#A11 INTEGERS 12A (2012): John Selfridge Memorial Issue

ON A PARTITION PROBLEM OF CANFIELD AND WILF

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Received: 5/30/11, Revised: 4/18/12, Accepted: 6/30/12, Published: 10/12/12

Abstract

Let A and M be nonempty sets of positive integers. A partition of the positive integer n with parts in A and multiplicities in M is a representation of n in the form $n = \sum_{a\in A} m_a a$ where $m_a \in M \cup \{0\}$ for all $a \in A$, and $m_a \in M$ for only finitely many a. Denote by $p_{A,M}(n)$ the number of partitions of n with parts in A and multiplicities in M. It is proved that there exist infinite sets A and M of positive integers whose partition function $p_{A,M}$ has weakly superpolynomial but not superpolynomial growth. The counting function of the set A is $A(x) = \sum_{a\in A, a\leq x} 1$. It is also proved that $p_{A,M}$ must have at least weakly superpolynomial growth if M is infinite and $A(x) \gg \log x$.

–To the memory of John Selfridge

1. Partition Problems With Restricted Multiplicities

Let $\mathbf{N}$ denote the set of positive integers and let A be a nonempty subset of $\mathbf{N}$. A *partition of* n *with parts in* A is a representation of n in the form

$$n = \sum_{a\in A} m_a a$$

where $m_a \in \mathbf{N} \cup \{0\}$ for all $a \in A$, and $m_a \in \mathbf{N}$ for only finitely many a. The *partition function* $p_A(n)$ counts the number of partitions of n with parts in A. If $\gcd(A) = d > 1$, then $p_A(n) = 0$ for all n not divisible by d, and so $p_A(n) = 0$ for infinitely many positive integers n. If $p_A(n) \geq 1$ for all sufficiently large n, then $\gcd(A) = 1$.

[1]The work of M.B.N. was supported in part by a PSC-CUNY Research Award.

If $A = \{a_1, \ldots, a_k\}$ is a set of k relatively prime positive integers, then Schur [8] proved that

$$p_A(n) \sim \frac{n^{k-1}}{(k-1)!a_1 a_2 \cdots a_k}. \tag{1}$$

Nathanson [6] gave a simpler proof of the more precise result:

$$p_A(n) = \frac{n^{k-1}}{(k-1)!a_1 a_2 \cdots a_k} + O\left(n^{k-2}\right). \tag{2}$$

An arithmetic function is a real-valued function whose domain is the set of positive integers. An arithmetic function f has *polynomial growth* if there is a positive integer k and an integer $N_0(k)$ such that $1 \leq f(n) \leq n^k$ for all $n \geq N_0(k)$. Equivalently, f has polynomial growth if

$$\limsup_{n\to\infty} \frac{\log f(n)}{\log n} < \infty.$$

We shall call an arithmetic function *nonpolynomial* or *weakly superpolynomial* if it does not have polynomial growth. Thus, the function f is weakly superpolynomial if for every positive integer k there are infinitely many positive integers n such that $f(n) > n^k$, or, equivalently, if

$$\limsup_{n\to\infty} \frac{\log f(n)}{\log n} = \infty.$$

An arithmetic function f has *superpolynomial growth* if for every positive integer k we have $f(n) > n^k$ for all sufficiently large integers n. Equivalently,

$$\lim_{n\to\infty} \frac{\log f(n)}{\log n} = \infty.$$

In the following section we construct strictly increasing arithmetic functions that are weakly superpolynomial but not superpolynomial.

The asymptotic formula (1) implies the following result of Nathanson [5, Theorem 15.2, pp. 458–461].

