$ we get from (1) and (25) the dynamics of the state-error:

$$\begin{aligned}\tilde{x}(k+1) = {}& \{(A_o + \delta A) - \hat{K}(C_o + \delta C)\}\tilde{x}(k) \\ &+ \{\Delta A - \delta A - \hat{K}(\Delta C - \delta C)\}x(k) \\ &+ \{B_o - \hat{K}D_o\}w(k) + E_o x(k-\tau) \end{aligned} \tag{26}$$

Then from system (Σ_Δ) and (26), we obtain the dynamics of the filtering error $e(k)$:

$$\begin{aligned}(\Sigma_{\Delta e})\ \xi(k+1) :={}& \begin{bmatrix} x(k+1) \\ \tilde{x}(k+1) \end{bmatrix} \\ ={}& \{A_a + L_a\Delta_1(t)M_a\}\xi(k) + E_a\xi(k-\tau) \\ &+ B_a w(k), \quad \xi(0) = \xi_o \end{aligned} \tag{27}$$

$$\begin{aligned} e(t) &= H_a\xi(k) \\ &= [0 \quad H_o]\xi(k) \end{aligned} \tag{28}$$

where

$$A_a = \begin{bmatrix} A_o & 0 \\ -\delta A + \hat{K}\delta C & A_o + \delta A - \hat{K}(C_o + \delta C) \end{bmatrix}, \; L_a = \begin{bmatrix} L_1 \\ L_1 - \hat{K}L_2 \end{bmatrix} \tag{29}$$

$$E_a = \begin{bmatrix} E_o & 0 \\ E_o & 0 \end{bmatrix}, \; B_a = \begin{bmatrix} B_o \\ B_o - \hat{K}D_o \end{bmatrix}, \; \xi_o = \begin{bmatrix} \alpha_o \\ \alpha_o \end{bmatrix}, \; M_a = [M \quad 0] \tag{30}$$

THEOREM 2 *Given a prescribed level of noise attenuation* $\gamma > 0$ *and a matrix* $0 < \mathcal{Q} = \mathcal{Q}^t \in \Re^{2n\times 2n}$. *If for some scalar* $\mu > 0$ *there exists a matrix* $0 < \mathcal{P} = \mathcal{P}^t \in \Re^{2n\times 2n}$ *satisfying the ARI:*

$$\begin{aligned} &A_a^t\{\mathcal{P}^{-1} - E_a\mathcal{Q}^{-1}E_a^t - \gamma^{-2}B_aB_a^t - \mu L_aL_a^t\}^{-1} \\ &\quad \times A_a - \mathcal{P} + H_a^tH_a + \mu^{-1}M_a^tM_a + \mathcal{Q} < 0 \end{aligned} \tag{31}$$

then the robust $\mathcal{H}_\infty$*-estimation problem for the system* $(\Sigma_{\Delta e})$ *is solvable with estimator (25) and yields.*

$$||e(k)||_2 < \gamma||w(k)||_2 \tag{32}$$

Proof By Theorem 1 and Remark 2, system $(\Sigma_{\Delta e})$ is QS with disturbance attenuation γ if given a matrix $0 < \mathcal{Q} = \mathcal{Q}^t \in \Re^{2n\times 2n}$ there exists a matrix $0 < \mathcal{P} = \mathcal{P}^t \in \Re^{2n\times 2n}$ satisfying

$$\begin{bmatrix} -\mathcal{P}+\mathcal{Q} & 0 & 0 & H_a^t & A_{\Delta a}^t \\ 0 & -\gamma^2 I & 0 & 0 & B_a^t \\ 0 & 0 & -\mathcal{Q} & 0 & E_a^t \\ H_a & 0 & 0 & -I & 0 \\ A_{\Delta a} & B_a & E_a & 0 & -\mathcal{P}^{-1} \end{bmatrix} < 0 \tag{33}$$

