INTEGERS: ELECTRONIC JOURNAL OF COMBINATORIAL NUMBER THEORY 5(3) (2005), #A12

FILTERS AND SUBGROUPS ASSOCIATED WITH HARTMAN MEASURABLE FUNCTIONS

Gabriel Maresch [1, 2]
Institute of Discrete Mathematics and Geometry,
Technical University Vienna, 1040 Vienna, Austria
gabriel.maresch@tuwien.ac.at

Received: 1/20/05, Revised: 6/8/05, Accepted: 8/12/05, Published: 9/8/05

Abstract

A bounded function $\varphi : G \to \mathbb{C}$ on an LCA group G is called Hartman measurable if it can be extended to a Riemann integrable function $\varphi^* : X \to \mathbb{C}$ on some group compactification (ι_X, X), i.e. on a compact group X such that $\iota_X : G \to X$ is a continuous homomorphism with image $\iota_X(G)$ dense in X and $\varphi = \varphi^* \circ \iota_X$. The concept of Hartman measurability of functions is a generalization of Hartman measurability of sets, which was introduced - with different nomenclature - by S. Hartman to treat number theoretic problems arising in diophantine approximation and equidistribution. We transfer certain results concerning Hartman sets to this more general setting. In particular we assign to each Hartman measurable function φ a filter $\mathcal{F}(\varphi)$ on G and a subgroup $\Gamma(\varphi)$ of the dual $\hat{G}$ and show how these objects encode information about the involved group compactification. We present methods how this information can be recovered.

Key words: almost periodic function, Hartman measurable function, Hartman set, filter, Fourier series.

2000 Mathematics Subject Classification: 37A45, 43A60, (11K70)

[1]The author would like to thank the FWF for financial support through grant S8312
[2]This paper is also available on arXiv.org

1 Introduction

1.1 Motivation:

In [5] the investigation of finitely additive measures in number theoretic context led to the concept of a Hartman measurable subset $H \subseteq G$ of a discrete abelian group G. By definition, H is Hartman measurable if it is the preimage $H = \iota_X^{-1}(M)$ of a continuity set $M \subseteq X$ in a group compactification (ι_X, X) of G. This, more explicitly, means that $\iota_X : G \to C$ is a group homomorphism with $\iota(G)$ dense in the compact group X and that $\mu_X(\partial M) = 0$. Here ∂M denotes the topological boundary of M and μ_X the normalized Haar measure on X. By putting $m_G(H) = \mu_X(M)$ one can define a finitely additive measure on the Boolean set algebra of all Hartman measurable sets in G. For the special case $G = \mathbb{Z}$ a Hartman set $H \subseteq \mathbb{Z}$, by identification with its characteristic function, can be considered to be a two-sided infinite binary sequence, called a Hartman sequence. Certain number theoretic, ergodic and combinatorial aspects of Hartman sequences have been studied in [9] and [10], while [12] presents a method to reconstruct the group compactification (ι_X, X) for given H.

In order to benefit from powerful tools from functional and harmonic analysis it is desirable to study appropriate generalizations of Hartman measurable sets by replacing their characteristic functions by complex valued functions not only taking the values 0 and 1. The natural definition of a Hartman measurable function $\varphi : G \to \mathbb{C}$ is the requirement $\varphi^* \circ \iota_X$, where (ι_X, X) is a group compactification of G and φ^* is integrable in the Riemann sense, i.e. its points of discontinuity form a null set with respect to the Haar measure on X. This definition is equivalent to the one of R-almost periodicity, introduced in [6] by S. Hartman. The investigation of the space $\mathcal{H}(G)$ of all Hartman measurable functions on G is the content of [8]. Here we are going to transfer ideas from [12] into this context. Thus our main topic is to describe (ι_X, X) only in terms of φ. In particular we establish further connections to Fourier analysis. The natural framework for our investigation is that of LCA (locally compact abelian) groups.

1.2 Content of the paper

After the introduction we collect in section 2 the necessary preliminary definitions and facts about Hartman measurable sets and functions.

Section 3 treats the following situation: Given a Hartman measurable function $\varphi : G \to X$ on an LCA group G, we know by the very definition of Hartman measurability that there is some group compactification (ι_X, X) of G such that $\varphi = \varphi^* \circ \iota_X$ for some Riemann integrable function $\varphi^* : X \to \mathbb{C}$. We say that φ can be realized in (ι_X, X) resp. by φ^*. It is easy to see that in this case φ can be realized as well on any "bigger" compactification $(\iota_{\tilde{X}}, \tilde{X})$. The notion of "bigger" and "smaller" is made more precise in the next section.