Theorem 1. *If A is an infinite set of integers and* $\gcd(A) = 1$, *then $p_A(n)$ has superpolynomial growth.*

Canfield and Wilf [2] studied the following variation of the classical partition problem. Let A and M be nonempty sets of positive integers. A *partition of n with parts in A and multiplicities in M* is a representation of n in the form

$$n = \sum_{a\in A} m_a a$$

where $m_a \in M \cup \{0\}$ for all $a \in A$, and $m_a \in M$ for only finitely many a. The associated partition function $p_{A,M}(n)$ counts the number of partitions of n with parts in A and multiplicities in M. Note that $p_{A,M}(0) = 1$ and $p_{A,M}(n) = 0$ for all $n < 0$.

Let A and M be infinite sets of positive integers such that $p_{A,M}(n) \geq 1$ for all sufficiently large n. Canfield and Wilf ("Unsolved problem 1" in [2]) asked if the partition function $p_{A,M}(N)$ must have weakly superpolynomial growth. The question can be rephrased as follows: Do there exist infinite sets A and B of positive integers such that $p_{A,M}(n) \geq 1$ for all sufficiently large n and the partition function $p_{A,M}(N)$ has polynomial growth? This beautiful problem is still unsolved.

The goal of this paper is to construct infinite sets A and M of positive integers such that the partition function $p_{A,M}(N)$ is weakly superpolynomial but not superpolynomial.

2. Weakly Superpolynomial Functions

Polynomial and superpolynomial growth functions were first studied in connection with the growth of finitely and infinitely generated groups (cf. Milnor [4], Grigorchuk and Pak [3], Nathanson [7]). Growth functions of infinite groups are always strictly increasing, but even strictly increasing functions that do not have polynomial growth are not necessarily superpolynomial.

We note that an arithmetic function f is weakly superpolynomial but not superpolynomial if and only if

$$\limsup_{n\to\infty} \frac{\log f(n)}{\log n} = \infty$$

and

$$\liminf_{n\to\infty} \frac{\log f(n)}{\log n} < \infty.$$

In this section we construct a strictly increasing arithmetic function that is weakly superpolynomial but not polynomial.

Let $(n_k)_{k=1}^{\infty}$ be a sequence of positive integers such that $n_1 = 1$ and

$$n_{k+1} > 2n_k^k$$

for all $k \geq 1$. We define the arithmetic function

$$f(n) = n_k^k + (n - n_k) \qquad \text{for } n_k \leq n < n_{k+1}.$$

This function is strictly increasing because

$$n_k^k - n_k \leq n_{k+1}^{k+1} - n_{k+1}$$

for all $k \geq 1$. We have

$$\lim_{k\to\infty} \frac{\log f(n_k)}{\log n_k} = \lim_{k\to\infty} \frac{k \log n_k}{\log n_k} = \infty$$

and so

$$\limsup_{n\to\infty} \frac{\log f(n)}{\log n} = \infty.$$

Therefore, the function f does not have polynomial growth.

For every positive integer n there is a positive integer k such that $n_k \leq n < n_{k+1}$. Then $f(n) = n + n_k^k - n_k \geq n$ and so

$$\liminf_{n\to\infty} \frac{\log f(n)}{\log n} \geq 1. \tag{3}$$

The inequalities

$$f(n_{k+1} - 1) = n_k^k + (n_{k+1} - 1 - n_k) < \frac{3n_{k+1}}{2}$$

and

$$n_{k+1} - 1 > \frac{n_{k+1}}{2}$$

imply that

$$1 < \frac{\log f(n_{k+1} - 1)}{\log(n_{k+1} - 1)} < \frac{\log(3n_{k+1}/2)}{\log(n_{k+1}/2)} = 1 + \frac{\log 3}{\log(n_{k+1}/2)}$$

and so

$$\lim_{k\to\infty} \frac{\log f(n_{k+1} - 1)}{\log(n_{k+1} - 1)} = 1.$$

Therefore,

$$\liminf_{n\to\infty} \frac{\log f(n)}{\log n} \leq 1. \tag{4}$$

Combining (3) and (4), we obtain

$$\liminf_{n\to\infty} \frac{\log f(n)}{\log n} = 1.$$

Thus, the function f has weakly superpolynomial but not superpolynomial growth.

