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LECTURE HALL PARTITIONS AND THE WREATH PRODUCTS $C_k \wr S_n$

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Abstract

It is shown that statistics on the wreath product groups, $C_k \wr S_n$, can be interpreted in terms of natural statistics on lecture hall partitions. Lecture hall theory is applied to prove distribution results for statistics on $C_k \wr S_n$. Finally, some new statistics on $C_k \wr S_n$ are introduced, inspired by lecture hall theory, and their distributions are derived.

1. Introduction

The purpose of this note is to show that statistics on the wreath product $C_k \wr S_n$ of a cyclic group C_k, of order k, and the symmetric group S_n, can be interpreted in terms of natural statistics on lecture hall partitions. We demonstrate that lecture hall theory can be used to prove results about the distribution of statistics on $C_k \wr S_n$. We introduce some new statistics on $C_k \wr S_n$, inspired by lecture hall partitions, including a quadratic version of "flag-major index", and prove distribution results for these statistics.

The paper is organized as follows. In Section 2, we define the **s**-lecture hall partitions and state a few useful results. Section 3 is devoted to statistics of interest on the wreath product groups and a very brief discussion of what is known. Section 4 introduces **s**-inversion sequences, which will be used to relate statistics on $C_k \wr S_n$ to statistics on lecture hall partitions.

Section 5 describes a bijection between $(k, 2k, \ldots, nk)$-inversion sequences and

$C_k \wr S_n$ that allows statistics to be translated from one domain to another.

Section 6 reviews recent work of Savage-Schuster [13] relating inversion sequences to lecture hall partitions. This work was developed with the intention of extending work on permutation statistics to a more general setting.

Section 7 is the heart of the paper. We prove there a theorem which allows us to apply the tools of Section 6 to $C_k \wr S_n$. This contains our main results relating statistics such as descent, flag-major index and flag-inversion number to statistics on lecture hall partitions, also proving an Euler-Mahonian distribution result.

In Section 8 we define a new statistic "lhall" on $C_k \wr S_n$ and derive its surprisingly nice distribution.

In Section 9, we are led to define a distorted version of the descent statistic on $C_k \wr S_n$, that reveals an even closer connection to lecture hall partitions.

A few words about notation: $\mathbb{Z}$ is the set of integers, $\mathbb{R}$ the set of real numbers, S_n the set of permutations of n elements; $[\,j\,] = \{1, 2, \ldots, j\}$, where $[\,0\,] = \emptyset$; $[\,n\,]_q = (1-q^n)/(1-q)$; and for $x = (x_1, x_2, \ldots, x_n)$, $|x| = x_1 + x_2 + \cdots + x_n$.

2. Lecture Hall Partitions

For a sequence $\mathbf{s} = \{s_i\}_{i \geq 1}$ of positive integers, the $\mathbf{s}$-*lecture hall partitions* are the elements of the set

$$\mathbf{L}_n^{(\mathbf{s})} \;=\; \left\{ \lambda \in \mathbb{Z}^n \;\middle|\; 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \cdots \leq \frac{\lambda_n}{s_n} \right\}.$$

For example, $(0,1,3,4) \in \mathbf{L}_n^{(1,2,3,4)}$, but $(0,1,3,4) \notin \mathbf{L}_n^{(1,3,5,7)}$, since $3/5 > 4/7$.

The original lecture hall partitions $\mathbf{L}_n = \mathbf{L}_n^{(1,2,\ldots,n)}$ were introduced by Bousquet-Mélou and Eriksson in [3], where they showed that

$$\sum_{\lambda \in \mathbf{L}_n} y^{|\lambda|} = \prod_{i=1}^{n} \frac{1}{1-y^{2i-1}}. \tag{1}$$

In [4] they proved the following refinement, which will be useful in the present work.

Theorem 1. The Refined Lecture Hall Theorem [4]: *For any nonnegative integer n,*

$$\sum_{\lambda \in \mathbf{L}_n} q^{|\lceil \lambda \rceil|} y^{|\lambda|} \;=\; \prod_{i=1}^{n} \frac{1+qy^i}{1-q^2y^{n+i}}, \tag{2}$$

where $\lceil \lambda \rceil = (\lceil \lambda_1/1 \rceil\,,\ \lceil \lambda_2/2 \rceil\,,\ \ldots,\ \lceil \lambda_n/n \rceil)$.

If the largest part in a lecture hall partition in $\mathbf{L}_n$ is constrained, we have the following.

Theorem 2. [8, 13] *For integers $n \geq 1$ and $t \geq 0$,*

$$\sum_{\lambda \in \mathbf{L}_n;\ \lambda_n \leq tn} q^{|\lceil \lambda \rceil|} \ = \ [\, t+1\,]_q^n \, . \tag{3}$$

For example, when $n = 3$ and $t = 1$, the set $\{\lambda \in \mathbf{L}_3 \mid \lambda_3 \leq 3\}$ has the eight elements:

$$\{(0,0,0),(0,0,1),(0,0,2),(0,0,3),(0,1,2),(0,1,3),(0,2,3),(1,2,3)\}$$

and

$$\sum_{\lambda \in \mathbf{L}_3;\ \lambda_3 \leq 3} q^{\lceil \lambda_1/1 \rceil + \lceil \lambda_2/2 \rceil + \lceil \lambda_3/3 \rceil} = 1 + 3q + 3q^2 + q^3 = [\,2\,]_q^3 \, .$$

3. Statistics on $C_k \wr S_n$

An element $\pi \in S_n$ is a bijection $\pi : [\,n\,] \to [\,n\,]$ and we write $\pi = (\pi_1, \ldots, \pi_n)$, to mean that $\pi(i) = \pi_i$. A *descent* in $\pi \in S_n$ is a position $i \in [\,n-1\,]$ such that $\pi_i > \pi_{i+1}$. The set of all descents of π is $\operatorname{Des} \pi$ and $\operatorname{des} \pi = |\operatorname{Des} \pi|$. The *inversion* number of π is

$$\operatorname{inv} \pi \ = |\{(i,j) \mid 1 \leq i < j \leq n \text{ and } \pi_i > \pi_j\}|.$$

For example, if $\pi = (5,4,1,3,2)$, then $\operatorname{Des} \pi = \{1,2,4\}$, $\operatorname{des} \pi = 3$ and $\operatorname{inv} \pi = 8$.

For positive integers k and n, we view $C_k \wr S_n$ combinatorially as a set of pairs (π, σ):

$$C_k \wr S_n = \{(\pi, \sigma) \mid \pi \in S_n,\ \sigma \in \{0, 1, \ldots, k-1\}^n\}.$$

We use the notation π^σ to denote (π, σ) and write

$$\pi^\sigma \ = \ (\pi_1^{\sigma_1}, \pi_2^{\sigma_2}, \ldots, \pi_n^{\sigma_n}) \ = \ ((\pi_1, \ldots, \pi_n), (\sigma_1, \ldots, \sigma_n)) \ = \ (\pi, \sigma).$$

Statistics on $C_k \wr S_n$ (or k-colored permutations or k-indexed permutations) have been studied by many, starting with Reiner's work on signed permutations [12], followed by independent work of Brenti [5] and Steingrímsson [14] on the more general wreath products. Pairs of "(descent, major index)" statistics have been found, satisfying relations like Carlitz's q-Eulerian polynomials, starting with work of Adin, Brenti, and Roichman [1]. There have very recently been many new and exciting discoveries, including [7, 10, 9, 2]. It is remarkable the many variations in the definitions of the statistics, even when they give the same distribution.