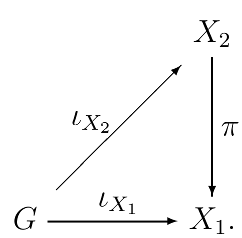
In particular every Hartman measurable function can be realized in the maximal group compactification of G, the Bohr compactification (ι_b, bG). The question arises if there is a realization of φ in a group compactification that is as "small" as possible. If a Hartman measurable function φ possesses a so called aperiodic realization then the group compactification on which this aperiodic realization can be obtained is minimal (Theorem 1). This approach works for arbitrary Hartman measurable φ if one allows "almost realizations", i.e. if one demands $\varphi = \varphi^* \circ \iota_X$ almost everywhere with respect to the finitely additive Hartman measure m_G on G rather than $\varphi = \varphi^* \circ \iota_X$ everywhere on G (Theorem 2). Whenever φ is even almost periodic one can guarantee $\varphi = \varphi^* \circ \iota_X$ everywhere on G. The group compactification on which the minimal realization of φ occurs is unique up to equivalence of group compactifications. It can be obtained by a method involving filters on G similar to that presented in [12].

The content of section 4 is motivated by the following reasoning: Every group compactification of the LCA group G corresponds to a (discrete) subgroup Γ of the dual $\hat{G}$ in such a way that it is equivalent to the group compactification (ι_Γ, C_Γ) defined by $\iota_\Gamma : g \to (\chi(g))_{\chi \in \Gamma}$, $C_\Gamma := \overline{\iota_\Gamma(G)} \leq \mathbb{T}^\Gamma$. If (ι_X, X) is a group compactification admitting an aperiodic almost realization of the Hartman measurable function φ, the corresponding subgroup $\Gamma \leq \hat{G}$ contains all characters χ such that the corresponding Fourier coefficient $m_G(\varphi \cdot \overline{\chi})$ does not vanish. If φ is almost periodic or if φ can be realized on a finite dimensional compactification this result is sharp in the sense that the subgroup Γ is minimal with the above property (Theorem 3). For general Hartman measurable functions the situation is more difficult. This is discussed and illustrated by an example.

Section 5 summarizes the main results and includes an illustrating diagram.

2 Preliminaries and Notation

Throughout this paper G denotes always an LCA (locally compact abelian) group. For group compactifications of G let us write $(\iota_{X_1}, X_1) \leq (\iota_{X_2}, X_2)$ iff there is a continuous group homomorphism $\pi : X_2 \to X_1$ such that the diagram



commutes. In this situation we say that (ι_{X_1}, X_1) is covered by (ι_{X_2}, X_2). If (ι_{X_1}, X_1) is covered by (ι_{X_2}, X_2) (via π_1) and (ι_{X_2}, X_2) is covered by (ι_{X_1}, X_1) (via π_2) then (ι_{X_1}, X_1) and (ι_{X_2}, X_2) are called equivalent. In this case compactness of X_1 and X_2 implies that π_1 and π_2 are both topological and algebraic isomorphisms between X_1 and X_2. " $\leq$ " is a partial order on the class of group compactifications modulo equivalence. The maximal element with respect to this order is (ι_b, bG), the Bohr compactification of the topological group G. Recall that $AP(G)$, the set of almost periodic functions on G, is isometrically isomorphic to $C(bG)$, the set of continuous functions on bG. The mapping $\iota_b^* : C(bG) \to AP(G)$, defined via $f \mapsto f \circ \iota_b$, is an isometry. Note that this is just a different way to characterize those continuous functions on G, which can be extended to continuous functions on bG. This definition (which is best suited for our purposes) is equivalent to the notion of almost periodicity established by Bohr resp. Bochner.

For a locally compact abelian (LCA) group G let us denote by $\hat{G}$ the set of all continuous homomorphisms $\chi : G \to \mathbb{T} \cong \mathbb{R}/\mathbb{Z}$. We will occasionally identify $\mathbb{T}$ with the unimodular group $\{z \in \mathbb{C} : |z| = 1\}$. This will cause no confusion.

$\hat{G}$ endowed with the compact-open topology is an LCA group in its own right. $\hat{G}$ is the Pontryagin dual of G. Every subset $\Gamma \subseteq \hat{G}$ induces a group compactification (ι_Γ, C_Γ) of G via $\iota_\Gamma(g) := (\chi(g))_{\chi\in\Gamma} \in \mathbb{T}^\Gamma$ and $C_\Gamma := \overline{\iota_\Gamma(G)} \leq \mathbb{T}^\Gamma$

One can show that every group compactification (ι_X, X) of an LCA group G is equivalent to the group compactification induced by the subgroup $\{\iota_X \circ \eta : \eta \in \hat{X}\} \leq \hat{G}$ (Theorem 26.13 in [7]). Thus group compactifications of LCA groups can be described by subgroups of the dual and vice versa.