3. Weakly Superpolynomial Partition Functions

Theorem 2. *Let a be an integer, $a \geq 2$, and let $A = \{a^i\}_{i=0}^{\infty}$. Let M be an infinite set of positive integers such that M contains $\{1, 2, \ldots, a-1\}$ and no element of M is divisible by a. Then $p_{A,M}(n) \geq 1$ for all $n \in \mathbf{N}$, and $p_{A,M}(n) = 1$ for all $n \in A$. In particular, the partition function $p_{A,M}$ does not have superpolynomial growth.*

Proof. Every positive integer n has a unique a-adic representation, and so $p_{A,M}(n) \geq 1$ for all $n \in \mathbf{N}$.

We shall prove that, for every positive integer r, the only partition of a^r with parts in A and multiplicities in M is $a^r = 1 \cdot a^r$. If there were another representation, then it could be written in the form

$$a^r = \sum_{i=1}^{k} m_i a^{j_i}$$

where $k \geq 2$, $m_i \in M$ for $i = 1, \ldots, k$, and $0 \leq j_1 < j_2 < \cdots < j_k < r$. Then

$$a^{r-j_1} = m_1 + a \sum_{i=2}^{k} m_i a^{j_i - j_1 - 1}.$$

We have $j_i - j_1 - 1 \geq 0$ for $i = 2, \ldots, k$, and so m_1 is divisible by a, which is absurd. Therefore, $p_{A,M}(a^r) = 1$ for all $r \geq 0$. It follows that

$$\liminf_{n \to \infty} \frac{\log p_{A,M}(n)}{\log n} = \liminf_{r \to \infty} \frac{\log p_{A,M}(a^r)}{\log a^r} = 0$$

and so the partition function $p_{A,M}$ is not superpolynomial. □

Theorem 3. *Let A and M be infinite sets of positive integers. If $A(x) \geq c \log x$ for some $c > 0$ and all $x \geq x_0(A)$, then for every positive integer k there exist infinitely many integers n such that*

$$p_{A,M}(n) > n^k.$$

In particular, the partition function $p_{A,M}$ is weakly superpolynomial.

Proof. Let $x \geq 1$ and let

$$A(x) = \sum_{\substack{a \in A \\ a \leq x}} 1 \qquad \text{and} \qquad M(x) = \sum_{\substack{m \in M \\ m \leq x}} 1$$

denote the counting functions of the sets A and M, respectively. If $n \leq x$ and $n = \sum_{a \in A} m_a a$ is a partition of n with parts in A and multiplicities in $M \cup \{0\}$, then $a \leq x$ and $m_a \leq x$, and so

$$\max \{p_{A,M}(n) : n \leq x\} \leq \sum_{n \leq x} p_{A,M}(n) \leq (M(x) + 1)^{A(x)}. \tag{5}$$

Conversely, if the integer n can be represented in the form $n = \sum_{a \in A} m_a a$ with $a \leq x$ and $m_a \leq x$, then $n \leq x^2 A(x) \leq x^3$ and so

$$\sum_{n \leq x^2 A(x)} p_{A,M}(n) \geq (M(x) + 1)^{A(x)} > M(x)^{A(x)}.$$

Choose an integer n_x such that $n_x \leq x^2 A(x)$ and

$$p_{A,M}(n_x) = \max\left\{p_{A,M}(n) : n \leq x^2 A(x)\right\}.$$

Inequality (5) implies that

$$p_{A,M}(n_x) \leq \left(M\left(x^2 A(x)\right) + 1\right)^{A\left(x^2 A(x)\right)}. \tag{6}$$

Moreover,

$$M(x)^{A(x)} < \sum_{n \leq x^2 A(x)} p_{A,M}(n) \leq \left(x^2 A(x) + 1\right) p_{A,M}(n_x) \leq 2x^3 p_{A,M}(n_x).$$