We start with a fairly standard definition of *descent*. The *descent set* of $\pi^\sigma \in C_k \wr S_n$ is

$$\operatorname{Des} \pi^\sigma \ = \ \{i \in \{0, 1, \ldots, n-1\} \mid \sigma_i < \sigma_{i+1}, \text{ or } \sigma_i = \sigma_{i+1} \text{ and } \pi_i > \pi_{i+1}\}, \tag{4}$$

with the convention that $\pi_0 = \sigma_0 = 0$.

We will consider the following statistics defined on $C_k \wr S_n$.

$$\begin{aligned}
\operatorname{des} \pi^\sigma &= |\operatorname{Des} \pi^\sigma| \\
\operatorname{comaj} \pi^\sigma &= \sum_{i \in \operatorname{Des} \pi^\sigma} (n-i) \\
\operatorname{fmaj} \pi^\sigma &= k \operatorname{comaj} \pi^\sigma - \sum_{i=1}^{n} \sigma_i \\
\operatorname{finv} \pi^\sigma &= \operatorname{inv} \pi + \sum_{i=1}^{n} i\sigma_i.
\end{aligned}$$

As an example, for $\pi^\sigma = (5^1, 4^1, 1^0, 3^0, 2^2) \in C_3 \wr S_5$, we have $\operatorname{Des} \pi^\sigma = \{0, 1, 4\}$; $\operatorname{des} \pi^\sigma = 3$; $\operatorname{comaj} \pi^\sigma = 10$; $\operatorname{fmaj} \pi^\sigma = 26$; and $\operatorname{finv} \pi^\sigma = 21$. Note that this definition of fmaj differs a bit from those appearing elsewhere, even among those who define the descent set as in (4) ([1, 7]).

Using lecture hall theory, we will show, among other things:

$$\sum_{\pi^\sigma \in C_k \wr S_n} q^{\operatorname{fmaj} \pi^\sigma} = \sum_{\pi^\sigma \in C_k \wr S_n} q^{\operatorname{finv} \pi^\sigma}, \tag{5}$$

$$\sum_{t \geq 0} [\,kt+1\,]_q^n \, x^t = \frac{\sum_{\pi^\sigma \in C_k \wr S_n} q^{\operatorname{fmaj} \pi^\sigma} x^{\operatorname{des} \pi^\sigma}}{\prod_{i=0}^{n} (1 - xq^{ki})}, \tag{6}$$

$$\sum_{\lambda \in \mathbf{L}_n} q^{|\lceil \lambda \rceil|} x^{\lceil \lambda_n / (kn) \rceil} = \frac{\sum_{\pi^\sigma \in C_k \wr S_n} q^{\operatorname{fmaj} \pi^\sigma} x^{\operatorname{des} \pi^\sigma}}{\prod_{i=1}^{n} (1 - xq^{ki})}. \tag{7}$$

Relations of the form (6), for general k, have been found only recently, starting with Chow and Mansour [7] and Hyatt [10], sometimes with slightly different definitions of Des or fmaj. Our intention here is to highlight our methods, which are quite novel, and which allow us to prove new results like (7).

4. Statistics on s-Inversion Sequences

The connection between statistics on $C_k \wr S_n$ and statistics on lecture hall partitions will be made via statistics on *inversion sequences.*

Given a sequence $\mathbf{s} = \{s_i\}_{i \geq 1}$ of positive integers, and positive integer n, the set $\mathbf{I}_n^{(\mathbf{s})}$ of $\mathbf{s}$-*inversion sequences* is defined by

$$\mathbf{I}_n^{(\mathbf{s})} = \{(e_1, \ldots, e_n) \in \mathbb{Z}^n \mid 0 \leq e_i < s_i \text{ for } 1 \leq i \leq n\}.$$

The familiar "inversion sequences" associated with permutations are the elements of $\mathbf{I}_n^{(\mathbf{s})}$ for $\mathbf{s} = (1, 2, \ldots, n)$.

The *ascent set* of an inversion sequence $e \in \mathbf{I}_n^{(\mathbf{s})}$ is the set

$$\mathrm{Asc}\, e \;=\; \left\{ i \in \{0, 1, \ldots, n-1\} \;\middle|\; \frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}} \right\},$$

with the convention that $e_0 = 0$. For example, as an element of $\mathbf{I}_5^{(3,6,9,12,15)}$, the inversion sequence $e = (1, 3, 2, 2, 13)$ has the ascent set $\mathrm{Asc}\, e = \{0, 1, 4\}$.

The following statistics on $\mathbf{I}_n^{(\mathbf{s})}$ were defined in [13]:

$$\begin{aligned}
\mathrm{asc}\, e &= \big|\mathrm{Asc}\, e\big|, \\
\mathrm{amaj}\, e &= \sum_{i \in \mathrm{Asc}\, e} (n - i), \\
|e| &= \sum_{i=1}^{n} e_i, \\
\mathrm{lhp}\, e &= -|e| + \sum_{i \in \mathrm{Asc}\, e} (s_{i+1} + \ldots + s_n)\,.
\end{aligned}$$

For $e = (1, 3, 2, 2, 13) \in \mathbf{I}_5^{(3,6,9,12,15)}$, we have $\mathrm{asc}\, e = 3$; $\mathrm{amaj}\, e = 10$; $|e| = 21$; and $\mathrm{lhp}\, e = 81$.

In this paper, our focus is the sequence $\mathbf{s} = (k, 2k, \ldots, nk)$, where k is a positive integer. Let $\mathbf{I}_{n,k} = \mathbf{I}_n^{(k,2k,\ldots,nk)}$. We will require two new statistics on $\mathbf{I}_{n,k}$:

$$\begin{aligned}
\mathbf{N}(e) &= \sum_{j=1}^{n} \left\lfloor \frac{e_j}{j} \right\rfloor; \\
\mathrm{lfmaj}\, e &= k\, \mathrm{amaj}\, e - \mathbf{N}(e).
\end{aligned}$$

For $e = (1, 3, 2, 2, 13) \in \mathbf{I}_5^{(3,6,9,12,15)}$, $\mathbf{N}(e) = 4$ and $\mathrm{lfmaj}\, e = 26$.

5. From Statistics on $C_k \wr S_n$ to Statistics on $\mathbf{I}_{n,k}$

We will make use of the following bijection between S_n and $\mathbf{I}_{n,1}$ which was proved in [13] to have the required properties.