By Fact 1, inequality (33) holds if and only if

$$\begin{bmatrix} -\mathcal{P} + \mathcal{Q} + \mu^{-1} M_a^t M_a & 0 & 0 & H_a^t & A_a^t \\ 0 & -\gamma^2 I & 0 & 0 & B_a^t \\ 0 & 0 & -\mathcal{Q} & 0 & E_a^t \\ H_a & 0 & 0 & -I & 0 \\ A_a & B_a & E_a & 0 & -\mathcal{P}^{-1} \end{bmatrix}$$

$$+ \begin{bmatrix} M_a^t \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Delta^t(k) [0 \quad 0 \quad 0 \quad 0 \quad L_a^t]$$

$$+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ L_a \end{bmatrix} \Delta(k) [M_a \quad 0 \quad 0 \quad 0 \quad 0] < 0 \tag{34}$$

By a well-known result in [11], inequality (34) is equivalent to

$$\begin{bmatrix} -\mathcal{P} + \mathcal{Q} + \mu^{-1} M_a^t M_a & 0 & 0 & H_a^t & A_a^t \\ 0 & -\gamma^2 I & 0 & 0 & B_a^t \\ 0 & 0 & -\mathcal{Q} & 0 & E_a^t \\ H_a & 0 & 0 & -I & 0 \\ A_a & B_a & E_a & 0 & -\mathcal{P}^{-1} \end{bmatrix}$$

$$+ \begin{bmatrix} \mu^{-1/2} M_a^t \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} [\mu^{-1/2} M_a \quad 0 \quad 0 \quad 0 \quad 0]$$

$$+\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \mu^{1/2}L_a \end{bmatrix}[0 \quad 0 \quad 0 \quad 0 \quad \mu^{1/2}L_a^t] < 0 \tag{35}$$

for some $\mu > 0$. Rearranging, we get

$$\begin{bmatrix} -\mathcal{P}+\mathcal{Q}+\mu^{-1}M_a^tM_a & 0 & 0 & H_a^t & A_a^t \\ 0 & -\gamma^2 I & 0 & 0 & B_a^t \\ 0 & 0 & -\mathcal{Q} & 0 & E_a^t \\ H_a & 0 & 0 & -I & 0 \\ A_a & B_a & E_a & 0 & -[\mathcal{P}^{-1}-\mu L_aL_a^t] \end{bmatrix} < 0 \tag{36}$$

Application of Facts 1, 2 to the AMI (36) yields the ARI (31). □

Remark 5 It should be observed that Theorem 2 establishes an LMI-feasibility condition for the robust $\mathcal{H}_\infty$-estimation problem associated with system (Σ_Δ) which requires knowledge about the nominal matrices of the system as well as the structural matrices of the uncertainty. In this way, it provides a partial solution to the $\mathcal{H}_\infty$-estimation under consideration.

To facilitate further development, we introduce

$$\begin{aligned} \mathcal{R}_{11} &= \mathcal{S}_1 M^t[\mu I - M\mathcal{S}_1 M^t]^{-1} M\mathcal{S}_1, \\ \mathcal{R}_{12} &= \mathcal{S}_2 H_o^t[I - H_o\mathcal{S}_2 H_o^t]^{-1} H_o\mathcal{S}_2 \end{aligned} \tag{37}$$

$$\begin{aligned} \mathcal{R}_1 = [\mathcal{R}_{11}+\mathcal{S}_1] + [\mathcal{R}_{11}+\mathcal{S}_1]E_o\{\mathcal{Q}_1 - E_o^t[\mathcal{R}_{11}+\mathcal{S}_1+\mathcal{R}_{12}+\mathcal{S}_2]E_o\}^{-1} \\ \times E_o^t[\mathcal{R}_{11}+\mathcal{S}_1] \end{aligned} \tag{38}$$