The system $\Sigma(G) \subseteq \mathfrak{P}(G)$ of all Hartman measurable sets on G, i.e. the system of all preimages $\iota_b^{-1}(M)$ of μ_b-continuity sets in the Bohr compactification (ι_b, bG) of G, is a Boolean set algebra and enjoys the property that there exists a unique translation invariant finitely additive probability measure m_G on $\Sigma(G)$: $m_G(\iota_X^{-1}(M^*)) = \mu_X(M^*)$. For details we refer to [5].

Let us denote by Δ the symmetric difference of sets and by τ_g the translation operator on an abelian group defined by $\tau_g(h) := g + h$. We introduce two mappings:

- for a Hartman measurable set M denote by $d_M : G \to [0,1]$ the mapping $g \mapsto m_G(M\Delta\tau_g M)$,
- for a μ_X-continuity set M^* on some group compactification (ι_X, X) denote by $d_{M^*} : X \to [0,1]$ the mapping $g \mapsto \mu_X(M^*\Delta\tau_g M^*)$.

Note that the mapping d_{M^*} (and similarly the mapping d_M) can be used to define a translation invariant pseudometric by letting $\rho_{M^*}(g,h) := d_{M^*}(g - h)$. The set of zeros $\{g : d_{M^*}(g) = 0\}$ is always a closed subgroup. We will denote this subgroup by $\ker d_{M^*}$.

Now consider sets of the form $F(M,\varepsilon) := \{g \in G : d_M(g) < \varepsilon\}$ and denote by $\mathcal{F}(M)$ the filter on G generated by $\{F(M,\varepsilon) : \varepsilon > 0\}$, i.e. the set of all $F \subseteq G$ such that there exists an $\varepsilon > 0$ with $F(M,\varepsilon) \subseteq F$. When we have a realization M^* of M on some group compactification (ι_X, X) we can transfer the topological data encoded in the neighborhood filter of the unit 0_X in X to G by considering the pullback induced by ι_X.

To be precise: Let (ι_X, X) be a group compactification and $\mathfrak{U}(X, 0_X)$ the filter of all neighborhoods of the unit 0_X in X. By $\mathfrak{U}_{(\iota_X,X)}$ we denote the filter on G generated by $\iota_X^{-1}(\mathfrak{U}(X, 0_X))$. Note that if the mapping ι_X is one-one, $\iota_X^{-1}(\mathfrak{U}(X, 0_X))$ is already a filter.

For $\mathbb{Z}$, the group of integers, Theorem 2 in [12] states that for any Hartman set $M \subseteq \mathbb{Z}$ there is a group compactification (ι_X, X) such that $\mathcal{F}(M)$ and $\mathfrak{U}(X, 0_X)$ coincide. Hence the filter $\mathcal{F}(M)$ on $\mathbb{Z}$ contains much information about the group compactification (ι_X, X): *If $M \subseteq \mathbb{Z}$ is a Hartman measurable set and (ι_X, X) is a group compactification of the integers such that M can be realized on X via the continuity set M^* then $H = \ker d_{M^*}$ is a closed subgroup of X and $\mathcal{F}(M) = \mathfrak{U}_{(\pi_H \circ \iota_X, X/H)}$, for $\pi_H : X \to X/H$ the canonical quotient mapping.*

In what follows we need to generalize this result to arbitrary (LCA) groups. This poses no problem since the proof given in [12] for $G = \mathbb{Z}$ applies verbatim to an arbitrary topological group.

Recall that for a filter $\mathcal{F} \subseteq \mathfrak{P}(X)$ on some set X and a function $\varphi : X \to \mathbb{C}$ the filter-limit $\mathcal{F}\text{–}\lim_{x\in X} \varphi(x)$ is defined to be the unique $\lambda \in \mathbb{C}$ such that $\forall\varepsilon > 0$ we have $\{x \in X : |\varphi(x) - \lambda| < \varepsilon\} \in \mathcal{F}$. In [12] the filter $\mathcal{F} = \mathcal{F}(M)$ is also used to define the subgroup $\mathrm{Sub}(M)$ of $\mathbb{T}$ consisting of all those elements α such that $\mathcal{F}\text{–}\lim_{n\in\mathbb{Z}} \lfloor n\alpha \rfloor = 0$ (or equivalently: $\mathcal{F}\text{–}\lim_{n\in\mathbb{Z}} e^{2\pi i n\alpha} = 1$).

All three objects - filter, compactification and subgroup - carry the same information regarding a fixed Hartman set M. It is interesting to note that any subgroup of a compact abelian group G can be written as $\{g \in G : \mathcal{F}\text{–}\lim_{\chi\in\hat{G}} \chi(g) = 1\}$ for some filter $\mathcal{F}$ on $\hat{G}$ (cf. [2]).

