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SUB-RAMSEY NUMBERS FOR ARITHMETIC PROGRESSIONS AND SCHUR TRIPLES

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Abstract

For a given positive integer k, $sr(m,k)$ denotes the minimal positive integer such that every coloring of $[n]$, $n \geq sr(m,k)$, that uses each color at most k times, yields a rainbow $AP(m)$; that is, an m-term arithmetic progression, all of whose terms receive different colors. We prove that $sr(3,k) = \frac{17}{8}k + O(1)$ and, for $m > 1$ and $k > 1$, that $sr(m,k) = \Omega(m^2 k)$, improving the previous bounds of Alon, Caro, and Tuza from 1989. Our new lower bound on $sr(m,2)$ immediately implies that for $n \leq \frac{m^2}{2}$, there exists a mapping $\phi : [n] \to [n]$ without a fixed point such that for every $AP(m)$ $\mathcal{A}$ in $[n]$, the set $\mathcal{A} \cap \phi(\mathcal{A})$ is not empty. We also propose the study of sub-Ramsey–type problems for linear equations other than $x + y = 2z$. For a given positive integer k, we define $ss(k)$ to be the minimal positive integer n such that every coloring of $[n]$, $n \geq ss(k)$, that uses each color at most k times, yields a rainbow solution to the Schur equation $x + y = z$. We prove that $ss(k) = \lfloor \frac{5k}{2} \rfloor + 1$.

Key words: rainbow arithmetic progressions, sub-Ramsey problems, Schur triples

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1. Introduction

Let $\mathbb{N}$ denote the set of positive integers, and for $i, j \in \mathbb{N}$, $i \leq j$, let $[i, j]$ denote the set $\{i, i+1, \ldots, j\}$ (with $[n]$ abbreviating $[1, n]$ as usual). A k-term arithmetic progression, $k \in \mathbb{N}$, is a set of the form $\{a + (i-1)\, d : i \in [k]\}$, for some $a, d \in \mathbb{N}$, and will be abbreviated as $AP(k)$ throughout. The classical result of van der Waerden [vW27, GRS90] states that for all natural numbers m and k there is an integer $n_0 = n_0(m, k)$, such that every k-coloring of $[n]$, $n \geq n_0$, contains a monochromatic $AP(m)$. This statement was further generalized to sets of positive upper density in the celebrated work of Szemerédi [Sz75]. Canonical versions of van der Waerden's theorem were discovered by Erdős and others [E87].

Given a coloring of $\mathbb{N}$, a set $S \subseteq \mathbb{N}$ is called *rainbow* if all elements of S are colored with different colors. In [JL+03], Jungić et al. considered a rainbow counterpart of van der Waerden's theorem, and proved that every 3-coloring of $\mathbb{N}$ with the upper density of each color greater than 1/6 contains a rainbow $AP(3)$. Improving on their methods and some extensions [JR03], Axenovich and Fon-Der-Flaass [AF04] proved the following "finite" version of this result.

Theorem 1 (Conjectured in [JL+03], proved in [AF04].) Given $n \geq 3$, every partition of $[n]$ into three color classes $\mathcal{R}$, $\mathcal{G}$, and $\mathcal{B}$ with $\min(|\mathcal{R}|, |\mathcal{G}|, |\mathcal{B}|) > r(n)$, where

$$r(n) := \begin{cases} \lfloor (n+2)/6 \rfloor & \text{if } n \not\equiv 2 \pmod 6 \\ (n+4)/6 & \text{if } n \equiv 2 \pmod 6 \end{cases} \tag{1}$$

contains a rainbow $AP(3)$.

Theorem 1 is the best possible. It is interesting to note that similar statements about the existence of rainbow $AP(k)$ in k-colorings of $[n]$, $k \geq 4$, do not hold [AF04, CJR].

In lay terms, Axenovich and Fon-Der-Flaass showed that sufficiently *large* color classes in a 3-coloring imply the existence of a rainbow $AP(3)$. In this paper, we are interested in conditions that guarantee the existence of rainbow patterns when color classes have *small* cardinality. A notable distinction between these two approaches is that in the latter case the number of colors can be greater than the number of elements in the particular pattern.

This setup was first studied by Alon, Caro and Tuza in [ACT89], where for a given $k \in \mathbb{N}$, they defined sub-k-colorings as colorings in which every color class has size at most k. For given k, $m \in \mathbb{N}$, they introduced the sub-k-Ramsey number $sr(m, k)$ as the minimum integer $n_0 = n_0(m, k)$ such that every sub-k-coloring of $[n]$, $n \geq n_0$, yields a rainbow $AP(m)$. They proved that for every $m \geq 3$, $k \geq 2$,

$$\frac{1}{6} \frac{(k-1)m(m-1)}{\log{(k-1)m}} - k + 1 \leq sr(m, k) \leq (1 + o(1)) \frac{24}{13} (k-1)(m-1)^2 \log{(k-1)(m-1)},$$

where the factor of $1 + o(1)$ approaches 1 as $m \to \infty$. Also, if m is fixed and k grows, they proved that

$$sr(m, k) \leq (1 + o(1))\frac{1}{2}m(m-1)^2(k-1).$$

For $k = 2$, we improve on their lower bound by constructing a coloring that has already been used in [JL+03] to prove a lower bound for a related problem concerning rainbow arithmetic progressions in equinumerous colorings.

Theorem 2 *For $m \geq 3$, $sr(m, 2) > \lfloor \frac{m^2}{2} \rfloor$.*

Motivated by [EH58] and [AC86], Caro [C87] proved that for every positive integer m, there is a minimum integer $n = n_0(m)$ such that for every $\phi : [n] \to [n]$ without a fixed point, there is an $AP(m)$ $\mathcal{A}$ satisfying: $\phi(i) \notin \mathcal{A}$ for $i \in \mathcal{A}$. Moreover, he showed that $\frac{c_1 m^2}{\log m} \leq n_0(m) \leq m^2(\log m)^{\frac{c_2 \log m}{\log \log m}}$ for some absolute constants c_1 and c_2. In [ACT89], Alon et al. applied the same methods they had used to bound $sr(m, k)$ to drastically improve the earlier bounds on $n_0(m)$. They proved that for every m,

$$\frac{m(m-1)}{3 \log m} + O(1) \leq sr(m, 3) - 1 \leq n_0(m) \leq (1 + o(1))\frac{48}{13}m^2 \log m.$$

Since $sr(m, k)$ is an increasing function in both m and k, then in particular, $sr(m, 2) \leq sr(m, 3)$. Therefore, Theorem 2 implies the following improvement on the lower bound for $n_0(m)$ for all m:

Corollary 1 *For all positive integers m, $n_0(m) \geq \lfloor \frac{m^2}{2} \rfloor$.*

Furthermore, we prove the following theorem, which together with the fact that $sr(3, k) = \Omega(k)$ and $sr(m, 2) = \Omega(m^2)$ implies $sr(m, k) = \Omega(m^2 k)$ for all integers m and k with $m > 2$ and $k > 1$.[2]

Theorem 3 *Let $k \geq 3$ and $m \geq 46$ be integers and set $a = \lfloor \frac{k}{3} \rfloor$ and $l = \lfloor \frac{m-1}{9} \rfloor$. Then $sr(m, k) > 3(l^2 + l)a$.*

The exact determination of the asymptotic behavior of $sr(m, k)$ appears to be difficult. In the case of $AP(3)$, i.e. for $m = 3$, the above mentioned upper bounds of Alon et al. [ACT89] yield $sr(3, k) \leq (1 + o(1))6k$. They provided a sharper estimate:

$$\text{as } k \text{ grows, } 2k \leq sr(3, k) \leq (4.5 + o(1))k.$$

In what follows, we use $sr(k)$ to denote the sub-k-Ramsey number $sr(3, k)$. Using methods developed in [JL+03, AF04], we determine $sr(k)$ for $k > 603$.

[2] In the trivial cases, we have $sr(1, k) = 1$, $sr(2, k) = k + 1$, and $sr(m, 1) = m$.