$$\begin{aligned} \mathcal{R}_3 = [\mathcal{R}_{12}+\mathcal{S}_2] + [\mathcal{R}_{12}+\mathcal{S}_2]E_o\{\mathcal{Q}_1 - E_o^t[\mathcal{R}_{11}+\mathcal{S}_1+\mathcal{R}_{12}+\mathcal{S}_2]E_o\}^{-1} \\ \times E_o^t[\mathcal{R}_{12}+\mathcal{S}_2] \end{aligned} \tag{39}$$

$$\begin{aligned} \mathcal{R}_2 = -\{[\mathcal{R}_{11}+\mathcal{S}_1] - [\mathcal{R}_{11}+\mathcal{S}_1]E_o\mathcal{Q}_3^{-1}E_o^t[\mathcal{R}_{11}+\mathcal{S}_1]\} \\ \times E_o\mathcal{Q}_1^{-1}E_o^t\mathcal{R}_3 \end{aligned} \tag{40}$$

$$\mathcal{Q}_3 = \mathcal{Q}_1^{-1} - E_o^t[\mathcal{R}_{11} + \mathcal{S}_1]E_o \tag{41}$$

$$\mathcal{G}_1 = \mu L_1^t L_1 + \gamma^{-2} B_o^t B_o + A_o^t \mathcal{R}_2^t A_o,$$
$$\mathcal{G}_2 = \mu L_2 L_1^t + \gamma^{-2} D_o B_o^t + C_o \mathcal{R}_2^t A_o \tag{42}$$

for some matrices $0 < \mathcal{S}_1 = \mathcal{S}_1^t, 0 < \mathcal{S}_2 = \mathcal{S}_2^t, 0 < \mathcal{Q}_1 = \mathcal{S}_1^t$ and $0 < \mathcal{Q}_2 = \mathcal{S}_2^t$. Accordingly we define the matrices:

$$\delta A = \mathcal{G}_1 A_o^{-t}\{\mathcal{R}_1 - \mathcal{R}_2^t\}^{-1}, \quad \delta C = \mathcal{G}_2 A_o^{-t}\{\mathcal{R}_1 - \mathcal{R}_2^t\}^{-1} \tag{43}$$

$$\hat{A} = A_o + \delta A, \ \hat{C} = C_o + \delta C \tag{44}$$

$$\mathcal{T} = \delta C\{\mathcal{R}_3 - \mathcal{R}_2\}\hat{A}^t + C_o \mathcal{R}_3 \hat{A}^t - C_o \mathcal{R}_2^t \delta A^t + \delta C\{\mathcal{R}_1 - \mathcal{R}_2^t\}\delta A^t \tag{45}$$

$$\mathcal{Z} = \hat{C}\mathcal{R}_3\hat{C}^t - \hat{C}\mathcal{R}_2^t \delta\hat{C}^t - \delta C \mathcal{R}_2 \bar{C}^t + \delta C \mathcal{R}_1 \delta C^t \tag{46}$$

It is important to note that the indicated inverses in (37)–(38) exist in view of Fact 2 and the selection of matrices $0 < \mathcal{Q}_1$ and $0 < \mathcal{Q}_2$. Observe in (42) using (37)–(39) that $(\mathcal{R}_1 - \mathcal{R}_2^t) > 0$. The next theorem establishes the main result.

THEOREM 3 *Consider the augmented system* $(\Sigma_{\Delta e})$ *for some* $\gamma > 0$ *and given matrices* $0 < \mathcal{Q}_1 = \mathcal{Q}_1^t \in \Re^{n\times n}$ *and* $0 < \mathcal{Q}_2 = \mathcal{Q}_2^t \in \Re^{n\times n}$. *If for some scalar* $\mu > 0$ *there exist matrices* $0 < \mathcal{S}_1 = \mathcal{S}_1^t \in \Re^{n\times n}$ *and* $0 < \mathcal{S}_2 = \mathcal{S}_2^t \in \Re^{n\times n}$ *satisfying the ARIs*