Lemma 1. *For positive integer n, the mapping $\phi : S_n \to \mathbf{I}_{n,1}$ defined by $\phi(\pi) = t = (t_1, t_2, \ldots, t_n)$, where*

$$t_i \;=\; |\{j \in [i-1] \mid \pi_j > \pi_i\}|$$

is a bijection satisfying both $\mathrm{Des}\, \pi \;=\; \mathrm{Asc}\, t$ *and* $\mathrm{inv}\, \pi \;=\; |t|$.

For example, if $\pi = (5,4,1,3,2)$ then $t = \phi(\pi) = (0,1,2,2,3) \in \mathbf{I}_{5,1}$. Checking the statistics, $\operatorname{Des}\pi = \{1,2,4\} = \operatorname{Asc} t$ and $\operatorname{inv}\pi = 8 = |t|$.

Noting that, as sets, $\mathbf{I}_{n,k}$ and $C_k \wr S_n$ have the same cardinality, we set up a bijection which translates statistics from one domain to the other in a useful way.

Theorem 3. *For each pair of integers (n,k) with $n \geq 1$, $k \geq 1$, there is a bijection*

$$\Theta: \quad C_k \wr S_n \longrightarrow \mathbf{I}_{n,k}$$

with the following properties. If $\Theta(\pi^\sigma) = e = (e_1,\ldots,e_n)$ then

$$\operatorname{Asc} e = \operatorname{Des}\pi^\sigma \tag{8}$$

$$\mathbf{N}(e) = \sum_{i=1}^{n} \sigma_i \tag{9}$$

$$\operatorname{lfmaj} e = \operatorname{fmaj}\pi^\sigma \tag{10}$$

$$e_n = n(\sigma_n + 1) - \pi_n \tag{11}$$

$$|e| = \operatorname{inv}\pi + \sum_{i=1}^{n} i\sigma_i = \operatorname{finv}\pi^\sigma. \tag{12}$$

Proof. Define Θ by

$$e = \Theta(\pi_1^{\sigma_1}, \pi_2^{\sigma_2}, \ldots, \pi_n^{\sigma_n}) = (\sigma_1 + t_1,\ 2\sigma_2 + t_2,\ \ldots,\ n\sigma_n + t_n),$$

where $(t_1, t_2, \ldots, t_n) = \phi(\pi)$, as in Lemma 1.

For example, for $\pi^\sigma = (5^1, 4^1, 1^0, 3^0, 2^2) \in C_3 \wr S_5$, $t = \phi(5,4,1,3,2) = (0,1,2,2,3)$, so we get $e = \Theta(\pi^\sigma) = (1,3,2,2,13)$. Note that properties (8) through (12) hold for this example:

$$\begin{aligned}
\operatorname{Asc} e &= \{0,1,4\} = \operatorname{Des}\pi^\sigma \\
N(e) &= 4 = 1+1+0+0+4 = |\sigma| \\
\operatorname{lfmaj} e &= 26 = \operatorname{fmaj}\pi^\sigma \\
e_5 &= 13 = 5(\sigma_5 + 1) - \pi_5 \\
|e| &= 21 = \operatorname{finv}\pi^\sigma.
\end{aligned}$$

Clearly, $\Theta(\pi^\sigma) \in \mathbf{I}_{n,k}$. Since $C_k \wr S_n$ and $\mathbf{I}_{n,k}$ have the same cardinality, to show that Θ is a bijection, it suffices to show that Θ is onto. Let $e = (e_1,\ldots,e_n) \in \mathbf{I}_{n,k}$. Define $\sigma = (\sigma_1, \ldots \sigma_n)$ by $\sigma_i = \lfloor e_i/i \rfloor$. Then $\sigma \in \{0,1,\ldots,k-1\}^n$. Define $t = (t_1, \ldots t_n)$ by $t_i = e_i - i\sigma_i$. Then $t \in \mathbf{I}_{n,1}$. Finally, let $\pi = \phi^{-1}(t) \in S_n$. Then $\pi^\sigma \in C_k \wr S_n$ and $\Theta^{-1}(e) = \pi^\sigma$.

To prove properties (8) through (12), observe first that $t_n = n - \pi_n$, so property (11) holds. It is clear from the definition of Θ that (12) is true. Also, note that $\lfloor e_i/i \rfloor = \sigma_i$ since $0 \leq t_i < i$ and property (9) holds. So property (10) will follow once we prove (8). By Lemma 1, since $t = \phi(\pi)$, we know that $\operatorname{Asc} t = \operatorname{Des} \pi$, so it remains to show $\operatorname{Asc} e = \operatorname{Des} \pi^\sigma$.

Note first that $e_1 = \sigma_1 + t_1 = \sigma_1$, since $t_1 = 0$. So,

$$0 \in \operatorname{Des} \pi^\sigma \iff \sigma_1 > 0 \iff e_1 > 0 \iff 0 \in \operatorname{Asc} e.$$

For $1 \le i \le n$, $i \in \operatorname{Asc} e$ if and only if

$$\begin{aligned} 0 \; < \; \frac{e_{i+1}}{k(i+1)} - \frac{e_i}{ki} &= \frac{(i+1)\sigma_{i+1} + t_{i+1}}{k(i+1)} - \frac{i\sigma_i + t_i}{ki} \\ &= \frac{i(i+1)(\sigma_{i+1} - \sigma_i) + it_{i+1} - (i+1)t_i}{ki(i+1)} \\ &= \frac{\Delta_i}{ki(i+1)}, \end{aligned}$$

where

$$\Delta_i = i(i+1)(\sigma_{i+1} - \sigma_i) + it_{i+1} - (i+1)t_i.$$

So, $i \in \operatorname{Asc} e$ if and only if $\Delta_i > 0$.

If $\sigma_i = \sigma_{i+1}$ then

$$\Delta_i > 0 \iff it_{i+1} - (i+1)t_i > 0 \iff i \in \operatorname{Asc} t \iff i \in \operatorname{Des} \pi \iff i \in \operatorname{Des} \pi^\sigma.$$

For the remaining cases, note that since $0 \le t_{i+1} \le i$ and $0 \le t_i \le i - 1$,

$$i(i+1)(\sigma_{i+1} - \sigma_i) - i^2 + 1 \;\le\; \Delta_i \le i(i+1)(\sigma_{i+1} - \sigma_i) + i^2.$$

If $\sigma_i \neq \sigma_{i+1}$, then $i \in \operatorname{Des} \pi^\sigma$ if and only if $\sigma_i < \sigma_{i+1}$. But if $\sigma_i < \sigma_{i+1}$, then

$$\Delta_i \;\ge\; i(i+1) - i^2 + 1 \; = i + 1 > 0,$$

so $i \in \operatorname{Asc} e$. And if $\sigma_i > \sigma_{i+1}$ then

$$\Delta_i \;\le\; -i(i+1) + i^2 \; = -i \le 0$$

and $i \notin \operatorname{Asc} e$. This completes the proof. □

6. Lecture Hall Polytopes and s-Inversion Sequences

The **s**-*lecture hall polytope* was introduced in [13], for an arbitrary sequence $\mathbf{s} = \{s_i\}_{i \ge 1}$ of positive integers, as

$$\mathbf{P}_n^{(\mathbf{s})} = \left\{ \lambda \in \mathbb{R}^n \;\middle|\; 0 \le \frac{\lambda_1}{s_1} \le \frac{\lambda_2}{s_2} \le \cdots \le \frac{\lambda_n}{s_n} \le 1 \right\}.$$