Theorem 4 *For $k \geq 603$, $sr(k)$ is the the least positive integer n such that $k < \frac{8n+\epsilon(n)}{17}$ where $\epsilon(n)$ is defined by*

$n \mod 17$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\epsilon(n)$	0	-8	1	10	2	11	3	-5	4	-4	5	-3	6	-2	7	-1	8

In particular,

$$sr(k) = \frac{17}{8}k + O(1) \ .$$

A set $\{x < y < z\}$ of integers is an arithmetic progression of length three if and only if $x + z = 2y$. Hence, one can define sub-Ramsey problems for other linear equations. A classical candidate is the Schur equation $x+y=z$ [S16]. Arguably, the first result in Ramsey theory is due to Schur, who, in 1916, proved that for every k and sufficiently large n, every k-coloring of $[n]$ contains a monochromatic solution to the equation $x + y = z$. More than seven decades later, building up on the previous work of Alekseev and Savchev, E. and G. Szekeres (see [JL+03] and references therein), Schönheim [S90] proved the following rainbow counterpart, which is clearly an analogue of Theorem 1.

Theorem 5 ([S90]) For every $n \geq 3$, every partition of $[n]$ into three color classes $\mathcal{R}$, $\mathcal{G}$, and $\mathcal{B}$ with $\min(|\mathcal{R}|, |\mathcal{G}|, |\mathcal{B}|) > n/4$, contains a rainbow solution to the equation $x + y = z$. The term $n/4$ cannot be improved.

For a given positive integer k, let $ss(k)$ denote the minimal number such that every coloring of $[n]$, $n \geq ss(k)$, that uses each color at most k times, yields a rainbow solution to the equation $x + y = z$. We prove the following theorem.

Theorem 6 *For all positive integers k, $ss(k) = \lfloor \frac{5k}{2} \rfloor + 1$.*

The paper is organized as follows. In Section 2, we construct a coloring that settles Theorem 2 and hence Corollary 1. In Section 3, we constructively prove Theorem 3. In Section 4, we use Theorem 1 and prove a somewhat surprising claim that, in order to prove good bounds on $sr(k)$, it suffices to only consider sub-k-colorings with three colors. Furthermore, we relate our problem to the problem of finding good bounds on $\sigma(n)$, the minimum integer k such that there is a sub-k-coloring of $[n]$ with three colors and no rainbow $AP(3)$. In Section 5, we provide lower and upper bounds on $\sigma(n)$, which in turn imply Theorem 4. In Section 6, we prove lemmata that together imply Theorem 6. In Section 7, we propose new sub-Ramsey–type problems, while surveying the current state of rainbow Ramsey theory.

2. Proof of Theorem 2

We construct a coloring c of $[\lfloor \frac{m^2}{2} \rfloor]$ that uses each color exactly twice and prove that it does not contain a rainbow $AP(m)$. Define a *j-block* B_j ($j \in \mathbb{N}$) to be the sequence $12\ldots j12\ldots j$, where the *left half* and the *right half* of the block are naturally defined. For $a \in \mathbb{Z}$, let $B_j + a$ be the sequence $(a+1)(a+2)\ldots(a+j)(a+1)(a+2)\ldots(a+j)$. Define $B_j^- = B_j - \binom{j+1}{2}$ and $B_i^+ = B_i + \binom{i}{2}$. If $m = 2l+1$ is odd, define the coloring c of $[2l^2+2l]$ in the following way (bars denote endpoints of the blocks):

$$|B_l^-|\ldots|B_j^-|\ldots|B_2^-||B_1^-||B_1^+||B_2^+|\ldots|B_i^+|\ldots|B_l^+|.$$

If $m = 2l$ is even, define the coloring c of $[2l^2]$ in the following way (bars denote endpoints of the blocks):

$$|B_{l-1}^-|\ldots|B_j^-|\ldots|B_2^-||B_1^-||B_1^+||B_2^+|\ldots|B_i^+|\ldots|B_l^+|.$$

We only show the proof of Theorem 2 in the case when m is odd (since the case when m is even is essentially the same). Note that the coloring c uses each of the l^2+l colors exactly twice (the colors are integers from the interval $[1-\binom{l+1}{2}, \binom{l+1}{2}]$). Now, we show that the coloring c of $[2l^2+2l]$ contains no rainbow $AP(2l+1)$. The key observation is that a rainbow AP with length greater than l and difference d cannot contain elements from opposite halves of any block B_j^- (or B_j^+) where d is a factor of j. Fix a longest rainbow AP $\mathcal{A}$ and let d denote its difference. If $d=1$, then the length of $\mathcal{A}$ is $\le l$. If $d > l$, then the length of $\mathcal{A}$ is $\le 2l$. If $1 < d \le l$, then $\mathcal{A}$ is one of the following three types:

(1) $\mathcal{A}$ is contained in $|B_d^-|\ldots|B_j^-|\ldots|B_2^-||B_1^-||B_1^+||B_2^+|\ldots|B_i^+|\ldots|B_d^+|$. Then $\mathcal{A}$ intersects neither the left half of B_d^- nor the right half of B_d^+. Therefore, the length of $\mathcal{A}$ is at most $1 + \frac{2d^2-1}{d} < 2d+1 \le 2l+1$.

(2) $\mathcal{A}$ is contained in $|B_{(j+1)d}^-||B_{(j+1)d-1}^-|\ldots|B_{jd}^-|$ or in $|B_{jd}^+||B_{jd+1}^+|\ldots|B_{(j+1)d}^+|$, where $(j+1)d \le l$. Assume the first case occurs (both cases are handled the same way). Then $\mathcal{A}$ intersects neither the left half of $B_{(j+1)d}^-$ nor the right half of B_{jd}^-. Therefore, the length of $\mathcal{A}$ is at most

$$1 + \frac{(2j+1)d^2-1}{d} < (2j+1)d + 1 \le 2l+1.$$

(3) $\mathcal{A}$ is contained in $|B_l^-||B_{l-1}^-|\ldots|B_{jd+1}^-||B_{jd}^-|$ or in $|B_{jd}^+||B_{jd+1}^+|\ldots|B_{l-1}^+||B_l^+|$, where $l - jd < d$. We note that $1 < d \le jd \le l$. Assume the first case occurs (both cases are handled the same way). Then $\mathcal{A}$ does not intersect the right half of B_{jd}^-. Therefore, since $jd \ge l-d+1$, the length of $\mathcal{A}$ is at most

$$\begin{aligned} 1 + \frac{1}{d}(l(l+1) - j^2d^2 - 1) &\le 1 + \frac{2ld - l - d^2 + 2d - 2}{d} = 2l + 1 - \frac{l + d^2 - 2d + 2}{d} \\ &< 2l + 1 - \frac{d^2 - d}{d} = 2l + 1 - (d-1) \le 2l\ . \end{aligned}$$

3. Proof of Theorem 3

We construct a coloring c of $[3a(l^2+l)]$ that uses each color exactly $3a$ times and prove that it does not contain a rainbow $AP(9l+1)$. As we did in the proof for the case $k=2$, we construct a block coloring where each color appears in only one block.

For each j, let C_j denote the sequence of aj terms such that the i^{th} term equals $\lceil \frac{i}{a} \rceil$. Notice that C_j consists of j constant strings of length a. For $j \in \mathbb{N}$, let B_j be the sequence of $3aj$ terms that consists of 3 copies of C_j. The *beginning third*, *middle third*, and *last third* of B_j, which are all copies of C_j, are naturally defined. Notice that in the sequence B_j, there are exactly $3a$ terms equal to i for each $i \in [1, j]$.

For $j \in \mathbb{N}$ and $n \in \mathbb{Z}$, we define a block $B_j + n$ as the sequence obtained by adding n to each term of B_j. Define the block sequences $B_j^- = B_j - \binom{j+1}{2}$ and $B_j^+ = B_j + \binom{j}{2}$. Finally, define the coloring c of $[3a(l^2+l)]$ in the following way (bars denote endpoints of the blocks):

$$|B_l^-|\dots|B_j^-|\dots|B_2^-||B_1^-||B_1^+||B_2^+|\dots|B_i^+|\dots|B_l^+|.$$

Note that each color appears in one block only. Since each color is used exactly $3a$ times, then c is a sub-k-coloring. Now, we show that the coloring c contains no rainbow $AP(9l+1)$.