$$A_o \mathcal{R}_1 A_o^t - \mathcal{S}_1 + \mathcal{Q}_1 + \mu L_1 L_1^t + \gamma^{-2} B_o B_o^t < 0 \tag{47}$$

$$\hat{A}\mathcal{R}_3\hat{A}^t - \mathcal{S}_2 + \delta A \mathcal{R}_1 \delta A^t + \mathcal{Q}_2 - \hat{A}\mathcal{R}_2^t \delta A^t - \delta A \mathcal{R}_2 \hat{A}^t - \mathcal{T}^t \mathcal{Z}^{-1}\mathcal{T} < 0 \tag{48}$$

then the robust $\mathcal{H}_\infty$*-estimation problem for the system* $(\Sigma_{\Delta e})$ *is solvable with the estimator*

$$\hat{x}(k+1) = \hat{A}\hat{x}(k) + \mathcal{T}^t \mathcal{Z}^{-1}[y(k) - \hat{C}\hat{x}(k)] \tag{49}$$

which yields

$$||e(k)||_2 < \gamma||w(k)||_2 \tag{50}$$

Proof Given a matrix of $0 < \mathcal{Q} = \mathcal{Q}^t \in \Re^{2n\times 2n}$ and by Theorem 2, it follows that there exists a matrix $0 < \mathcal{P} = \mathcal{P}^t \in \Re^{2n\times 2n}$ that satisfies the ARI (31). From the results of [6], it follows that (31) holds if and only if there exists a matrix $0 < \mathcal{S} = \mathcal{S}^t \in \Re^{2n\times 2n}$ satisfying

$$\begin{aligned}\Xi(\mathcal{S}) := A_a\{\mathcal{S}^{-1} - E_a\mathcal{Q}^{-1}E_a^t - \mu^{-1}M_a^tM_a - H_a^tH_a\}^{-1}A_a^t \\ - \mathcal{S} + \mu L_aL_a^t + \gamma^{-2}B_aB_a^t + \mathcal{Q} < 0\end{aligned} \tag{51}$$

Define

$$\mathcal{S} = \begin{bmatrix} \mathcal{S}_1 & 0 \\ 0 & \mathcal{S}_2 \end{bmatrix}, \quad \mathcal{Q} = \begin{bmatrix} \mathcal{Q}_1 & 0 \\ 0 & \mathcal{Q}_2 \end{bmatrix} \tag{52}$$

Expansion of (50) using (29)–(30) and (51) yields:

$$\Xi(\mathcal{S}) := \begin{bmatrix} \Xi_1(\mathcal{S}) & \Xi_2(\mathcal{S}) \\ \Xi_2^t(\mathcal{S}) & \Xi_3(\mathcal{S}) \end{bmatrix} \tag{53}$$

where

$$\Xi_1(\mathcal{S}) = A_o\mathcal{R}_1A_o^t - \mathcal{S}_1 + \mathcal{Q}_1 + \mu L_1L_1^t + \gamma^{-2}B_oB_o^t \tag{54}$$

$$\begin{aligned}\Xi_2(\mathcal{S}) = A_o\mathcal{R}_1(-\delta A + \hat{K}\delta C)^t + \mu L_1L_1^t - \mu L_1L_2^t\hat{K}^t + \gamma^{-2} - \gamma^{-2}B_oD_o^t\hat{K}^t \\ + A_o\mathcal{R}_2[A_o + \delta A - \hat{K}(C_o + \delta C)]^t\end{aligned} \tag{55}$$

$$\begin{aligned}\Xi_3(\mathcal{S}) = [A_o + \delta A - \hat{K}(C_o + \delta C)]\mathcal{R}_3[A_o + \delta A - \hat{K}(C_o + \delta C)]^t \\ + [A_o + \delta A - \hat{K}(C_o + \delta C)]\mathcal{R}_2^t(-\delta A + \hat{K}\delta C)^t \\ + (-\delta A + \hat{K}\delta C)\mathcal{R}_1(-\delta A + \hat{K}\delta C)^t - \mathcal{S}_2 + \mathcal{Q}_2 \\ + \mu(L_1 - \hat{K}L_2)(L_1 - \hat{K}L_2)^t \\ + \gamma^{-2}(B_o - \hat{K}D_o)(B_o - \hat{K}D_o)^t\end{aligned} \tag{56}$$