We transfer these concepts into our more general context. To that cause we need the following definitions. Recall that a bounded function f on a group compactification (ι_X, X) is called Riemann integrable iff the set $\mathrm{disc}(f)$ of points of discontinuity is a μ_X-null set, for μ_X the normalized Haar measure on X. Let us denote the set of all such functions by $R_{\mu_X}(X)$ or, simply, $R(X)$. We use the following characterization, a proof of which can be found in [11].

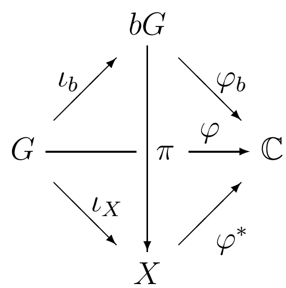
Proposition 1. *Let X be a compact space and μ_X a finite positive Borel measure on X. For a bounded real-valued μ_X-measurable function f the following assertions are equivalent:*

1. *f is Riemann integrable.*
2. *For every $\varepsilon > 0$ there exist continuous functions g_ε and h_ε such that $g_\varepsilon \leq f \leq h_\varepsilon$ and $\int_X (h_\varepsilon - g_\varepsilon) d\mu_X < \varepsilon$.*

Let φ be a function defined on a topological group G and (ι_b, bG) the Bohr compactification of G. We call a function φ_b defined on bG an extension (or realization) of φ iff $\varphi = \varphi_b \circ \iota_b$. For example: The set of almost periodic functions on G coincides with the set of those functions that can be extended to continuous functions on the Bohr compactification.

Definition 2. *Let (ι_b, bG) be the Bohr compactification of the topological group G. We call a bounded function φ on G Hartman measurable iff φ can be extended to a Riemann integrable function φ_b on bG. The set of Hartman measurable functions $\{\varphi^* \circ \iota_b : \varphi^* \in R_{\mu_b}(bG)\}$ is denoted by $\mathcal{H}(G)$.*

Given a Hartman measurable function φ, we say that φ^* *realizes* φ if φ^* is a Riemann integrable function defined on some group compactification (ι_X, X) such that $\varphi = \varphi^* \circ \iota_X$, cf. the diagram below:



In this situation we also say that φ can be realized on (ι_X, X). Most of this paper is devoted to the task of finding a minimal group compactification on which a given $\varphi \in \mathcal{H}(G)$ can be realized. Note that $\varphi^* \in R(X)$ implies $\varphi_b = \varphi^* \circ \pi \in R(bG)$ (cf. [8]).

3 Filters associated with Hartman measurable functions

By definition every $\varphi \in \mathcal{H}(G)$ has a realization on bG by a Riemann integrable function $\varphi^* \in R_{\mu_b}(bG)$. The mapping

$$d_{\varphi^*} : x \mapsto \|\varphi^* - \tau_x \varphi^*\|_1 := \int_{bG} |\varphi^* - \tau_x \varphi^*| d\mu$$

is continuous (cf. [3], Corollary 2.32). This implies that $d_\varphi := d_{\varphi^*} \circ \iota_b$ is an almost periodic function on G.

The finitely additive invariant measure m_G can be extended to an invariant mean on $\mathcal{H}(G)$, i.e. to an invariant and non-negative normalized linear functional on $\mathcal{H}(G)$. It will cause no confusion if we denote this invariant mean again by m_G (cf. [8]). Thus we can also write $d_\varphi(g) = m_G(|\varphi - \tau_g \varphi|)$. It is then obvious to define $F(\varphi, \varepsilon) := \{g \in G : d_\varphi(\tau_g \varphi) < \varepsilon\}$ and denote by $\mathcal{F}(\varphi)$ the filter on G generated by $\{F(\varphi, \varepsilon) : \varepsilon > 0\}$.

In the LCA setting, we can apply the tools developed in [12] to conclude a functional analogue of Theorem 2 in [12].

Definition 3. *Let $\varphi \in \mathcal{H}(G)$ be realized by φ^* on the group compactification (ι_X, X). φ^* is called an aperiodic realization iff* $\ker d_{\varphi^*} := \{x \in X : \|\varphi^* - \tau_x \varphi^*\|_1 = 0\} = \{0_X\}$.

Theorem 1. *Let $\varphi \in \mathcal{H}(G)$ be realized by φ^* on the group compactification (ι_X, X). Then $\mathcal{F}(\varphi) \subseteq \mathfrak{U}_{(\iota_X, X)}$. Furthermore $\mathcal{F}(\varphi) = \mathfrak{U}_{(\iota_X, X)}$ if φ^* is an aperiodic realization.*

Proof. Suppose $\varphi = \varphi^* \circ \iota_X$ with $\varphi^* \in R_{\mu_X}(X)$ for a group compactification (ι_X, X). For any set $A \in \mathcal{F}(\varphi)$ there exists $\varepsilon > 0$ such that $d_\varphi(x) < \varepsilon$ implies $x \in A$. Using almost periodicity of d_φ, i.e. continuity of d_{φ^*}, we find a neighborhood $U \in \mathfrak{U}(X, 0_X)$ such that $d_{\varphi^*}(U) \subseteq [0, \varepsilon)$. For every $x \in \iota_X^{-1}(U)$ we have $d_\varphi(x) < \varepsilon$. Consequently $\iota_X^{-1}(U) \subseteq A \in \mathfrak{U}_{(\iota_X, X)}$ and hence $\mathcal{F}(\varphi) \subseteq \mathfrak{U}_{(\iota_X, X)}$.