Let $\mathcal{A} = \{x + id | \ i \in [0, s-1]\}$ be a maximal rainbow progression, i.e., if $x - d$ or $x + sd$ belong to $[3a(l^2+l)]$ then they are colored by one of the colors used to color $\mathcal{A}$.

We say that $\mathcal{A}$ *goes through* block B_j^+ (or B_j^-), $j \in [l-1]$, if there are $p, r \in [0, s-1]$ with the property that $\{x + id | \ i \in [p, r]\} \subseteq B_j^+$ and $\{x + (p-1)d, x + (r+1)d\} \cap B_j^+ = \emptyset$.

The key observation is that $\mathcal{A}$ cannot go through any block B_j^- or B_j^+ if $d \le ja$ and a multiple of d belongs to the interval $[(j-\frac{1}{2})a, (j+\frac{1}{2})a]$. Suppose the opposite, let $t \in [(j-\frac{1}{2})a, (j+\frac{1}{2})a]$ be a multiple of d and let $\mathcal{A}$ go through B_j^+ or B_j^-. Without loss of generality, $\mathcal{A}$ goes through B_j^+. Since $d \le ja$, then there is a term $x + id$ of $\mathcal{A}$ that is in the middle third of the block B_j^+, and then either $x + id - t$ or $x + id + t$ is the same color as $x + id$, which contradicts the fact that $\mathcal{A}$ is rainbow.

If $d \le a$, then by the key observation $\mathcal{A}$ cannot go through any block and therefore must lie in two consecutive blocks. Since any two consecutive blocks contain less than $2l$ colors, then the length of $\mathcal{A}$ is less than $2l$.

If $d > a$, then by the key observation, the rainbow AP $\mathcal{A}$ with difference d does not go through any block B_j^+ or B_j^- with $j = \lceil \frac{de}{a} - \frac{1}{2} \rceil$ and e an integer satisfying $e > 1$. So either $\mathcal{A}$ is contained in $\lceil \frac{d}{a} \rceil + 1$ consecutive blocks or lies in

$$|B_b^-|\dots|B_2^-||B_1^-||B_1^+||B_2^+|\dots|B_b^+|,$$

where $b = \min(l, \lceil \frac{2d}{a} - \frac{1}{2} \rceil)$. In the former case, the length of $\mathcal{A}$ is less than $1 + (\lceil \frac{d}{a} \rceil + 1)\frac{3la}{d} < 9l+1$. In the latter case, the length of $\mathcal{A}$ is less than $1 + \frac{2\sum_{j=1}^{b} 3ia}{d} = 1 + \frac{3b(b+1)a}{d} < 1 + \frac{15}{2}(l+1) \le 9l+1$ since $ab < \frac{5d}{2}$, $b + 1 \le l + 1$, and $l \ge 5$ (in view of $m \ge 46$).

4. Proof of Theorem 4: A Reduction to 3-colorings

As we mentioned in the introduction, the number of colors in a sub-k-coloring can be greater than three. In the following lemma we show that it is enough to consider only sub-k-colorings with three colors.

Lemma 1 *Let n, k, $r \in \mathbb{N}$ be such that $n \geq 21$, $k \leq \frac{n}{2} - \frac{13}{6}$, and $r \geq 3$. For every sub-k-coloring c of $[n]$ with r colors and no rainbow $AP(3)$ there exists a sub-k-coloring $\overline{c}$ of $[n]$ with three colors and no rainbow $AP(3)$, such that for all $i, j \in [n]$*

$$c(i) = c(j) \Rightarrow \overline{c}(i) = \overline{c}(j).$$

Proof. Let $C_1, C_2, \ldots, C_r$ be the color classes of a sub-k-coloring c of $[n]$ with $k \leq \frac{n}{2} - \frac{13}{6}$ and $r \geq 3$. Suppose that c contains no rainbow $AP(3)$. Without loss of generality, assume that $|C_1| \geq |C_2| \geq \ldots \geq |C_r|$. Then Theorem 1 implies that $|C_3| \leq \frac{n+4}{6}$. Indeed, otherwise $|C_1| \geq |C_2| > \frac{n+4}{6}$ and $|\cup_{i=3}^{r} C_i| > \frac{n+4}{6}$ imply that there is an $AP(3)$ with terms from C_1, C_2, and C_i for some $i \in [3, r]$.

Suppose $|C_2| \leq \frac{n+4}{6}$. Let $s = \min\left\{j : \left|\cup_{i=1}^{j} C_i\right| > \frac{n+4}{6}\right\}$. If $s = 1$, then $|\cup_{i=1}^{s} C_i| = |C_1| \leq k \leq \frac{n}{2} - \frac{13}{6}$, and if $s > 1$, then $|\cup_{i=1}^{s} C_i| = \left|\cup_{i=1}^{s-1} C_i\right| + |C_s| \leq \frac{n+4}{6} + \frac{n+4}{6} = \frac{n+4}{3}$. In either case, we have $|\cup_{i=1}^{s} C_i| \leq \frac{n}{2} - \frac{13}{6}$. Let $t = \min\left\{j : \left|\cup_{i=s+1}^{j} C_i\right| > \frac{n+4}{6}\right\}$. Since $t \geq 2$ and $|C_2| \leq \frac{n+4}{6}$, we have $\left|\cup_{i=s+1}^{t} C_i\right| \leq \frac{n+4}{3}$. It follows that $|[n] \setminus \cup_{i=1}^{t} C_i| \geq n - \frac{n}{2} + \frac{13}{6} - \frac{n+4}{3} = \frac{n+5}{6}$. Therefore, by Theorem 1, the 3-coloring with color classes $\cup_{i=1}^{s} C_i$, $\cup_{i=s+1}^{t} C_i$, and $[n] \setminus (\cup_{i=1}^{t} C_i)$ yields a rainbow $AP(3)$, that clearly implies the existence of a rainbow $AP(3)$ in the original coloring c. This contradicts our assumptions.

Since $k \geq |C_1| \geq |C_2| > \frac{n+4}{6}$ it follows that $|\cup_{i=3}^{r} C_i| \leq \frac{n+4}{6}$, else Theorem 1 implies there is a rainbow $AP(3)$, a contradiction. Then, we define $\overline{c}$ of $[n]$ to be the 3-coloring given by color classes C_1, C_2, and $\cup_{i=3}^{r} C_i$. Clearly, $\overline{c}$ is a sub-k-coloring with no rainbow $AP(3)$, as required. □

For $n \in \mathbb{N}$, we define $\sigma(n)$ as the minimum positive integer k such that there is a sub-k-coloring of $[n]$ with three colors and no rainbow $AP(3)$.

We will prove in Proposition 2 that

$$\sigma(n) = \frac{8n + \epsilon(n)}{17} \leq \frac{n}{2} - \frac{13}{6}$$

for $n \geq 1280$, where $\epsilon(n)$ is as defined in the statement of Theorem 4.

Note $k < \sigma(sr(k))$ holds trivially, while Lemma 1 implies $sr(k)$ is the minimal such integer, provided $k \leq \frac{sr(k)-1}{2} - \frac{13}{6}$ and $sr(k) - 1 \geq 21$. However, if $sr(k) \geq 1280$, then $k + 1 \leq \sigma(sr(k)) \leq \frac{sr(k)}{2} - \frac{13}{6}$ follows from Proposition 2, whence both these conditions hold.

Finally, if $sr(k) < 1280$, then $k < \sigma(1280)$. Therefore, it follows from Proposition 2 that for $k \geq \sigma(1280) = \frac{8 \cdot 1280+11}{17} = 603$, we have that $sr(k)$ is the least positive integer n such that $k < \sigma(n)$. Hence, Theorem 4 follows from Proposition 2.