For internal stability with ℓ_2-bound, it is required that $\Xi(\mathcal{S}) < 0$. Necessary and sufficient conditions to achieve this are

$$\Xi_1(\mathcal{S}) < 0, \quad \Xi_2(\mathcal{S}) = 0, \quad \Xi_3(\mathcal{S}) < 0 \tag{57}$$

It is readily seen from (53) that the condition $\Xi_1(\mathcal{S}) < 0$ is equivalent to the ARI (46). Using (42) in (54) and arranging terms, we reach that the condition $\Xi_2(\mathcal{S}) < 0$ is satisfied. Finally, from (43)–(45) and using the 'completion of squares' argument with some standard algebraic manipulations we therefore conclude that the Kalman gain is given by $\hat{K} = \mathcal{T}^t \mathcal{Z}^{-1}$ the ARI (47) corresponds to $\Xi_3(\mathcal{S}) < 0$. □

Two important special cases follow

COROLLARY 2 *Consider the uncertain discrete system without delay*

$$\begin{aligned} x(k+1) &= [A_o + \Delta A(k)]x(k) + B_o w(k) \\ &= A_\Delta(k)x(k) + B_o w(k) \end{aligned} \tag{58}$$

$$\begin{aligned} y(k) &= [C_o + \Delta C(k)]x(k) + D_o w(k) \\ &= C_\Delta(k)x(k) + D_o w(k) \end{aligned} \tag{59}$$

$$z(k) = H_o x(k) \tag{60}$$

If for some $\gamma > 0$ and a scalar $\mu > 0$ there exist matrices $0 < \mathcal{S}_1 = \mathcal{S}_1^t \in \Re^{n \times n}$ and $0 < \mathcal{S}_2 = \mathcal{S}_2^t \in \Re^{n \times n}$ satisfying the ARIs

$$A_o[\mathcal{R}_{11} + \mathcal{S}_1]A_o^t - \mathcal{S}_1 + \mathcal{Q}_1 + \mu L_1 L_1^t + \gamma^{-2} B_o B_o^t < 0 \tag{61}$$

$$\hat{A}[\mathcal{R}_{12} + \mathcal{S}_2]\hat{A}^t + \delta A[\mathcal{R}_{11} + \mathcal{S}_1]\delta A^t - \mathcal{S}_2 - \mathcal{T}_1^t \mathcal{Z}_1^{-1} \mathcal{T}_1 < 0 \tag{62}$$

then the robust $\mathcal{H}_\infty$-estimation problem is solvable with the estimator

$$\hat{x}(k+1) = \hat{A}\hat{x}(k) + \mathcal{T}_1^t \mathcal{Z}_1^{-1}[y(k) - \hat{C}\hat{x}(k)] \tag{63}$$

which yields

$$||e(k)||_2 < \gamma ||w(k)||_2 \tag{64}$$

and where

$$\mathcal{G}_3 = \mu L_1^t L_1 + \gamma^{-2} B_o^t B_o, \quad \mathcal{G}_4 = \mu L_2^t L_1 + \gamma^{-2} D_o B_o^t$$
$$\delta A = \mathcal{G}_3 A_o^{-t} \mathcal{R}_4^{-1}, \quad \delta C = \mathcal{G}_4 A_o^{-t} \mathcal{R}_4^{-1}$$
$$\hat{A} = A_o + \delta A, \quad \hat{C} = C_o + \delta C$$
$$\mathcal{T}_1 = \hat{C}\mathcal{R}_5 \hat{A}^t + \delta C \mathcal{R}_4 \hat{A}^t, \qquad \mathcal{Z}_1 = \hat{C}\mathcal{R}_3 \hat{C}^t + \delta C \mathcal{R}_4 \delta C^t$$