$\mathbf{P}_n^{(\mathbf{s})}$ is a convex, simplicial polytope with the $n+1$ vertices:

$$(0,0,\ldots,0),(s_1,s_2,\ldots,s_n),(0,s_2,\ldots,s_n),(0,0,s_3,\ldots,s_n),\ldots,(0,0,\ldots,0,s_n),$$

all with integer coordinates. The t-th *dilation* of $\mathbf{P}_n^{(\mathbf{s})}$ is the polytope

$$t\mathbf{P}_n^{(\mathbf{s})} \;=\; \left\{\lambda \in \mathbb{R}^n \;\middle|\; 0 \le \frac{\lambda_1}{s_1} \le \frac{\lambda_2}{s_2} \le \cdots \le \frac{\lambda_n}{s_n} \le t\right\}.$$

A multivariate function, $\mathbf{f}_n^{(\mathbf{s})}(t;q,y,z)$, was used in [13] to enumerate lattice points in $t\mathbf{P}_n^{(\mathbf{s})}$ according to statistics significant in the theory of lecture hall partitions:

$$\mathbf{f}_n^{(\mathbf{s})}(t;q,y,z) \;=\; \sum_{\lambda \in t\mathbf{P}_n^{(\mathbf{s})} \cap \mathbb{Z}^n} q^{|\lceil \lambda \rceil_{\mathbf{s}}|} y^{|\lambda|} z^{|\,\epsilon^+(\lambda)\,|},$$

where

$$\lceil \lambda \rceil_{\mathbf{s}} \;=\; \left(\left\lceil \frac{\lambda_1}{s_1}\right\rceil, \left\lceil \frac{\lambda_2}{s_2}\right\rceil, \ldots, \left\lceil \frac{\lambda_n}{s_n}\right\rceil\right), \tag{13}$$

$$\epsilon^+(\lambda) \;=\; \left(s_1\left\lceil \frac{\lambda_1}{s_1}\right\rceil - \lambda_1,\; s_2\left\lceil \frac{\lambda_2}{s_2}\right\rceil - \lambda_2,\; \ldots,\; s_n\left\lceil \frac{\lambda_n}{s_n}\right\rceil - \lambda_n\right). \tag{14}$$

The following theorems show the connection between statistics on $\mathbf{s}$-inversion sequences and statistics on $\mathbf{s}$-lecture hall partitions.

Theorem 4. ([13]) *For any sequence* $\mathbf{s}$ *of positive integers, and any positive integer* n*,*

$$\sum_{t \ge 0} \mathbf{f}_n^{(\mathbf{s})}(t;q,y,z)\, x^t \;=\; \frac{\sum_{e \in \mathbf{I}_n^{(\mathbf{s})}} x^{\operatorname{asc} e}\, q^{\operatorname{amaj} e}\, y^{\operatorname{lhp} e}\, z^{|\,e\,|}}{\prod_{i=0}^{n}(1 - xq^{n-i}y^{s_{i+1}+\cdots+s_n})}. \tag{15}$$

Theorem 5. ([13]) *For any sequence* $\mathbf{s}$ *of positive integers, and any positive integer* n*,*

$$\sum_{\lambda \in \mathbf{L}_n^{(\mathbf{s})}} q^{|\lceil \lambda \rceil|_{\mathbf{s}}} y^{|\lambda|} z^{|\,\epsilon^+(\lambda)\,|} x^{\lceil \lambda_n/s_n \rceil} \;=\; \frac{\sum_{e \in \mathbf{I}_n^{(\mathbf{s})}} x^{\operatorname{asc} e}\, q^{\operatorname{amaj} e}\, y^{\operatorname{lhp} e}\, z^{|\,e\,|}}{\prod_{i=0}^{n-1}(1 - xq^{n-i}y^{s_{i+1}+\cdots+s_n})}. \tag{16}$$

7. Lecture Hall Partitions and the Inversion Sequences $\mathbf{I}_{n,k}$

In order to apply the results of the previous section to the problem of interest, we need an analog of Ifmaj on $\mathbf{I}_{n,k}$ for lecture hall partitions.

First observe that the following sets of lecture hall partitions are all the same:

$$\mathbf{L}_n \;=\; \mathbf{L}_n^{(1,2,\ldots,n)} \;=\; \mathbf{L}_n^{(2,4,\ldots,2n)} \;= \mathbf{L}_n^{(3,6,\ldots,3n)} \;= \ldots.$$

However, the lecture hall polytopes $\mathbf{P}_{n,k}$ defined by

$$\mathbf{P}_{n,k} \;=\; \left\{ \lambda \in \mathbb{R}^n \;\middle|\; 0 \le \frac{\lambda_1}{k} \le \frac{\lambda_2}{2k} \le \cdots \le \frac{\lambda_n}{nk} \le 1 \right\}$$

are different for different k. On the other hand, the following dilations *are* the same

$$t\,\mathbf{P}_{n,k} \;=\; kt\,\mathbf{P}_{n,1}, \tag{17}$$

a fact we will exploit. Furthermore,

$$kt\,\mathbf{P}_{n,1} \cap \mathbb{Z}^n = \{\lambda \in \mathbf{L}_n \mid \lambda_n \le ktn\}.$$

Since the definitions (13) and (14) depend on the sequence $\mathbf{s} = (k, 2k, \ldots, nk)$, we will make the dependence explicit in the notation. For $\lambda \in \mathbf{L}_n$ and $k \ge 1$, let:

$$\lceil \lambda \rceil_k \;=\; \left(\left\lceil \frac{\lambda_1}{k} \right\rceil, \left\lceil \frac{\lambda_2}{2k} \right\rceil, \ldots, \left\lceil \frac{\lambda_n}{nk} \right\rceil \right); \tag{18}$$

$$\epsilon_k^+(\lambda) \;=\; \left(k\left\lceil \frac{\lambda_1}{k} \right\rceil - \lambda_1,\; 2k\left\lceil \frac{\lambda_2}{2k} \right\rceil - \lambda_2,\; \ldots,\; nk\left\lceil \frac{\lambda_n}{nk} \right\rceil - \lambda_n \right); \tag{19}$$

$$\eta_k(\lambda) \;=\; k\lceil \lambda \rceil_k - \lceil \lambda \rceil. \tag{20}$$

Note: for $\lambda \in \mathbf{L}_n$,

$$\lceil \lambda \rceil_1 = \lceil \lambda \rceil,$$

where $\lceil \lambda \rceil$ was defined in Theorem 1.

We now show that the new statistic η_k on $\mathbf{L}_n$ corresponds to the statistic $\mathbf{N}$ on $\mathbf{I}_{n,k}$.