Suppose that $\varphi^* \in R_{\mu_X}(X)$ is aperiodic, i.e. $d_{\varphi^*}(x) = 0$ iff $x = 0_X$, the unit in X. Let $A \in \mathfrak{U}_{(\iota_X, X)}$ be arbitrary; w.l.o.g. we can assume $A \supseteq \iota_X^{-1}(U)$ for an open neighborhood $U \in \mathfrak{U}(X, 0_X)$. Due to the continuity of d_{φ^*} and compactness of X we have $d_{\varphi^*}(x) \geq \varepsilon > 0$ for $x \in X \setminus U^\circ$. This implies $\iota_X(\{g \in G : d_\varphi(g) < \varepsilon\}) \subseteq U$ and hence $\{g \in G : d_\varphi(g) < \varepsilon\} \subseteq \iota_X^{-1}(U) \subseteq A \in \mathcal{F}(\varphi)$. Thus $\mathfrak{U}_{(\iota_X, X)} \subseteq \mathcal{F}(\varphi)$ and consequently $\mathfrak{U}_{(\iota_X, X)} = \mathcal{F}(\varphi)$. ☐

Definition 4. *Let $\varphi \in \mathcal{H}(G)$ and let (ι_X, X) be a group compactification of G. A function $\psi^* \in R_{\mu_X}(X)$ is called an almost realization of φ iff $m_G(|\varphi - \psi|) = 0$ for $\psi := \psi^* \circ \iota_X$ and m_G the unique invariant mean on $\mathcal{H}(G)$.*

Theorem 2. *Every $\varphi \in \mathcal{H}(G)$ has an aperiodic almost realization on some group compactification (ι_X, X). If $\varphi^* : X \to \mathbb{C}$ is any aperiodic almost realization of φ then $\mathcal{F}(\varphi) = \mathfrak{U}_{(\iota_X, X)}$.*

Proof. We only have to prove that an aperiodic almost realization exists, the rest follows from Theorem 1. Let φ^* be a realization of φ on X. The reader will easily check that $H := \ker d_{\varphi^*} = \{x \in X : d_{\varphi^*}(x) = 0\}$ is a closed subgroup of the compact abelian group X.

Weil's formula for continuous functions on quotients (Theorem 3.22 in [3]) states that there exists $\alpha > 0$ such that for every $f \in C(X)$

$$\int_{X/H} \Big(\underbrace{\int_H f(s+t) d\mu_H(t)}_{= {}^\flat f(s)} \Big) d\mu_{X/H}(s) \quad = \quad \alpha \int_X f(u) d\mu_X(u) \tag{1}$$

holds. This implies that the canonical mapping ${}^\flat : C(X) \to C(X/H)$, $f \to {}^\flat f$ defined by ${}^\flat f(s + H) = \int_H f(s+t) d\mu_H(t)$ satisfies $\|{}^\flat f\|_1 \leq \alpha \|f\|_1$. We rescale the Haar measure on H such that $\alpha = 1$. Thus we can extend ${}^\flat$ to a continuous linear operator $L^1(X) \to L^1(X/H)$. Furthermore positivity of ${}^\flat$ enables us to extend ${}^\flat$ to a mapping defined on $R_{\mu_X}(X)$ in the following way:

According to Proposition 1 $f \in R_{\mu_X}(X)$ implies that there are $g_n, h_n \in C(X)$ such that $g_n \leq f \leq h_n$ and $\|h_n - g_n\|_1 \to 0$ as $n \to \infty$. Thus every function $\tilde{f}$ on X/H satisfying

$$f_\bullet := \sup_{n \geq 0} {}^\flat g_n \leq \tilde{f} \leq \inf_{n \geq 0} {}^\flat h_n =: f^\bullet$$

is in $R_{\mu_{X/H}}(X/H)$. Note that $f_\bullet$ and $f^\bullet$ are μ_H-measurable and coincide μ_H-a.e.; to define ${}^\flat f$ we pick any function $\tilde{f}$ satisfying $f_\bullet \leq \tilde{f} \leq f^\bullet$. Then Weil's formula (1) will still be valid, regardless of the particular choice of the g_n, h_n and ${}^\flat f$.