Proof Follows from Theorem 3 by setting $E_o \equiv 0$ and $\mathcal{Q}_1 = \mathcal{Q}_2 = 0$ in (37)–(45) and observing that $\mathcal{R}_2 \equiv 0$. □

COROLLARY 3 *Consider the system*

$$x(k+1) = A_o x(k) + B_o w(k) + E_o x(k-\tau) \tag{65}$$

$$y(k) = C_o x(k) + D_o w(k) \tag{66}$$

$$z(k) = H_o x(k) \tag{67}$$

for some $\gamma > 0$ *and given matrices* $0 < \mathcal{Q}_1 = \mathcal{Q}_1^t \in \Re^{2n \times 2n}$ *and* $0 < \mathcal{Q}_2 = \mathcal{Q}_2^t \in \Re^{2n \times 2n}$. *If there exist matrices* $0 < \mathcal{S}_1 = \mathcal{S}_1^t \in \Re^{n \times n}$ *and* $0 < \mathcal{S}_2 = \mathcal{S}_2^t \in \Re^{n \times n}$ *satisfying the ARIs*

$$A_o \mathcal{R}_4 A_o^t - \mathcal{S}_1 + \mathcal{Q}_1 + \gamma^{-2} B_o B_o^t < 0 \tag{68}$$

$$\hat{A}\mathcal{R}_3 \hat{A}^t + \delta A \mathcal{R}_4 \delta A^t - \mathcal{S}_2 + \mathcal{Q}_2 - \hat{A}\mathcal{R}_5^t \delta A^t - \delta A \mathcal{R}_5 \hat{A} - \mathcal{T}_2^t \mathcal{Z}_2^{-1} \mathcal{T}_2 < 0 \tag{69}$$

then the robust $\mathcal{H}_\infty$*-estimation problem is solvable with the estimator*

$$\hat{x}(k+1) = \hat{A}\hat{x}(k) + \mathcal{T}_2^t \mathcal{Z}_2^{-1}[y(k) - \hat{C}\hat{x}(k)] \tag{70}$$

which yields

$$||e(k)||_2 < \gamma ||w(k)||_2 \tag{71}$$

and where

$$\mathcal{R}_4 = \mathcal{S}_1 - \mathcal{S}_1 E_o \{\mathcal{Q}_1 - E_o^t [\mathcal{S}_1 + \mathcal{R}_{12} + \mathcal{S}_2] E_o\}^{-1} E_o^t \mathcal{S}_1$$
$$\mathcal{R}_5 = -\{\mathcal{S}_1 - \mathcal{S}_1 E_o \mathcal{Q}_3^{-1} E_o^t \mathcal{S}_1\} E_o \mathcal{Q}_1^{-1} E_o^t \mathcal{R}_3$$

$$
\begin{aligned}
\mathcal{G}_5 &= \gamma^{-2}B_o^t B_o + A_o^t \mathcal{R}_5^t A_o \\
\mathcal{G}_6 &= \gamma^{-2}D_o B_o^t + C_o \mathcal{R}_5^t A_o \\
\delta A &= \mathcal{G}_5 A_o^{-t}\{\mathcal{R}_3 - \mathcal{R}_5^t\}^{-1}, \quad \delta C = \mathcal{G}_6 A_o^{-t}\{\mathcal{R}_3 - \mathcal{R}_5^t\}^{-1} \\
\hat{A} &= A_o + \delta A, \quad \hat{C} = C_o + \delta c \\
\mathcal{T}_2 &= \delta C\{\mathcal{R}_4 - \mathcal{R}_5\}\hat{A}^t + C_o \mathcal{R}_4 \hat{A}^t - C_o \mathcal{R}_5^t \delta A^t + \delta C\{\mathcal{R}_3 - \mathcal{R}_5^t\}\delta A^t \\
\mathcal{Z}_2 &= \hat{C}\mathcal{R}_4\hat{C}^t - \hat{C}\mathcal{R}_5^t\delta\hat{C}^t - \delta C\mathcal{R}_5\bar{C}^t + \delta C\mathcal{R}_3\delta C^t
\end{aligned}
$$