5. Proof of Theorem 4: Bounds on $\sigma(n)$

For a given 3-coloring $c : [a, b] \to \{R, B, G\}$ let $\mathcal{R}$, $\mathcal{B}$, and $\mathcal{G}$ denote sets of elements of $[a, b]$ colored with R, B, and G, respectively. First, we determine an upper bound for $\sigma(n)$.

Proposition 1 *For all $n \in \mathbb{N}$, $\sigma(n) \leq \frac{8n+\epsilon(n)}{17} \leq \frac{8n+11}{17}$ where $\epsilon(n)$ is as defined in the statement of Theorem 4.*

Proof. We define a 3-coloring $c : \mathbb{N} \to \{R, G, B\}$ by

$$c(n) = \begin{cases} G & \text{if } n \equiv 0 \pmod{17} \\ R & \text{if } n \equiv 1, 2, 4, 8, 9, 13, 15, 16 \pmod{17} \\ B & \text{if } n \equiv 3, 5, 6, 7, 10, 11, 12, 14 \pmod{17}. \end{cases}$$

The coloring c is periodic with a period 17. We claim that c contains no rainbow $AP(3)$. Otherwise, let $\{i, j, k\}$ be an $AP(3)$ with $i + k = 2j$. If $c(j) = G$, then $i + k \equiv 0 \pmod{17}$, which implies $c(i) = c(k)$. If $c(i) = G$, then $2j \equiv k \pmod{17}$. It is not difficult to check that in this case $c(j) = c(2j) = c(k)$.

It is easily noted what interval of length x, where $0 \leq x < 17$ and $x \equiv n \pmod{17}$, minimizes the maximum number of integers colored by R or B. In fact, in all but the case $x = 3$ and $x = 5$, the estimate given by the pigeonhole principle is attainable. Calling this minimum $y(x)$, it follows that $\sigma(n) \leq \frac{8(n-x)}{17} + y(x)$, and the bound in terms of $\epsilon(n)$ follows by computing $y(x)$. □

Next, we prove a lower bound for $\sigma(n)$. We will do so through a sequence of lemmas. We start with some definitions from [JL+03, JR03]. Given a 3-coloring c of $[n]$ with colors R(ed), B(lue), and G(reen), we say that $X \in \{R, B, G\}$ is a *dominant color* if for every two consecutive elements of $[n]$ that are colored with different colors, one of them is colored with X. We say that $Y \in \{R, B, G\}$ is a *recessive color* if there are no two consecutive elements of $[n]$ colored with Y.

Lemma 2 ([JR03]) In every 3-coloring $c : [n] \to \{R, B, G\}$ with no rainbow $AP(3)$, one of the colors must be dominant and another color must be recessive.

Without loss of generality, let R be a dominant color and let G be a recessive color. The set $g_1 < g_2 < \ldots < g_s$ of all elements of $[n]$ colored by G divide $[n]$ naturally into subsegments, called *blocks*, of the form $I_i = [g_i, g_{i+1} - 1]$, for $1 \le i \le s-1$, $I_s = [g_s, n]$, and, if $g_1 \ne 1$, $I_0 = [1, g_1 - 1]$. Clearly, each block I_i, $1 \le i \le s$, contains a single element colored by G.

Our goal is to show the following.

Proposition 2 *If $n \ge 1280$, then $\sigma(n) = \frac{8n+\epsilon(n)}{17}$.*

If B is a recessive color, then, since R is dominant and G is recessive, in every pair of consecutive integers in $[n]$, at least one of them is color R. This implies that $|\mathcal{R}| \ge \lfloor \frac{n}{2} \rfloor \ge \frac{8n+11}{17}$ for $n \ge 39$. Therefore, in the rest of the proof of Proposition 2, we can assume that B is not a recessive color.

We note that, in this setting, R, a dominant color, cannot be recessive. Otherwise, since all three colors are used, there will be a rainbow $AP(3)$ with difference 1.

Next, we prove that G, the unique recessive color, is sparse.

Lemma 3 $g_{i+1} - g_i > 3$ *for* $1 \le i \le s-1$.

Proof. Suppose there exists $i \in [s-1]$ such that $g_{i+1} = g_i + 2$. Note that the fact that G is recessive and R is dominant implies $c(g_i+1) = R$. Since B is not recessive there exists $j \in [n]$ such that $c(j) = c(j+1) = B$. Fix j so that there is no other occurrence of consecutive elements colored with B between $j+1$ and g_i, if $j+1 < g_i$; or between g_{i+1} and j if $j > g_{i+1}$.

If $g_i \equiv j \pmod 2$, then the following $AP(3)$s: $\{g_i, \frac{g_i+j}{2}, j\}$, $\{g_i+1, \frac{g_i+j}{2}+1, j+1\}$, and $\{g_i+2, \frac{g_i+j}{2}+1, j\}$ are not rainbow, so $c\left(\frac{g_i+j}{2}\right) \in \{G, B\}$ and $c\left(\frac{g_i+j}{2}+1\right) = B$. This contradicts either our choice of j or our assumption that R is the dominant color. If $g_i \not\equiv j \pmod 2$, then the following $AP(3)$s: $\{g_i, \frac{g_i+j+1}{2}, j+1\}$, $\{g_i+1, \frac{g_i+1+j}{2}, j\}$, and $\{g_i+2, \frac{g_i+j+3}{2}, j+1\}$ are not rainbow, so we have that $c\left(\frac{g_i+j+1}{2}\right) = B$ and $c\left(\frac{g_i+j+3}{2}\right) \in \{G, B\}$, which, as above, contradicts our assumptions.

Therefore, $g_{i+1} - g_i > 2$ for all i.

Now, suppose there is $i \in [s-1]$ such that $g_{i+1} = g_i + 3$. Since R is dominant and c has no rainbow $AP(3)$, we have $c(g_i+1) = c(g_i+2) = R$. As above, we choose j with $c(j) = c(j+1) = B$, that is the closest to either g_i from the left or g_{i+1} from the right.

If $g_i \equiv j \pmod 2$, then the following $AP(3)$s: $\{g_i, \frac{g_i+j}{2}, j\}$, $\{g_i+1, \frac{g_i+j}{2}+1, j+1\}$, and $\{g_i+3, \frac{g_i+j}{2}+2, j+1\}$ cannot be rainbow, so we have $c\left(\frac{g_i+j}{2}\right) \in \{G, B\}$, $c\left(\frac{g_i+j}{2}+2\right) \in \{G, B\}$, and $c\left(\frac{g_i+j}{2}+1\right) = R$.[3] Since there are no two elements colored with G that are one place apart and since c has no rainbow AP(3), we have that $c\left(\frac{g_i+j}{2}\right) = \left(\frac{g_i+j}{2}+2\right) = B$.

If $g_i \equiv \frac{g_i+j}{2} \pmod 2$, then from the fact that $\left\{g_i, \frac{g_i+(g_i+j)/2}{2}+1, \frac{g_i+j}{2}+2\right\}$ and $\{g_i+2, \frac{g_i+(g_i+j)/2}{2}+1, \frac{g_i+j}{2}\right\}$ are not rainbow, it follows that $c\left(\frac{g_i+(g_i+j)/2}{2}+1\right) = B$. At the same time, since $\left\{g_i, \frac{g_i+(g_i+j)/2}{2}, \frac{g_i+j}{2}\right\}$ is not rainbow, then $c\left(\frac{g_i+(g_i+j)/2}{2}\right) \in \{G, B\}$. However,

$$\left\{c\left(\frac{g_i+(g_i+j)/2}{2}\right), c\left(\frac{g_i+(g_i+j)/2}{2}+1\right)\right\} \subseteq \{G, B\}$$

contradicts our choice of j or our assumption that R is the dominant color.

If $g_i \not\equiv \frac{g_i+j}{2} \pmod 2$, then the fact that the following $AP(3)$s: $\left\{g_i+3, \frac{g_i+(g_i+j)/2+1}{2}+1, \frac{g_i+j}{2}\right\}$ and $\left\{g_i+3, \frac{g_i+(g_i+j)/2+1}{2}+2, \frac{g_i+j}{2}+2\right\}$ are not rainbow implies that

$$\left\{c\left(\frac{g_i+(g_i+j)/2+1}{2}+1\right), c\left(\frac{g_i+(g_i+j)/2+1}{2}+2\right)\right\} \subseteq \{G, B\},$$

which is a contradiction as above.