Since φ^* is Riemann integrable on X, there exist functions $\varphi_n \in C(X)$ such that $\|\varphi^* - \varphi_n\|_1 \to 0$. Note that $d_{\varphi_n} \to d_{\varphi^*}$ even uniformly on X:

$$|d_{\varphi_n}(s) - d_{\varphi^*}(s)| = \Big| \|\tau_s\varphi_n - \varphi_n\|_1 - \|\tau_s\varphi^* - \varphi^*\|_1 \Big| \leq 2\|\varphi_n - \varphi^*\|_1 \to 0.$$

Using the continuity of ${}^\flat$ as a mapping on $L^1(X)$ the same argument also shows that $|d_{{}^\flat\varphi^*}(s+H) - d_{{}^\flat\varphi_n}(s+H)| \leq 2\|\varphi_n - \varphi^*\|_1 \to 0$ uniformly on X/H. Suppose $d^*_{{}^\flat\varphi}(s+H) = 0$. Then

$$d_{\varphi^*}(s) = \lim_{n\to\infty} d_{\varphi_n}(s) = \lim_{n\to\infty} d_{{}^\flat\varphi_n}(s+H) = 0$$

implies $s \in H$, i.e. $s + H = 0_X + H \in X/H$. So ${}^\flat\varphi^*$ is aperiodic.

We show that φ^* being a realization of φ implies that ${}^\flat\varphi^*$ is an almost realization of φ. By definition $t \in H$ iff $A_t := \{s \in X : \varphi^*(s+t) = \varphi^*(s)\}$ has μ_X-measure 1. Applying Weil's formula (1) to the function $f = 1\!\!1_{A_t} \in L^1(X)$ gives

$$\int_{X/H} {}^\flat f d\mu_{X/H} = \int_{X/H} {}^\flat 1\!\!1_{A_t}(s+H) d\mu_{X/H}(s+H) = \int_X f d\mu_X = 1. \qquad (2)$$

Plugging the definition of ${}^\flat$ into (2) we get $\mu_{X/H}$-a.e. the identity

$${}^\flat 1\!\!1_{A_t}(s+H) = \int_H 1\!\!1_{A_t}(s+u) d\mu_H(u) = 1.$$

So for every $t \in H$ and $\mu_{X/H}$-a.e. $s+H$ we know that the set $\{u \in H : \varphi^*(s+t+u) \neq \varphi^*(s+u)\}$ is a μ_H-null set. This means

$$\tau_t(\tau_s{\varphi^*}_{|H}) = \tau_s{\varphi^*}_{|H} \quad \mu_H\text{-a.e.}$$

Thus $\tau_s\varphi^*$ is constant μ_H-a.e. on H and for $\mu_{X/H}$ almost all $s+H$ we have

$${}^\flat\varphi^*(s+H) \;=\; \int_H \tau_s\varphi^*(t) d\mu_H(t) = \int_H \varphi(s)^* d\mu_H(t) = \varphi^*(s).$$

Let $\pi_H : X \to X/H$ be the quotient mapping onto the group compactification $(\iota_{X/H}, X/H)$. Let $\psi^* := {}^\flat\varphi^* \circ \pi_H$. Since ${}^\flat\varphi^*$ is Riemann integrable on X/H it is an elementary fact that ψ^* is Riemann integrable on X (cf. [8]). Once again, Weil's formula (1) together with the fact that the Haar measure on the quotient X/H is given by $\mu_{X/H} = \pi_H^{-1}(\mu_X)$ implies $\psi^* = \varphi^*$ μ_X-a.e. Thus the function ψ defined by

$$\psi := \psi^* \circ \iota_X = {}^\flat\varphi^* \circ \iota_{X/H}$$

satisfies $m_G(|\varphi - \psi|) = \|\varphi^* - \psi^*\|_1 = 0$ for the unique invariant mean m_G. Thus ψ^* is the required almost realization of φ. □

Corollary 5. *Every $\varphi \in AP(G)$ has an aperiodic realization on some group compactification (ι_X, X).*

Proof. We use the notation from Theorem 2. If φ is almost periodic then φ^* is continuous. Consequently ${}^\flat\varphi^*$ and $\psi^* := {}^\flat\varphi^* \circ \pi$ are also continuous. Since these functions coincide μ_X-a.e. they coincide everywhere on X. This implies that φ^* is constant on H-cosets and ${}^\flat\varphi^*(s+H) = \varphi^*(s)$ for all $s+H \in X/H$. So φ^* is truly a realization of φ, not only an almost realization. □

This Corollary is a special case of Følner's "Main Theorem for Almost Periodic Functions", for a detailed treatment cf. [4].