Remark 6 It is interesting to observe that Corollary 2 gives an AMI-based version of the results in [4]. Corollary 3 presents an $\mathcal{H}_\infty$ filter for a class of discrete-time systems with unknown state-delay. Both Corollary 2 and Corollary 3 are new results for state estimation of time-delay systems.

5 EXAMPLE

Consider a discrete-time system of the type (1)–(3) with

$$
A_o = \begin{bmatrix} 0.67 & 0.087 \\ 0 & 1.105 \end{bmatrix}, \quad E_o = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad B_o = \begin{bmatrix} 0.096 \\ 0.316 \end{bmatrix}
$$

$$
C_o = [0.5 \quad 0.5], \quad H_o = [1 \quad 2], \quad D_o = 1, \quad L_1 = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \quad L_2 = 0.4, \quad M = [0.2 \quad 0.3]
$$

We select $\mathcal{Q}_1$ and $\mathcal{Q}_2$ as

$$
\mathcal{Q}_1 = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}, \quad \mathcal{Q}_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}
$$

and solve the ARIs (47)–(48) using a sequential computational scheme. This scheme is initialized by dropping out the nonlinear terms in (47)–(48) and solving the resulting linear inequalities to yield an initial feasible solution. Then by continuously injecting the solutions into the actual inequalities, it has been found that a satisfactory feasible solution can be obtained after few iterations. In one case with an accuary of 10^{-5}, the result of computations are:

$$\mathcal{S}_1 = \begin{bmatrix} 9.0601 & -0.4380 \\ -0.4380 & 13.0666 \end{bmatrix}, \quad \mathcal{S}_2 = \begin{bmatrix} 140.5205 & 10.6250 \\ 10.6250 & 125.4550 \end{bmatrix}$$

for $\mu = 0.25$ where the associated matrices are given by:

$$\mathcal{R}_1 = \begin{bmatrix} 1.1417 & -1.0963 \\ -1.0963 & 1.1733 \end{bmatrix}, \quad \mathcal{R}_2 = \begin{bmatrix} 1.3145 & -0.4566 \\ -0.4566 & 0.9339 \end{bmatrix},$$

$$\mathcal{R}_3 = \begin{bmatrix} 1.2225 & -1.0433 \\ -1.0433 & 0.7652 \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} 0.3237 & 0.6978 \\ -1.3861 & -0.2980 \end{bmatrix},$$

$$\mathcal{T}^t = \begin{bmatrix} -9.6089 \\ -8.7517 \end{bmatrix}, \quad \hat{C}^t = \begin{bmatrix} -0.2041 \\ -0.1840 \end{bmatrix}, \quad \mathcal{Z} = -25.5851$$

Hence, from (49) the $\mathcal{H}_\infty$ estimator is described by:

$$\hat{x}(k+1) = \begin{bmatrix} 0.3237 & 0.6978 \\ -1.3861 & -0.2980 \end{bmatrix} \hat{x}(k) + \begin{bmatrix} 0.3756 \\ 0.3421 \end{bmatrix} [y(k) - [-0.2041 - 0.1840]\hat{x}(k)]$$

6 CONCLUSIONS

For a class of discrete-time systems with real time-varying norm-bounded parameteric uncertainties and unknown state-delay, this paper has

(1) developed complete results for robust stability with prescribed performance measure
(2) established a version of the discrete Bounded Real Lemma
(3) designed a linear state-estimator which provides robust stability with a guaranteed $\mathcal{H}_\infty$-performance for the estimation error irrespective of the parameteric uncertainties and unknown state-delays.

A numerical example has been worked out to illustrate the developed theory.

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