Theorem 6. *For positive integers n, k, let*

$$\mathbf{f}_{n,k}(t; q, y, z, w) \;=\; \sum_{\lambda \in t\mathbf{P}_{n,k}\; \cap \mathbb{Z}^n} q^{|\lceil \lambda \rceil_k|} y^{|\lambda|} z^{|\,\epsilon_k^+(\lambda)|}\, w^{|\,\eta_k(\lambda)\,|}. \tag{21}$$

Then

$$\sum_{t \ge 0} \mathbf{f}_{n,k}(t; q, y, z, w)\, x^t \;=\; \frac{\sum_{e \in \mathbf{I}_{n,k}} x^{\operatorname{asc} e}\, q^{\operatorname{amaj} e}\, y^{\operatorname{lhp} e}\, z^{|\,e\,|} w^{\mathbf{N}(e)}}{\prod_{i=0}^{n}(1 - xq^{n-i}y^{k(n(n+1)-i(i+1))/2})}. \tag{22}$$

Proof. If $w = 1$, this is just the case $\mathbf{s} = (k, 2k, \ldots, nk)$ of Theorem 4. To include w, we appeal to the combinatorial proof of (15) in Theorem 4 that was presented in [13]. In that proof, $\lambda \in (t\mathbf{P}_{n,k} \cap \mathbb{Z}^n)$ is associated with the inversion sequence $\epsilon_k^+(\lambda)$, which, by definition, is in $\mathbf{I}_{n,k}$. It suffices to check that $|\eta_k(\lambda)| = \mathbf{N}(\epsilon_k^+(\lambda))$:

$$\begin{aligned}
\mathbf{N}(\epsilon_k^+(\lambda)) &= \sum_{i=1}^{n} \left\lfloor \frac{ik\lceil \lambda_i/(ik)\rceil - \lambda_i}{i} \right\rfloor \\
&= \sum_{i=1}^{n} \lfloor k\lceil \lambda_i/(ik)\rceil - \lambda_i/i \rfloor \\
&= \sum_{i=1}^{n} (k\lceil \lambda_i/(ik)\rceil - \lceil \lambda_i/i\rceil) \\
&= |k\lceil \lambda\rceil_k - \lceil \lambda\rceil_1| \;=\; |\eta_k(\lambda)|.
\end{aligned}$$

□

The Ifmaj statistic is obtained by setting $q = q^k$ and $w = q^{-1}$ in Theorem 6.

Corollary 1. *For positive integers n, k,*

$$\sum_{t\geq 0} \sum_{\lambda \in kt\mathbf{P}_{n,1} \cap \mathbb{Z}^n} q^{|\lceil \lambda\rceil|} y^{|\lambda|} z^{|\epsilon_k^+(\lambda)|} x^t = \frac{\sum_{e\in \mathbf{I}_{n,k}} x^{\operatorname{asc} e} q^{\operatorname{Ifmaj} e} y^{\operatorname{lhp} e} z^{|e|}}{\prod_{i=0}^{n}(1 - xq^{k(n-i)} y^{k(n(n+1)-i(i+1))/2})} \tag{23}$$

Proof. With $q = q^k$ and $w = q^{-1}$, the numerator in the right-hand side of (22) becomes

$$x^{\operatorname{asc} e} q^{k \operatorname{amaj} e - \mathbf{N}(e)} y^{\operatorname{lhp} e} z^{|e|} = x^{\operatorname{asc} e} q^{\operatorname{Ifmaj} e} y^{\operatorname{lhp} e} z^{|e|}.$$

From (21), the left-hand side summand of (22) becomes

$$\begin{aligned}
\mathbf{f}_{n,k}(t; q^k, y, z, q^{-1}) &= \sum_{\lambda \in t\mathbf{P}_{n,k} \cap \mathbb{Z}^n} q^{k|\lceil \lambda\rceil_k| - |\eta_k(\lambda)|} y^{|\lambda|} z^{|\epsilon_k^+(\lambda)|} \\
&= \sum_{\lambda \in kt\mathbf{P}_{n,1} \cap \mathbb{Z}^n} q^{|\lceil \lambda\rceil|} y^{|\lambda|} z^{|\epsilon_k^+(\lambda)|}
\end{aligned}$$

by (18)-(20) and by (17). □

Corollary 2. *For positive integers n, k,*

$$\sum_{\lambda \in \mathbf{L}_n} q^{|\lceil \lambda\rceil|} y^{|\lambda|} z^{|\epsilon_k^+(\lambda)|} x^{\lceil \lambda_n/(nk)\rceil} = \frac{\sum_{e\in \mathbf{I}_{n,k}} x^{\operatorname{asc} e} q^{\operatorname{Ifmaj} e} y^{\operatorname{lhp} e} z^{|e|}}{\prod_{i=0}^{n-1}(1 - xq^{k(n-i)} y^{k(n(n+1)-i(i+1))/2})}. \tag{24}$$

Proof. For $t > 0$, let $H(t) = \sum_{\lambda\in kt\mathbf{P}_{n,1}\ \cap\mathbb{Z}^n} q^{|\lceil\lambda\rceil_1|} y^{|\lambda|} z^{|\,\epsilon_k^+(\lambda)|}$ from (23), with $H(0) = 1$. Then for $t > 0$, since

$$\left\{\lambda \in \mathbf{L}_n;\left\lceil\frac{\lambda_n}{nk}\right\rceil = t\right\} = \left\{\lambda \in \mathbf{L}_n;\left\lceil\frac{\lambda_n}{nk}\right\rceil \le t\right\} - \left\{\lambda \in \mathbf{L}_n;\left\lceil\frac{\lambda_n}{nk}\right\rceil \le t-1\right\}$$
$$= (kt\mathbf{P}_{n,1}\ \cap\mathbb{Z}^n) - (k(t-1)\mathbf{P}_{n,1}\ \cap\mathbb{Z}^n),$$

we have

$$\begin{aligned}\sum_{\lambda\in\mathbf{L}_n} q^{|\lceil\lambda\rceil|} y^{|\lambda|} z^{|\,\epsilon_k^+(\lambda)|}\, x^{\lceil\lambda_n/(nk)\rceil} &= \sum_{t\ge 0} x^t \sum_{\lambda\in\mathbf{L}_n;\lceil\frac{\lambda_n}{nk}\rceil = t} q^{|\lceil\lambda\rceil|} y^{|\lambda|} z^{|\,\epsilon_k^+(\lambda)|}\\ &= 1 + \sum_{t\ge 1}(H(t) - H(t-1))x^t\\ &= 1 + \sum_{t\ge 1} H(t)x^t - \sum_{t\ge 1} H(t-1)x^t\\ &= \sum_{t\ge 0} H(t)x^t - x\sum_{t\ge 0} H(t)x^t\\ &= (1-x)\sum_{t\ge 0} H(t)x^t.\end{aligned}$$

But $\sum_{t\ge 0} H(t)x^t$ is the left-hand side of (23), so we simply multiply the right-hand side of (23) by $(1-x)$ to complete the proof. □

We can now apply these results to the wreath product groups. First, we have the expected result that the pair $(\mathrm{des}\,,\mathrm{fmaj}\,)$ is Euler-Mahonian.