If $g_i \not\equiv j \pmod 2$, then the AP(3)s: $\{g_i, \frac{g_i+j+1}{2}, j+1\}$, $\{g_i+1, \frac{g_i+1+j}{2}, j\}$, and $\{g_i+3, \frac{g_i+j+1}{2}+1, j\}$ are not rainbow, so we have $c\left(\frac{g_i+j+1}{2}\right) = B$ and $c\left(\frac{g_i+j+1}{2}+1\right) \in \{G, B\}$, which again contradicts our assumptions.

Therefore, $g_{i+1} - g_i > 3$ for all i. □

Now, we have the following corollaries.

Corollary 2 *If $\{c(k), c(k+2)\} \subseteq \{B, G\}$ for some $k \in [n-2]$, then $c(k) = c(k+2) = B$.*

Corollary 3 *Each block I_i, $1 \le i \le s-1$, is of length of at least four.*

Note that Corollary 2 immediately implies the following property of c, which will be repeatedly used throughout the proof.

Corollary 4 *Every element colored with G is always followed and preceded by the string RR in c.*

In the rest of the proof of Proposition 2, we discuss two cases.

[3]Here, we have also used the definition of j.

Case 1. Each block I_j, $1 \leq j \leq s-1$, contains two consecutive elements colored with B.

We first observe that if I_j contains two consecutive elements colored with B then its size must be greater than 10. This easily follows from Corollary 4 and the fact that the coloring is rainbow $AP(3)$ free.

If $g_j + 3$ is blue then the initial part of I_j must be $GRRBR\_B\_R$, where $\_$ denotes an unknown color. If $g_j + 3$ is red then the initial part of I_j must be $GRRRR\_R\_R$. Because of the symmetry, the final part of I_j must either be $R\_B\_RBRR(G)$ or $R\_R\_RRRR(G)$, where (G) represents g_{j+1}. If the size of I_j is less than 17 then the initial and final parts of the block, as they are shown above, must overlap. This leads to only two possibilities for I_j (if $|I_j| \leq 20$): either I_j is of size 15 and looks like $GRRBRBBRRBBRBRR(G)$ or it is of size 17 and looks like $GRRBRBBBRRBBBRBRR(G)$. Both of these blocks have a very special "self-propagating" property that we use to determine $\mathcal{R}$, $\mathcal{B}$, and $\mathcal{G}$.

We describe this property with the following statement for the first mentioned block (the other case being almost identical and left to the reader).

Lemma 4 *If $c : [15l + r] \to \{B, G, R\}$, $l \geq 1$ and $1 \leq r \leq 15$, is a coloring without rainbow AP(3), with G recessive and R dominant, and such that the first 16 numbers are colored as GRRBRBBRRBBRBRRG, then for any $i \in [l]$ and any $j \in [2, 15]$ with $15i + j \leq 15l + r$, we have $c(15i + j) = c(j)$.*

Proof. Our proof is by induction on l. First, we establish the base case $l = 1$. Since $c(16) = G$, it follows from Corollary 4 that $c(17) = c(18) = R$. The $AP(3)$s $\{13, 16, 19\}$ and $\{11, 15, 19\}$ force $c(19) = B$, which in turn implies $c(20) = R$, due to $AP(3)$s $\{18, 19, 20\}$ and $\{16, 18, 20\}$ not being rainbow. Now, the $AP(3)$s $\{19, 20, 21\}$ and $\{11, 16, 21\}$ are not rainbow, so $c(21) = B$; while the $AP(3)$s $\{20, 21, 22\}$ and $\{16, 19, 22\}$ force $c(22) = B$. Since neither $\{1, 12, 23\}$ nor $\{15, 19, 23\}$ is rainbow, then $c(23) = R$. Continuing in this fashion, $\{22, 23, 24\}$ and $\{16, 20, 24\}$ force $c(24) = R$; while the fact that $\{21, 23, 25\}$ and $\{1, 13, 25\}$ are not rainbow implies $c(25) = B$. Since neither $\{24, 25, 26\}$ nor $\{16, 21, 26\}$ is rainbow, then $c(26) = B$. Further, $c(27) = R$, due to $AP(3)$s $\{23, 25, 27\}$ and $\{1, 14, 27\}$ not being rainbow. Next, the $AP(3)$s $\{26, 27, 28\}$ and $\{16, 22, 28\}$ force $c(28) = B$, which in turn implies $c(29) = R$, because of the $AP(3)$s $\{27, 28, 29\}$ and $\{1, 15, 29\}$. Finally, the $AP(3)$s $\{28, 29, 30\}$ and $\{16, 23, 30\}$ force $c(30) = R$; hence, for all $j \in [2, 15]$, $c(15 + j) = c(j)$, and Lemma 4 is true for $l = 1$.

Now suppose that the claim is true for some $l \geq 1$ and consider a coloring $c : [15(l + 1) + r] \to \{B, G, R\}$ with the properties listed in Lemma 4. By induction hypothesis, for all $i \in [l]$ and $j \in [2, 15]$, $c(15i + j) = c(j)$.

For $j \in [2, r]$, depending on the parity of $(l+1)+j$, either $\{1, \frac{15(l+1)+j+1}{2}, 15(l+1)+j\}$ or $\{16, \frac{15(l+1)+j+16}{2}, 15(l + 1) + j\}$ is an $AP(3)$. Since c is a coloring without rainbow $AP(3)$, it follows that $c(15(l+1)+j) = G$ or $c(15(l+1)+j) = c(j)$. However, assuming $c(15(l+1)+j') =$

$c(j')$ for $2 \le j' < j$, then the observations concerning the structure of the initial part of a block, as given after the start of Case 1, show that $c(15(l+1)+j) \ne G$. □

Now, back to the settings of Case 1; suppose that there is a block I_j of length 15. Going in both directions from that block, from Lemma 4, we see that the coloring of $[n]$ is almost completely determined, repeating the same 14-term sequence of Bs and Rs as described in Lemma 4. Let $r_1 \in [0, 14]$ be such that there is an element s with $c(s) = G$ and $s \equiv r_1 + 1 \pmod{15}$. Let $n = r_1 + 15l + r_2$, where l and r_2 are positive integers with $r_2 \le 15$. Since the 14-term sequence contains 8 Rs and 6 Bs, and at least half of the first r_1 elements and the last $r_2 - 1$ elements are colored by R, we have

$$\max\{\mathcal{R}, \mathcal{B}\} \ge 8l + \frac{r_1 + r_2 - 1}{2} = \frac{8n}{15} - \frac{r_1 + r_2}{30} - \frac{1}{2} \ge \frac{8n}{15} - \frac{43}{30} \ge \frac{8n + 11}{17}$$

for $n \ge 34$. Moreover, since $|I_j| \ge 15$ for all $1 \le j \le s - 1$, we have $s = |\mathcal{G}| < n/15 + 1$.

Since the block $GRRBRBBBRRBBBRBRR(G)$ is self-propagating (in the way described in Lemma 4 for the block $GRRBRBBRRBBRBRR(G)$), we get that if a coloring contains a block of length 17 then

$$\max\{\mathcal{R}, \mathcal{B}\} \ge \frac{8n + \epsilon(n)}{17}$$

where $\epsilon(n)$ is as defined before Proposition 1.

Finally, if each block I_j is of length greater than 20 for all $1 \le j \le s - 1$, we have $s = |\mathcal{G}| < \frac{n}{21} + 1$ and

$$\max\{|\mathcal{R}|, |\mathcal{B}|\} > \frac{n - \frac{n}{21} - 1}{2} = \frac{10n}{21} - \frac{1}{2} \ge \frac{8n + 11}{17}$$

for $n \ge 205$.

Case 2. There is a block with no two consecutive numbers colored with the non-recessive color B.