It follows that for all $x \geq x_0(A)$ we have

$$p_{A,M}(n_x) > \frac{M(x)^{A(x)}}{2x^3} \geq \frac{M(x)^{c\log x}}{2x^3}.$$

Let k be a positive integer. Because the set M is infinite, there exists $x_1(A,k) \geq x_0(A)$ such that, for all $x \geq x_1(A,k)$, we have

$$\log M(x) > \frac{\log 2}{c \log x} + \frac{3k+3}{c}$$

and so

$$p_{A,M}(n_x) > x^{3k} \geq n_x^k.$$

We shall iterate this process to construct inductively an infinite sequence of pairwise distinct positive integers $(n_{x_i})_{i=1}^{\infty}$ such that

$$p_{A,M}(n_{x_i}) > n_{x_i}^k \tag{7}$$

for all i. Let $r \geq 1$, and suppose that a finite sequence of pairwise distinct positive integers $(n_{x_i})_{i=1}^{r}$ has been constructed such that inequality (7) holds for $i = 1, \ldots, r$. Choose x_{r+1} so that

$$x_{r+1}^{3k} > \left(M\left(x_i^2 A(x_i)\right) + 1\right)^{A\left(x_i^2 A(x_i)\right)}$$

for all $i = 1, \ldots, r$, and let $n_{x_{r+1}}$ be the integer constructed according to procedure above. Applying inequality (6), we obtain

$$p\left(n_{x_i}\right) \leq \left(M\left(x_i^2 A(x_i)\right) + 1\right)^{A\left(x_i^2 A(x_i)\right)}$$

and so

$$p\left(n_{x_{r+1}}\right) > x_{r+1}^{3k} > p\left(n_{x_i}\right)$$

for $i = 1, \ldots, r$. It follows that $n_{x_{r+1}} \neq n_{x_i}$ for $i = 1, \ldots, r$. This completes the induction and the proof. □

Theorem 4. *Let a be an integer, $a \geq 2$, and let $A = \{a^i\}_{i=0}^{\infty}$. Let M be an infinite set of positive integers such that M contains $\{1, 2, \ldots, a-1\}$ and no element of M is divisible by a. The partition function $p_{A,M}$ is weakly superpolynomial but not superpolynomial.*

Proof. The counting function for the set $A = \{a^i\}_{i=1}^{\infty}$ is $A(x) = [\log x/\log a] + 1 > \log x/\log a$. By Theorem 3, the partition function $p_{A,M}$ is weakly superpolynomial. By Theorem 2, the partition function $p_{A,M}$ is not superpolynomial. This completes the proof. □

4. Open Problems

1. We repeat the original problem of Canfield and Wilf: Do there exist infinite sets A and B of positive integers such that $p_{A,M}(n) \geq 1$ for all sufficiently large n and the partition function $p_{A,M}(N)$ has polynomial growth?

2. By Theorem 3, if the partition function $p_{A,M}$ has polynomial growth, then the set A must have sub-logarithmic growth, that is, $A(x) \gg \log x$ is impossible.

   (a) Let $A = \{k!\}_{k=1}^{\infty}$. Does there exist an infinite set M of positive integers such that $p_{A,M}(n) \geq 1$ for all sufficiently large n and $p_{A,M}$ has polynomial growth?

   (b) Let $A = \left\{k^k\right\}_{k=1}^{\infty}$. Does there exist an infinite set M of positive integers such that $p_{A,M}(n) \geq 1$ for all sufficiently large n and $p_{A,M}$ has polynomial growth?

3. Let A be an infinite set of positive integers and let $M = \mathbf{N}$. Bateman and Erdős [1] proved that the partition function $p_A = p_{A,\mathbf{N}}$ is eventually strictly increasing if and only if $\gcd(A \setminus \{a\}) = 1$ for all $a \in A$. It would be interesting to extend this result to partition functions with restricted multiplicities: Determine a necessary and sufficient condition for infinite sets A and M of positive integers to have the property that $p_{A,M}(n) < p_{A,M}(n+1)$ or $p_{A,M}(n) \leq p_{A,M}(n+1)$ for all sufficiently large n.

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