Theorem 7. *For positive integers n, k,*

$$\sum_{t\ge 0} [\,kt+1\,]_q^n\ x^t = \frac{\sum_{\pi^\sigma\in C_k\wr S_n} q^{\mathrm{fmaj}\,\pi^\sigma} x^{\mathrm{des}\,\pi^\sigma}}{\prod_{i=0}^n (1 - xq^{ki})}.$$

Proof. Set $y = z = 1$ in (23). On the left-hand side, in the summand, we get

$$\sum_{\lambda\in kt\mathbf{P}_{n,1}\ \cap\mathbb{Z}^n} q^{|\lceil\lambda\rceil|}.$$

Since $kt\,\mathbf{P}_{n,1}\cap\mathbb{Z}^n = \{\lambda\in\mathbf{L}_n \mid \lambda_n \le ktn\}$, by Theorem 2,

$$\sum_{\lambda\in kt\mathbf{P}_{n,1}\ \cap\mathbb{Z}^n} q^{|\lceil\lambda\rceil|} = [\,kt+1\,]_q^n\,.$$

For the right-hand side, we get

$$\frac{\sum_{e\in\mathbf{I}_{n,k}} x^{\mathrm{asc}\,e} q^{\mathrm{Ifmaj}\,e}}{\prod_{i=0}^n (1 - xq^{k(n-i)})}.$$

Reindex the product in the denominator and for the numerator, use the fact that by Theorem 3, the distribution of $(\mathrm{des}\,,\mathrm{fmaj}\,)$ on $C_k \wr S_n$ is the same as the distribution of $(\mathrm{asc}\,,\mathrm{Ifmaj}\,)$ on $\mathbf{I}_{n,k}$. □

Now, to interpret the distribution $(\mathrm{des}\,,\mathrm{fmaj}\,,\mathrm{finv}\,)$ on $C_k \wr S_n$ in terms of lecture hall partitions, set $y = 1$ in (24) and use Theorem 3.

Theorem 8. *For positive integers n, k,*

$$\sum_{\lambda \in \mathbf{L}_n} q^{|\lceil\lambda\rceil|} z^{|\,\epsilon_k^+(\lambda)|}\, x^{\lceil \lambda_n/(nk)\rceil} \;=\; \frac{\sum_{\pi^\sigma \in C_k \wr S_n} q^{\mathrm{fmaj}\,\pi^\sigma} x^{\mathrm{des}\,\pi^\sigma} z^{\mathrm{finv}\,\pi^\sigma}}{\prod_{i=1}^n (1 - xq^{ki})}.$$

The implication of Theorem 8 for $z = 1$ is quite interesting. We have

$$\sum_{\lambda \in \mathbf{L}_n} q^{|\lceil\lambda\rceil|} x^{\lceil \lambda_n/(nk)\rceil} \;=\; \frac{\sum_{\pi^\sigma \in C_k \wr S_n} q^{\mathrm{fmaj}\,\pi^\sigma} x^{\mathrm{des}\,\pi^\sigma}}{\prod_{i=1}^n (1 - xq^{ki})}. \tag{25}$$

In the left-hand side of (25), the only dependence on k is in the exponent of x, in a statistic involving only the last part of λ. We take this further in Section 9.

8. A Lecture Hall Statistic on $C_k \wr S_n$

From the point of view of partition theory, the most important statistic for a lecture hall partition λ is the number $|\lambda| = \lambda_1 + \cdots + \lambda_n$ being partitioned. So, what does $|\lambda|$ correspond to on $C_k \wr S_n$?

In [6], a quadratic version of the major index was defined on S_n by $\mathrm{bin}\,\pi = \sum_{i \in \mathrm{Des}\,\pi} \binom{i+1}{2}$. In that spirit, we define "cobin" on $C_k \wr S_n$ by

$$\mathrm{cobin}\,\pi^\sigma \;=\; \sum_{i \in \mathrm{Des}\,\pi^\sigma} \left(\binom{n+1}{2} - \binom{i+1}{2} \right).$$

Now define the statistic "lhall" on $C_k \wr S_n$ by

$$\mathrm{lhall}\,\pi^\sigma \;=\; k\,\mathrm{cobin}\,\pi^\sigma - \mathrm{finv}\,\pi^\sigma.$$

Observe that under the bijection Θ of Theorem 3, if $e = \Theta(\pi^\sigma)$ then $\mathrm{lhall}\,\pi^\sigma = \mathrm{lhp}\,e$.

This can be seen as follows, since $|e| = \operatorname{finv}\pi^\sigma$ and $\operatorname{Asc} e = \operatorname{Des} e$:

$$\begin{aligned}
\operatorname{lhp} e &= -|e| + \sum_{i\in \operatorname{Asc} e} (k(i+1) + \cdots + kn) \\
&= -|e| + k \sum_{i\in \operatorname{Asc} e} \left(\binom{n+1}{2} - \binom{i+1}{2} \right) \\
&= -\operatorname{finv}\pi^\sigma + k \sum_{i\in \operatorname{Des} e} \left(\binom{n+1}{2} - \binom{i+1}{2} \right) \\
&= -\operatorname{finv}\pi^\sigma + k\operatorname{cobin}\pi^\sigma \\
&= \operatorname{lhall}\pi^\sigma .
\end{aligned}$$

The joint distribution of $(\operatorname{lhall}, \operatorname{fmaj})$ on $C_k \wr S_n$ has the following form.

Theorem 9. *For positive integers n, k,*

$$\begin{aligned}
\sum_{\pi^\sigma \in C_k \wr S_n} y^{\operatorname{lhall}\pi^\sigma} q^{\operatorname{fmaj}\pi^\sigma} &= \prod_{i=1}^{n} \frac{(1+qy^i)(1-q^{k(n+1-i)}y^{k(i+\cdots+n)})}{1-q^2y^{n+i}} \\
&= \prod_{i=1}^{\lceil n/2 \rceil} [\, k(2i-1)\,]_{qy^{n+1-i}} \prod_{i=1}^{\lfloor n/2 \rfloor} ([\,2\,]_{qy^i}\, [\,ki\,]_{q^2y^{2(n-i)+1}})
\end{aligned}$$

Proof. Under the bijection Θ of Theorem 3, if $e = \Theta(\pi^\sigma)$ then $\operatorname{lhall}\pi^\sigma = \operatorname{lhp} e$ and $\operatorname{fmaj}\pi^\sigma = \operatorname{Ifmaj} e$. So,

$$\sum_{\pi^\sigma \in C_k \wr S_n} y^{\operatorname{lhall}\pi^\sigma} q^{\operatorname{fmaj}\pi^\sigma} = \sum_{e \in \mathbf{I}_{n,k}} y^{\operatorname{lhp} e} q^{\operatorname{Ifmaj} e}.$$