Suppose I_j, $0 \le j \le s$, is the first block that contains two consecutive elements colored with B. Let $m \in I_j$ denote the smallest number k in I_j such that $c(k) = c(k+1) = B$. Next, we show that there cannot be three elements colored with G both before and after m.

Lemma 5 *If $m > g_3$, then $m > g_{s-2}$.*

Proof. Suppose this is not true and let $g_3 < m < g_{s-2}$. Then, there are u, v, x, and y such that $g_u < g_v < m < g_x < g_y$, $g_u \equiv g_v \pmod 2$, and $g_x \equiv g_y \pmod 2$.

If $2m - g_v + 2 \le n$, then $\{g_v, m, 2m - g_v\}$ and $\{g_v, m + 1, 2m - g_v + 2\}$ are $AP(3)$s that are not rainbow, and we have $\{c(2m - g_v), c(2m - g_v + 2)\} \subseteq \{G, B\}$. From Corollary 2 it follows that $c(2m - g_v) = c(2m - g_v + 2) = B$. Since $\{g_u, (2m - g_v + g_u)/2, 2m - g_v\}$ and $\{g_u, (2m - g_v + g_u + 2)/2, 2m - g_v + 2\}$ are $AP(3)$s that are not rainbow, it follows that $c((2m - g_v + g_u)/2) = c((2m - g_v + g_u)/2 + 1) = B$. However, since $g_u < g_v$, we have that $(2m - g_v + g_u)/2 < m$, which contradicts our choice of m. Therefore, $2m - g_v + 2 > n$.

If $2m - g_y \geq 1$, then both $2m - g_y$ and $2m - g_y + 2$ must be blue, whence $\frac{2m-g_y+g_x}{2} < m$ and $\frac{2m-g_y+g_x}{2} + 1$ must also both be blue (by the same arguments as used in the first part of the proof), which will contradict the minimality of m. Otherwise, $2m \leq g_y$, which combined with $2m - g_v \geq n - 1$, implies $n + 1 \leq n - 1 + g_v \leq g_y \leq n$, a contradiction. □

Case 2 naturally breaks into two subcases: (1) $m > g_3$, and (2) $m < g_3$.

First we deal with (1).

Let g_v be as defined in the proof of Lemma 5. The following lemma shows that B, although a non-recessive color, is sparse after m.

Lemma 6 *For every* $k \in [n-3]$, $\{c(k), c(k+1), c(k+2), c(k+3)\} \cap \{R\} \neq \emptyset$.

Proof. Suppose there exists $k \in [n-3]$ such that $c(k) = c(k+1) = c(k+2) = c(k+3) = B$. Let $k' \in \{k, k+1\}$ be such that $g_v \equiv k' \pmod 2$. Then $c\left(\frac{g_v+k'}{2}\right) = c\left(\frac{g_v+k'}{2} + 1\right) = B$. From the proof of Lemma 5, we have $2m - g_v + 2 > n$. From $k' \leq n - 3 < 2m - g_v + 2 - 3$, it follows that $\frac{g_v+k'}{2} < m$, which contradicts our choice of m.

We note that if $G \in \{c(k), c(k+1), c(k+2), c(k+3)\}$, then since all occurrences of G are preceded and followed by a string RR, it follows that $\{c(k), c(k+1), c(k+2), c(k+3)\} \cap \{R\} \neq \emptyset$. □

In order to prove the lower bound on $\sigma(n)$, claimed in Proposition 2, we need to dig deeper into the structure of the coloring c.

Lemma 7 $m \geq 2g_j - 1$.

Proof. Suppose $m < 2g_j - 1$. Then, $2g_j - m$, $2g_j - m - 1 \in [m]$, and $\{c(2g_j - m), c(2g_j - m - 1)\} \subseteq \{B, G\}$. Since R is dominant and G is recessive, we have $c(2g_j - m) = c(2g_j - m - 1) = B$, which is impossible because of our choice of m. □

Lemma 8 $|\{k \in [g_j + 1, 2g_j - 1] : c(k) = R\}| \geq |\{k \in [g_j - 1] : c(k) = R\}|$.

Proof. For every $k \in [g_j - 1]$ with $c(k) = R$, the element $2g_j - k$ of $[g_j + 1, 2g_j - 1]$ is colored with R, since the $AP(3)$ $\{k, g_j, 2g_j - k\}$ is not rainbow, and $[g_j + 1, 2g_j - 1] \subset I_j$ by Lemma 7. □

Since R is dominant and G is recessive and since there are no consecutive blue integers in $[2g_j-1, m-1]$ and since none of these integers is colored green (except possibly the integer 1 in the case $2g_j-1=g_j=1$), we obtain $|\{k\in[2g_j-1,m-1]:c(k)=R\}|\geq\frac{m-2g_j+1}{2}$. Furthermore, from Lemma 6, since both m and $m+1$ are colored B, it follows that $|\{k\in[m+2,n]:c(k)=R\}|\geq\frac{n-(m+2)}{4}$.

If $c(2g_j-1)\neq R$, using Lemma 8, we get:

$$|\mathcal{R}|\geq 2|\{k\in[g_j-1]:c(k)=R\}|+\frac{m-2g_j+1}{2}+\frac{n-m-2}{4},$$

which by Lemma 7 becomes:

$$|\mathcal{R}|\geq 2|\{k\in[g_j-1]:c(k)=R\}|+\frac{n}{4}-\frac{g_j}{2}-\frac{1}{4}.$$

If $c(2g_j-1)=R$ then the bound from Lemma 7 becomes strict and we consider the intervals $[1,2g_j-1]$, $[2g_j,m-1]$, and $[m+2,n]$ to get

$$|\mathcal{R}|\geq 2|\{k\in[g_j-1]:c(k)=R\}|+\frac{m-2g_j}{2}+\frac{n-m-2}{4},$$

which by the improved bound from Lemma 7 becomes:

$$|\mathcal{R}|\geq 2|\{k\in[g_j-1]:c(k)=R\}|+\frac{n}{4}-\frac{g_j}{2}-\frac{1}{2}.$$

By Corollary 3, each block I_i, $1\leq i\leq j-1$, has length at least four. Moreover, each block starts and ends with the string GRR or RR respectively, as observed in Corollary 4. Now, the definition of m implies

$$|\{k\in I_i:c(k)=R\}|\geq\frac{|I_i|}{2}+1,$$

for all $i\in[j-1]$, where $|I_i|$ denotes the length of the block I_i. Similarly, since $m>g_3$, $|\{k\in I_0:c(k)=R\}|\geq\frac{|I_0|}{2}$. Summing up these inequalities, we get

$$|\{k\in[g_j-1]:c(k)=R\}|=\sum_{i=0}^{j-1}|\{k\in I_i:c(k)=R\}|\geq\frac{g_j-1}{2}+(j-1),$$

since $\sum_{i=0}^{j-1}|I_i|=g_j-1$. Therefore,

$$|\mathcal{R}|\geq\frac{n}{4}+\frac{g_j}{2}+2j-\frac{7}{2}.$$

Since each block I_i, $1\leq i\leq j-1$, has length at least four, we have $g_j\geq 4j-3$. Thus, $|\mathcal{R}|\geq\frac{n}{4}+4j-5$. By Lemma 5, we have $j\geq s-2$ and $|\mathcal{R}|\geq\frac{n}{4}+4s-13$. Hence,

$$\max\{|\mathcal{R}|,|\mathcal{B}|\}\geq|\mathcal{R}|\geq\frac{n}{4}+4|\mathcal{G}|-13\geq\frac{n}{4}+4(n-2\max\{|\mathcal{R}|,|\mathcal{B}|\})-13.$$

It follows from here that

$$\max\{|\mathcal{R}|, |\mathcal{B}|\} \geq \frac{17n}{36} - \frac{13}{9} \geq \frac{8n+11}{17}$$

for $n \geq 1280$. Finally, we deal with the remaining subcase (2).