So, by Corollary 2 with $x = z = 1$,

$$\begin{aligned}
\frac{\sum_{\pi^\sigma \in C_k \wr S_n} y^{\operatorname{lhall}\pi^\sigma} q^{\operatorname{fmaj}\pi^\sigma}}{\prod_{i=0}^{n-1}(1-q^{k(n-i)}y^{k(n(n+1)-i(i+1))/2})} &= \frac{\sum_{e \in \mathbf{I}_{n,k}} y^{\operatorname{lhp} e} q^{\operatorname{Ifmaj} e}}{\prod_{i=0}^{n-1}(1-q^{k(n-i)}y^{k(n(n+1)-i(i+1))/2})} \\
&= \sum_{\lambda \in \mathbf{L}_n} y^{|\lambda|} q^{|\lceil \lambda \rceil|}.
\end{aligned}$$

Now apply Theorem 1 to get

$$\frac{\sum_{\pi^\sigma \in C_k \wr S_n} y^{\operatorname{lhall}\pi^\sigma} q^{\operatorname{fmaj}\pi^\sigma}}{\prod_{i=0}^{n-1}(1-q^{k(n-i)}y^{k(n(n+1)-i(i+1))/2})} = \prod_{i=1}^{n} \frac{1+qy^i}{1-q^2y^{n+i}}.$$

So,

$$\sum_{\pi^\sigma \in C_k \wr S_n} y^{\operatorname{lhall}\pi^\sigma} q^{\operatorname{fmaj}\pi^\sigma} = \prod_{i=1}^{n}(1-q^{k(n-i+1)}y^{k(n(n+1)-i(i+1))/2}) \prod_{i=1}^{n} \frac{1+qy^i}{1-q^2y^{n+i}},$$

which, after simplification, gives the theorem. □

Setting $y = 1$ in Theorem 9 and simplifying, we get

$$\sum_{\pi^\sigma \in C_k \wr S_n} q^{\mathrm{fmaj}\,\pi^\sigma} \;=\; \prod_{i=1}^{n} [\,ki\,]_q\,,$$

the same distribution as finv, Ifmaj, and $|e|$, as expected. But the statistic lhall itself also has a surprisingly simple distribution:

Theorem 10. *For positive integers n, k,*

$$\sum_{\pi^\sigma \in C_k \wr S_n} q^{\mathrm{lhall}\,\pi^\sigma} \;=\; \prod_{i=1}^{n} [\,ki\,]_{q^{2(n-i)+1}}\,.$$

Proof. Set $q = 1$ and $y = q$ in the proof of the Theorem 9, but apply (1) instead of (2) to get:

$$\begin{aligned}
\sum_{e \in \mathbf{I}_n^{(1,2,\ldots,n)}} q^{\mathrm{lhp}\,e} \;&=\; \prod_{i=1}^{n} \frac{1-q^{k(i+\cdots+n)}}{1-q^{2i-1}} \\
&=\; \prod_{i=1}^{\lfloor n/2 \rfloor} \frac{1-q^{k(2i-1)(n-i+1)}}{1-q^{2i-1}} \prod_{i=1}^{\lceil n/2 \rceil} \frac{1-q^{ki(2(n-i)+1)}}{1-q^{2(n-i)+1}} \\
&=\; \prod_{i=1}^{n} [\,ki\,]_{q^{2(n-i)+1}}\,.
\end{aligned}$$

□

9. Inflated Eulerian Polynomials for $C_k \wr S_n$

We showed in [11] how to obtain more refined information about the **s**-lecture hall partitions by considering the *rational lecture hall polytope* $\mathbf{R}_n^{(\mathbf{s})}$:

$$\mathbf{R}_n^{(\mathbf{s})} = \left\{ \lambda \in \mathbb{R}^n \;\middle|\; 0 \le \frac{\lambda_1}{s_1} \le \frac{\lambda_2}{s_2} \le \cdots \le \frac{\lambda_n}{s_n} \;\text{ and }\; \lambda_n \le 1 \right\}.$$

$\mathbf{R}_n^{(\mathbf{s})}$ is a convex simplicial polytope, whose vertices are

$$(0,0,\ldots,0)\,, \left(\frac{s_1}{s_n}, \frac{s_2}{s_n}, \ldots, \frac{s_n}{s_n}\right), \left(0, \frac{s_2}{s_n}, \ldots, \frac{s_n}{s_n}\right), \left(0, 0, \frac{s_3}{s_n}, \ldots, \frac{s_n}{s_n}\right), \ldots, \left(0, 0, \ldots, 0, \frac{s_n}{s_n}\right),$$

with rational (but not necessarily integer) coordinates. Let

$$\mathbf{g}_n^{(\mathbf{s})}(t; q, y, z) \;=\; \sum_{\lambda \in t\mathbf{R}_n^{(\mathbf{s})} \cap \mathbb{Z}^n} q^{|\lceil \lambda \rceil|_k} y^{|\lambda|} z^{|\,\epsilon_k^+(\lambda)\,|}. \qquad (26)$$

The following theorems were proved in [11]. These are analogs of Theorems 4 and 5.

Theorem 11. ([11]) *For any sequence* $\mathbf{s}$ *of positive integers, and positive integer* n*,*

$$\sum_{t\geq 0}\mathbf{g}_n^{(s)}(t;q,y,z)x^t \;=\; \frac{\sum_{e\in \mathbf{I}_n^{(\mathbf{s})}} q^{\operatorname{amaj} e}\, y^{\operatorname{lhp} e}\, z^{|\,e\,|}\, x^{s_n \operatorname{asc} e - e_n}}{(1-x)\prod_{i=0}^{n-1}(1-x^{s_n}q^{n-i}y^{s_{i+1}+\cdots+s_n})}.$$

Theorem 12. ([11]) *For any sequence* $\mathbf{s}$ *of positive integers, and positive integer* n*,*

$$\sum_{\lambda\in \mathbf{L}_n^{(\mathbf{s})}} q^{|\lceil\lambda\rceil|_k} y^{|\lambda|} z^{|\,\epsilon_k^+(\lambda)\,|} x^{\lambda_n} \;=\; \frac{\sum_{e\in \mathbf{I}_n^{(\mathbf{s})}} q^{\operatorname{amaj} e}\, y^{\operatorname{lhp} e}\, z^{|\,e\,|}\, x^{s_n \operatorname{asc} e - e_n}}{\prod_{i=0}^{n-1}(1-x^{s_n}q^{n-i}y^{s_{i+1}+\cdots+s_n})}.$$

We can specialize Theorems 11 and 12 to $\mathbf{s} = (k, 2k, \ldots, nk)$ and modify to track lfmaj as in Theorem 6 and its corollaries. We should expect something interesting because

$$\mathbf{R}_n^{(1,2,\ldots,n)} \;=\; \mathbf{R}_n^{(2,4,\ldots,2n)} \;=\; \mathbf{R}_n^{(3,6,\ldots,3n)} \;\ldots .$$

We get the following theorem, which is an analog of Theorem 6. The proof, which is analogous to that of Theorem 6, is omitted.

Theorem 13. *For positive integers* n, k*, let*

$$\mathbf{g}_{n,k}(t;q,y,z,w) \;=\; \sum_{\lambda\in t\mathbf{R}_n\,\cap\mathbb{Z}^n} q^{|\lceil\lambda\rceil_k|} y^{|\lambda|} z^{|\,\epsilon_k^+(\lambda)|}\, w^{|\,\eta_k(\lambda)\,|}. \tag{27}$$

Then

$$\sum_{t\geq 0}\mathbf{g}_{n,k}(t;q,y,z,w)\,x^t \;=\; \frac{\sum_{e\in \mathbf{I}_{n,k}} x^{kn\operatorname{asc} e - e_n}\, q^{\operatorname{amaj} e}\, y^{\operatorname{lhp} e}\, z^{|\,e\,|} w^{\mathbf{N}(e)}}{(1-x)\prod_{i=0}^{n-1}(1-x^{kn}q^{n-i}y^{k(n(n+1)-i(i+1))/2})}. \tag{28}$$

The following corollaries of Theorem 13 are analogs of Corollaries 1 and 2 with $y = z = 1$. Note that in the right-hand sides of the equations there is no dependence on k.

Corollary 3. *For positive integers* n, k*,*

$$\sum_{t\geq 0}\;\sum_{\lambda\in t\mathbf{R}_n\,\cap\mathbb{Z}^n} q^{|\lceil\lambda\rceil|} x^t \;=\; \frac{\sum_{e\in \mathbf{I}_{n,k}} x^{kn\operatorname{asc} e - e_n}\, q^{\operatorname{lfmaj} e}}{(1-x)\prod_{i=0}^{n-1}(1-x^{kn}q^{k(n-i)})}.$$

Corollary 4. *For positive integers* n, k*,*

$$\sum_{\lambda\in \mathbf{L}_n} q^{|\lceil\lambda\rceil|} x^{\lambda_n} \;=\; \frac{\sum_{e\in \mathbf{I}_{n,k}} x^{kn\operatorname{asc} e - e_n}\, q^{\operatorname{lfmaj} e}}{\prod_{i=0}^{n-1}(1-x^{kn}q^{k(n-i)})}.$$

Making use of Theorem 3 giving the correspondence between statistics on $\mathbf{I}_{n,k}$ and on $C_k \wr S_n$, we have the following analogs of Theorems 7 and 8. First, We need a result from [8]:

Lemma 2. ([8]) *For integers $t \geq 0$ and $n > 0$, let j and i be the unique integers satisfying $t = jn + i$ where $j \geq 0$ and $0 \leq i < n$. Then*

$$\sum_{\lambda \in t\mathbf{R}_n \cap \mathbb{Z}^n} q^{|\lceil \lambda \rceil|} \;=\; [\,j+1\,]_q^{n-i}\,[\,j+2\,]_q^{i}\,.$$

Theorem 14. *For positive integers n, k,*

$$\sum_{j\geq 0}\sum_{i=0}^{n-1} [\,j+1\,]_q^{n-i}\,[\,j+2\,]_q^{i}\; x^{nj+i} \;=\; \frac{\sum_{\pi^\sigma \in C_k \wr S_n} q^{\mathrm{fmaj}\,\pi^\sigma} x^{n(k\,\mathrm{des}\,\pi^\sigma - 1 - \sigma_n) + \pi_n}}{(1-x)\prod_{i=1}^{n}(1-x^{kn}q^{ki})}.$$

Proof. By Lemma 2,

$$\sum_{j\geq 0}\sum_{i=0}^{n-1} [\,j+1\,]_q^{n-i}\,[\,j+2\,]_q^{i}\; x^{nj+i} = \sum_{j\geq 0}\sum_{i=0}^{n-1} \sum_{\lambda \in (jn+i)\mathbf{R}_n \;\cap \mathbb{Z}^n} q^{|\lceil \lambda \rceil|} x^{jn+i}.$$

Since every $t \geq 0$ can be written uniquely as $t = jn + i$ for nonnegative integers j and i with $i < n$, the last expression can be rewritten as

$$\sum_{t\geq 0} \sum_{\lambda \in t\mathbf{R}_n \;\cap \mathbb{Z}^n} q^{|\lceil \lambda \rceil|} x^{t},$$

which, by Corollary 3, is equal to

$$\frac{\sum_{e \in \mathbf{I}_{n,k}} x^{kn\mathrm{asc}\,e - e_n}\, q^{\mathrm{Ifmaj}\,e}}{\prod_{i=0}^{n}(1-x^{kn}q^{k(n-i)})}.$$

Under the bijection Θ of Theorem 3, if $e = \Theta(\pi^\sigma)$ then $\mathrm{Ifmaj}\,e = \mathrm{fmaj}\,\pi^\sigma$, $\mathrm{asc}\,e = \mathrm{des}\,\pi^\sigma$, and $e_n = n(\sigma_n + 1) - \pi_n$. The result follows then, since

$$kn\,\mathrm{asc}\,e - e_n \;=\; kn\,\mathrm{des}\,\pi^\sigma - n(\sigma_n + 1) + \pi_n.$$

□

Theorem 15. *For any positive integers n, k,*

$$\sum_{\lambda \in \mathbf{L}_n} q^{|\lceil \lambda \rceil|} x^{\lambda_n} \;=\; \frac{\sum_{\pi^\sigma \in C_k \wr S_n} q^{\mathrm{fmaj}\,\pi^\sigma} x^{n(k\,\mathrm{des}\,\pi^\sigma - 1 - \sigma_n) + \pi_n}}{\prod_{i=1}^{n}(1-x^{kn}q^{ki})}.$$

Proof. Start from Corollary 4 and apply Theorem 3. □

(Note: There is no dependence on k in the left-hand side).

Let $Q_{n,k}(x)$ be the $q = 1$ specialization:

$$Q_{n,k}(x) = \sum_{\pi^\sigma \in C_k \wr S_n} x^{n(k \operatorname{des} \pi^\sigma - 1 - \sigma_n) + \pi_n}.$$

The $Q_{n,k}(x)$ are referred to as *inflated Eulerian polynomials* in [11]. To contrast the usual, Eulerian polynomials for $C_k \wr S_n$ are

$$E_{n,k}(x) = \sum_{\pi^\sigma \in C_k \wr S_n} x^{\operatorname{des} \pi^\sigma}.$$

It is interesting that $Q_{n,k}(x)$ is self-reciprocal, but in general $E_{n,k}(x)$ is not when $k > 2$.

10. Concluding Remarks

It is interesting from the results in Sections 7 - 9 that for fixed n, statistics on $C_k \wr S_n$ such as descent, flag-major index, and flag-inversion number appear naturally in the geometry of the *same* simplicial cone, $\mathbf{R}_n$, independent of k.

It would be interesting to see to what extent other statistics on $C_k \wr S_n$ can be interpreted in terms of lecture hall partitions. Different orderings on $C_k \wr S_n$ and different bijections $C_k \wr S_n \to \mathbf{I}_{n,k}$ would give different results.

Lecture hall partitions were discovered in the setting of affine Coxeter groups, and Theorem 1 was inspired by Bott's formula. It should be possible to trace through backwards to discover the algebraic significance of the statistic lhall, at least in the Coxeter groups $A_n = C_1 \wr S_n$ or $B_n = C_2 \wr S_n$ but we have not seen how to do this.

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