Let $m < g_3$. Let $t = \max\{k : c(k) = c(k+1) = B\}$. If $t < g_{s-2}$, then we apply the argument for the previous subcase to the coloring $\bar{c} : [n] \to \{R, B, G\}$ defined by $\bar{c}(i) = c(n+1-i)$. Let $r \in [s-2, s]$ be the greatest integer with the property that $t \geq g_r$. We need the following lemma.

Lemma 9 *Suppose $c(u) = c(u+1) = B$, $c(v) = c(x) = G$, and $c(y) = c(y+1) = B$, where $u < v < x < y$ are integers in $[n]$. Then, there are two consecutive elements in $[v+1, x-1]$ colored with B.*

Proof. Let $u' = \max\{k < v : c(k) = c(k+1) = B\}$, and $y' = \min\{k > x : c(k) = c(k+1) = B\}$. Note that $u' \geq u$ and $y' \leq y$. Without loss of generality, we can assume that $v - u' - 1 \leq y' - x$. Clearly, arithmetic progressions $\{u', v, 2v - u'\}$ and $\{u'+1, v, 2v-u'-1\}$ are not rainbow which implies, by Corollary 2, $c(2v-u'-1) = c(2v-u') = B$. If $2v-u' < x$, we have completed the proof. Otherwise, we have $2v - u' = (v - u' - 1) + (v+1) \leq (y' - x) + x = y'$, which contradicts our definition of y'. □

Thus, given two blocks, both with pairs of consecutive numbers colored with B, there is a block between them with a pair of consecutive numbers colored with B. This immediately implies that each of the blocks $I_j, I_{j+1}, \ldots, I_r$ contains a pair of consecutive numbers colored with B. Based on Case 1, we conclude that each of these blocks has length at least 21. From $|\mathcal{G}| \leq 1 + (r - j + 1) + 2 \leq 3 + \frac{n}{21}$, we get

$$\max\{|\mathcal{R}|, |\mathcal{B}|\} \geq \frac{n - \frac{n}{21} - 3}{2} = \frac{10n}{21} - \frac{3}{2} \geq \frac{8n+11}{17}$$

for $n \geq 384$.

Therefore for $n \geq 1280$, $\sigma(n) \geq \frac{8n+\epsilon(n)}{17}$, which with Proposition 1 completes the proof of Proposition 2.

6. Proof of Theorem 6

We call a coloring of $[n]$ *rainbow Schur-free* if it does not contain any rainbow solutions to equation $x + y = z$. In order to show the lower bound $ss(k) > \lfloor \frac{5k}{2} \rfloor$, we define the coloring $c : [n] \to \{R, B, G\}$ as follows:

$$c(i) := \begin{cases} R & \text{if } i \equiv 1 \text{ or } 4 \pmod 5 \\ B & \text{if } i \equiv 2 \text{ or } 3 \pmod 5 \\ G & \text{if } i \equiv 0 \pmod 5 \end{cases}$$

Clearly, c is rainbow Schur-free and each color class has at most $\lceil \frac{2n}{5} \rceil$ elements.

Now, let c denote an arbitrary rainbow Schur-free coloring of $[n]$. In the rest of the section, we establish properties of c that imply that one of the color classes has size at least $\frac{2n}{5}$. The tight upper bound $ss(k) \leq \lfloor \frac{5k}{2} \rfloor + 1$ immediately follows. Recall that in a coloring of $[n]$, a color X is called dominant if for every two consecutive integers with different colors, one of them is colored with X. Note that in every coloring that uses at least three colors, there is at most one dominant color. Also, recall that a color Y is called recessive if no two consecutive elements of $[n]$ receive color Y.

By the pigeonhole principle, we may assume that c uses at least three colors; so there is at most one dominant color. In fact, it is easy to conclude that color $R := c(1)$ is the unique dominant color. Indeed, if $c(1)$ is not dominant, then there exist integers i and $i+1$ such that the colors $c(1)$, $c(i)$, and $c(i+1)$ are all different. However, the set $\{1, i, i+1\}$ is then a rainbow solution to $x + y = z$, which contradicts our assumption on c. Furthermore, if all the colors that are not dominant are recessive, then for every pair of consecutive integers $1 \leq j < j+1 \leq n$, we have $c(j) = R$ or $c(j+1) = R$. Hence, the there are at least $\frac{n}{2} > \frac{2n}{5}$ elements colored with (the dominant color) R. Therefore, we may assume that at least one color in c is neither dominant nor recessive. As the following lemma shows, this color is necessarily unique as well.

Lemma 10 *There is at most one color neither dominant nor recessive.*

Proof. Suppose there are (at least) two colors in c that are not dominant and not recessive. Let $i, i+1, \ldots, i+k$ be the longest string of consecutive integers colored with such a color, which we denote by Y. Let $j, j+1$ be a string of two consecutive elements colored with Z, where Z denotes a non-dominant and non-recessive color other than Y. There are two possible cases depending on which of these two monochromatic strings comes first.

If $i + k < j$, then none of the integers in the string $j-i-k, j-i-k+1, \ldots, j-i+1$ can receive the dominant color R. Hence, all of them receive the same color, which is not dominant and is not recessive. However, the length of this string is $k+2$, which contradicts our choice of the string $i, i+1, \ldots, i+k$.

Similarly, if $i > j+1$, then none of the integers in the string $i-j-1, i-j, \ldots, i-j+k$ can receive the dominant color R. Hence, all of them receive the same color, which is not dominant and is not recessive. However, the length of this string is $k+2$, which again contradicts our choice of the string $i, i+1, \ldots, i+k$. □

Let B denote the unique color in c which is neither dominant nor recessive. Let N_c be the number of elements of $[n]$ that are not colored with R or B. Thus, these integers receive a non-dominant color that is recessive. As in Lemma 1, we can limit our consideration to 3-colorings. Define the 3-coloring $\bar{c}$ by $\bar{c}(i) = c(i)$, if $c(i) = R$ or B, and $\bar{c}(i) = G$ otherwise. We note that, for the coloring $\bar{c}$, R is dominant, B is neither dominant nor recessive and, by Lemma 10, G is recessive. Let $\mathcal{G} = \{g : g \in [n], \bar{c}(g) = G\}$. Then $\bar{c}$ is a rainbow Schur-free coloring of $[n]$ and $|\mathcal{G}| = N_c$. For $1 \le i \le |\mathcal{G}|$, let g_i denote the i^{th} smallest element of $\mathcal{G}$. Let $\mathcal{B} = \{b : b \in [n-1], c(b) = B, c(b+1) = B\}$. For $1 \le i \le |\mathcal{B}|$, let b_i denote the i^{th} smallest element of $\mathcal{B}$. If $b_1 > g_1$, then $c(b_1 - g_1) \ne R$ and $c(b_1 + 1 - g_1) \ne R$, so $b_1 - g_1 \in \mathcal{B}$ and $b_1 - g_1 < b_1$, a contradiction. Hence, $b_1 < g_1$. Since $c(g_1 - 1) = R$, then $1 < b_1 < b_1 + 1 < g_1 - 1 < g_1$, so $g_1 \ge 5$.

Next, we show that for $1 \le i \le |\mathcal{G}| - 1$, there exists $b' \in \mathcal{B}$ such that $g_i < b' < g_{i+1}$. Since $b_1 < g_1 \le g_i$, then there exists a largest element $b \in \mathcal{B}$ such that $b < g_i$. Since $c(g_i - b) \ne R$ and $c(g_i - b - 1) \ne R$, then $g_i - b - 1 \in \mathcal{B}$. However, then $c(g_{i+1} - (g_i - b)) \ne R$ and $c(g_{i+1} - (g_i - b - 1)) \ne R$, which implies that $b + g_{i+1} - g_i \in \mathcal{B}$. Since b is the largest element in $\mathcal{B}$ that is less than g_i, we have $b + g_{i+1} - g_i > g_i$. Defining $b' = b + g_{i+1} - g_i$, we obtain $b' \in \mathcal{B}$ such that $g_i < b' < g_{i+1}$.

Now, clearly, $c(g_i + 1) = c(g_{i+1} - 1) = R$, so $g_i < g_i + 1 < b' < b' + 1 < g_{i+1} - 1 < g_{i+1}$. Therefore, $g_{i+1} - g_i \ge 5$ for $1 \le i \le |\mathcal{G}| - 1$. Since $g_1 \ge 5$, then $|\mathcal{G}| \le \frac{n}{5}$. It immediately follows that in the coloring $\bar{c}$, as well as in c, we have at least $\frac{2n}{5}$ elements colored with R or B. We have completed the proof of Theorem 6.

7. Conclusion

We believe that our methods cannot be used for improving the upper bounds on $sr(m, k)$ in [ACT89], when $m > 3$. The main obstacle is the fact that there is no analogue of Theorem 1 for m-term arithmetic progressions, $m \ge 4$ (as shown in [AF04] for $m \ge 5$, and [CJR] for $m = 4$), that could be used as in Lemma 1.

Fox et al. [FMR] consider yet another partition-regular[4] equation, "the Sidon equation" $x + y = z + w$, which is a classical object in combinatorial number theory. They proved the following.

Theorem 7 ([FMR]) For every $n \ge 4$, every partition of $[n]$ into four color classes $\mathcal{R}$, $\mathcal{G}$, $\mathcal{B}$, and $\mathcal{Y}$, such that

$$\min\{|\mathcal{R}|, |\mathcal{B}|, |\mathcal{G}|, |\mathcal{Y}|\} > \frac{n+1}{6}$$

contains a rainbow solution of $x + y = z + w$. Moreover, this result is tight.

[4]For the definition of partition regularity, please refer to [GRS90].

For a given positive integer k, let $sd(k)$ denote the minimal number such that every coloring of $[n]$, $n \geq sd(k)$, that uses each color at most k times, yields a rainbow solution to equation $x + y = z + w$. We propose the following open problem.

Problem 1 *Determine $sd(k)$.*

We hope one could use Theorem 7 to prove a lemma similar to Lemma 1 and reduce Problem 1 to studying the minimal size of the largest color class in 4-colorings of $[n]$ without rainbow solutions to the above equations. Some structural results about such colorings are already provided in [FMR].

It is interesting to note that there are still no other existential rainbow-type results for partition regular equations other than the ones mentioned above. We are nowhere near the rainbow Rado-type characterization. For numerous open problems concerning the existence of rainbow subsets of integers in appropriate colorings of $[n]$ or $\mathbb{N}$, please refer to the survey [JRN05].

Both rainbow-Ramsey and sub-Ramsey problems have received considerable attention in graph theory. The sub-Ramsey number of a graph G, denoted by $sr(G, k)$, is the smallest integer n such that every edge-coloring of K_n, where each color is used at most k times, contains a rainbow subgraph isomorphic to G. Hell and Montellano [HM04] improved the bounds of Alspach et al. [AG+86], and proved that $sr(K_m, k)$ is $O(km^2)$ and $\Omega(m^{3/2})$. Hahn and Thomassen [HT86] show that $sr(P_m, k) = sr(C_m, k) = m$, when m is large enough with respect to k.[5] Results on sub-Ramsey number of stars and some other results dealing with existence of rainbow subgraphs in colorings with bounded color classes can be found in [AJMP03, ENR83, FHS87, FR93, LRW96].

Remark: After this work was originally submitted for publication, it came to our attention that Theorem 4 has been independently obtained by Maria Axenovich and Ryan Martin in [AM0x].

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References

[AC86] N. Alon, Y. Caro: Extremal problems concerning transformations of the set of edges of the complete graph, *Europ. J. Comb.* **7** (1986), 93–104.

[ACT89] N. Alon, Y. Caro, Zs. Tuza: Sub-Ramsey numbers for arithmetic progressions, *Graphs and Combinatorics* **5** (1989), 307–314.

[5] P_m and C_m denote the path and the cycle with m vertices, respectively.

[AJMP03] N. Alon, T. Jiang, Z. Miller and D. Pritikin: Properly colored subgraphs and rainbow subgraphs in edge-colorings with local constraints, *Random Structures and Algorithms* **23** (2003), 409-433.

[AG+86] B. Alspach, M. Gerson, G. Hahn, P. Hell: On sub-Ramsey numbers, *Ars Combinatoria* **22** (1986), 199–206.

[AM0x] M. Axenovich, R. Martin: Sub-Ramsey numbers for arithmetic progressions, to appear in Graphs and Combinatorics.

[AF04] M. Axenovich, D. Fon-Der-Flaass: On rainbow arithmetic progressions, *Electronic Journal of Combinatorics* **11** (2004), R1.

[C87] Y. Caro: Extremal problems concerning transformations of the edges of the complete hypergraphs, *J. Graph theory* **11** (1987), 25–37.

[CJR] D. Conlon, V. Jungić, R. Radoičić: On the existence of rainbow 4-term arithmetic progressions, manuscript.

[E87] P. Erdős: My joint work with Richard Rado, in *C. Whitehead (ed.) Surveys in Combinatorics 1987*, London Math. Soc. Lecture Notes Ser. Vol. 123, 53–80. Cambridge University Press 1987.

[EH58] P. Erdős, A. Hajnal: On the structure of set mappings, *Acta Math. Acad. Sci. Hung.* **9** (1958), 111–131.

[ENR83] P. Erdős, J. Nešetřil, V. Rödl: Some problems related to partitions of edges of a graph, in *Graphs and other combinatorial topics*, Teubner, Leipzig 1983, 54–63.

[FMR] J. Fox, M. Mahdian, R. Radoičić: Rainbow solutions to the Sidon equation, manuscript.

[FHS87] P. Fraisse, G. Hahn, D. Sotteau: Star sub-Ramsey numbers, *Annals of Discrete Mathematics* **34** (1987), 153–163.

[FR93] A. Frieze, B. Reed: Polychromatic Hamilton cycles, *Discrete Math.* **118** (1993), 69–74.

[GRS90] R. L. Graham, B. L. Rothschild, J. H. Spencer: Ramsey Theory, *John Wiley and Sons* 1990.

[HT86] G. Hahn, C. Thomassen: Path and cycle sub-Ramsey numbers and an edge-colouring conjecture, *Discrete Math.* **62** (1986), 29–33.

[HM04] P. Hell, J. J. Montellano: Polychromatic cliques, *Discrete Math.* **285** (2004), 319–322.

[JL+03] V. Jungić, J. Licht (Fox), M. Mahdian, J. Nešetřil, R. Radoičić: Rainbow arithmetic progressions and anti-Ramsey results, *Combinatorics, Probability, and Computing - Special Issue on Ramsey Theory* **12** (2003), 599–620.

[JR03] V. Jungić, R. Radoičić: Rainbow 3-term arithmetic progressions, *Integers, The Electronic Journal of Combinatorial Number Theory* **3** (2003), A18.

[JRN05] V. Jungić, J. Nešetřil, R. Radoičić: Rainbow Ramsey theory, *Integers, The Electronic Journal of Combinatorial Number Theory*, Proceedings of the Integers Conference 2003 in Honor of Tom Brown, **5(2)** (2005), A9.

[LRW96] H. Lefmann, V. Rödl, B. Wysocka: Multicolored subsets in colored hypergraphs, *Journal of Combinatorial Theory, Series A* **74** (1996) 209-248.

[S90] J. Schönheim: On partitions of the positive integers with no x, y, z belonging to distinct classes satisfying $x + y = z$, in *R. A. Mollin (ed.) Number theory (Proceedings of the First Conference of the Canadian Number Theory Association, Banff 1988)*, de Gruyter (1990), 515–528.

[S16] I. Schur: Über die Kongruenz $x^m + y^m \equiv z^m \bmod p$, *Jahresb. Deutsche Math. Verein* **25** (1916), 114–117.

[Sz75] E. Szemerédi: On sets of integers containing no k elements in arithmetic progression, *Acta Arithmetica* **27** (1975), 199–245.

[vW27] B. L. van der Waerden: Beweis einer Baudetschen Vermutung, *Nieuw Archief voor Wiskunde* **15** (1927), 212–216.