Remark: Note that for any given realization of a Hartman measurable function $\varphi \in \mathcal{H}(G)$ on a group compactification (ι_X, X) we can always assume that there exists an aperiodic almost realization of φ on a group compactification $(\tilde{\iota_X}, \tilde{X})$ with $(\tilde{\iota_X}, \tilde{X}) \leq (\iota_X, X)$. Since in [8] it is shown that every Hartman measurable function on an LCA group with separable dual has a realization on a metrizable group compactification, every Hartman measurable function on such a group has an aperiodic almost realization on a metrizable group compactification.

Lemma 6. *Let G be an LCA group and let (ι_X, X) be a group compactification. Then there exists a unique subgroup $\Gamma \leq \hat{G}$ such that (ι_Γ, C_Γ) and (ι_X, X) are equivalent. Furthermore (ι_X, X) is the supremum of all group compactifications (ι_γ, C_γ) such that $(\iota_\gamma, C_\gamma) \leq (\iota_X, X)$ (writing in short (ι_γ, C_γ) for $(\iota_{\langle\gamma\rangle}, C_{\langle\gamma\rangle})$).*

The mapping $(\iota_X, X) \mapsto C_\Gamma$ is a bijection between equivalence classes of group compactifications of G and subgroups of $\hat{G}$.

Proof. See Theorem 26.13 in [7]. □

Corollary 7. *Let $\varphi \in \mathcal{H}(G)$. Any two group compactifications (ι_{X_1}, X_1) and (ι_{X_2}, X_2) on which φ has an aperiodic almost realization are equivalent.*

Proof. By Theorem 1 we have $\mathfrak{U}_{(\iota_{X_1}, X_1)} = \mathcal{F}(\varphi) = \mathfrak{U}_{(\iota_{X_2}, X_2)}$. A straight forward generalization of Theorem 1 in [12] implies that the mapping

$$\Phi : \hat{G} \geq \Gamma \mapsto (\iota_\Gamma, C_\Gamma)$$

coincides with the composition of the mappings

$$\begin{aligned} \Sigma : (\iota_b, bG) \geq (\iota_X, X) &\mapsto \mathfrak{U}_{(\iota_X, X)}, \\ \Psi : \mathfrak{P}(G) \supset \mathcal{F} &\mapsto \{\chi \in \hat{G} : \mathcal{F}\text{–}\lim_{g \in G} \chi(g) = 0\}. \end{aligned}$$

Since Lemma 6 states that $\Phi = \Psi \circ \Sigma$ is invertible, Σ must be one-one. In particular $\mathfrak{U}_{(\iota_{X_1}, X_1)} = \mathfrak{U}_{(\iota_{X_2}, X_2)}$ implies that (ι_{X_1}, X_1) and (ι_{X_2}, X_2) are equivalent group compactifications. □

For the rest of this section assume that G is an LCA group with separable dual.

Corollary 8. *Every filter $\mathcal{F}(\varphi)$ with $\varphi \in \mathcal{H}(G)$ coincides with a filter $\mathfrak{U}_{(\iota_X,X)}$ for a metrizable group compactification (ι_X, X). If φ^* is an arbitrary realization of φ, say on the Bohr compactification bG, we can take $X \cong bG/\ker d_{\varphi^*}$.*

Corollary 9. *Hartman measurable functions induce exactly the filters coming from metrizable group compactifications.*

Proof. In Theorem 3 in [12] for every metrizable group compactification (ι_X, X) of the integers $G = \mathbb{Z}$, an aperiodic Hartman periodic function of the form $f = 1\!\!1_A$ is constructed. The same construction can be done in an arbitrary LCA group G as long as the dual $\hat{G}$ contains a countable and dense subset. This shows that any $\mathfrak{U}_{(\iota_X,X)}$ with metrizable X can be obtained already by Hartman measurable *sets*, i.e. by a filter $\mathcal{F}(\varphi)$ with $\varphi = 1\!\!1_A$. Since any Hartman measurable function on G can be realized on a metrizable group compactification (cf. [8]). Thus Theorem 1 implies that no filter $\mathcal{F}(\varphi)$ can coincide with $\mathfrak{U}_{(\iota_X,X)}$ for a non metrizable group compactification (ι, C). □

4 Subgroups associated with Hartman measurable functions

For Hartman measurable φ let us denote by $\Gamma(\varphi)$ the (countable) subgroup of $\hat{G}$ generated by the set

$$\operatorname{spec}\varphi := \{\chi \in \hat{G} : m_G(\varphi \cdot \overline{\chi}) \neq 0\}$$

of all characters with non vanishing Fourier coefficients. We will prove that $\Gamma = \Gamma(\varphi)$ determines a group compactification (ι_Γ, C_Γ) such that φ can be realized aperiodically on C_Γ. First we deal with almost periodic functions:

Proposition 10. *Let $\varphi \in AP(G)$ and (ι_X, X) a group compactification such that every character $\chi \in \Gamma(\varphi)$ has a representation $\chi = \eta \circ \iota_X$ with a continuous character $\eta \in \hat{X}$. Then every function $f \in \overline{span}\,\Gamma(\varphi) \subseteq AP(G)$ has a realization on (ι_X, X).*

Proof. This is essentially a reformulation of Theorem 5.7 in [1]. In fact the Stone-Weierstrass Theorem implies that $\overline{\operatorname{span}}\,\Gamma(\varphi) = \iota_\Gamma^*\ C(X)$. Furthermore $\varphi \in \overline{\operatorname{span}}\,\Gamma(\varphi)$, i.e. φ can be realized by some continuous $\varphi^* : X \to \mathbb{C}$. □

Proposition 11. *Let $\varphi \in AP(G)$ and (ι_Γ, C_Γ) the group compactification of G induced by the subgroup $\Gamma = \Gamma(\varphi) \leq \hat{G}$. Then for every continuous character $\psi \in \Gamma(\varphi)$ there exists a continuous $\psi^* : C_\Gamma \to \mathbb{C}$ such that $\psi = \psi^* \circ \iota_\Gamma$.*

Proof. Given the group compactification (ι_Γ, C_Γ), then the compact group C_Γ is by definition topologically isomorphic to $\overline{\{(\chi(g))_{\chi\in\Gamma} : g \in G\}} \leq \mathbb{T}^\Gamma$.

The restriction of each projection

$$\pi_{\chi_0} : C_\Gamma \leq \mathbb{T}^\Gamma \to \mathbb{T}, \quad (x_\chi)_{\chi\in\Gamma} \mapsto x_{\chi_0}$$

is a bounded character of C_Γ for each $\chi_0 \in \Gamma(\varphi)$. I.e. π_{χ_0} is an element of $\widehat{C_\Gamma}$. Thus $\chi_0 = \pi_{\chi_0} \circ \iota_\Gamma$ for each $\chi_0 \in \Gamma(\varphi)$ and we may apply Proposition 10 to obtain the assertion. □

Proposition 12. *Let $\varphi \in AP(G)$ and let (ι_X, X) be a group compactification of G such that φ can be realized by a continuous function $\varphi^* : X \to \mathbb{C}$. Then each continuous character $\chi \in \Gamma(\varphi)$ has a representation $\chi = \eta \circ \iota_\Gamma$ with $\eta \in \hat{X}$.*

Proof. Obviously it is enough to prove the assertion for a generating subset of $\Gamma(\varphi)$. Let $\chi \in \hat{G}$ be such that $m_G(\varphi \cdot \overline{\chi}) \neq 0$. Define a linear functional $m_\chi : C(X) \to \mathbb{C}$ via $\psi \mapsto m_\chi(\psi) = m_G((\psi \circ \iota_\Gamma) \cdot \overline{\chi})$. It is routine to check that m_χ is bounded and $\|m_\chi\| = 1$. Since X is compact the complex-valued mapping $\tilde{\eta} : X \mapsto m_\chi(\tau_x\varphi^*)$ is continuous on X (the mapping $x \mapsto \tau_x\varphi^*$ is continuous). For $g \in G$ we compute

$$\begin{aligned}
\tilde{\eta} \circ \iota_X(g) &= m_G((\tau_{\iota_X(g)}\varphi^* \circ \iota_X) \cdot \overline{\chi}) = m_G(\tau_g(\varphi^* \circ \iota_X) \cdot \overline{\chi}) \\
&= m_G((\varphi^* \circ \iota_X) \cdot \tau_{-g}\overline{\chi}) = m_G((\varphi^* \circ \iota_X) \cdot \chi(g)\overline{\chi}) \\
&= \chi(g) m_\chi(\varphi^*) = \chi(g)\tilde{\eta}(0).
\end{aligned}$$

Since $\tilde{\eta}(0) = m_\chi(\varphi^*) = m_G(\varphi \cdot \overline{\chi}) \neq 0$ we can define $\eta := \tilde{\eta}(0)^{-1}\tilde{\eta}$. The mapping $\eta : X \to \mathbb{T}$ is continuous and satisfies the functional equation

$$\eta(\iota_X(g) + \iota_X(h)) = \tilde{\eta}(0)^{-1}\tilde{\eta}(\iota_X(g) + \iota_X(h)) = \chi(g)\chi(h) = \eta(\iota_X(g))\eta(\iota_X(h))$$

on the dense set $\iota_X(G)$. Hence η is a bounded character on X and $\eta \circ \iota_X = \chi$. □

Corollary 13. *Let $\varphi \in \mathcal{H}(G)$ be realized by φ^* on the group compactification (ι_X, X). Then each $\chi \in \Gamma(\varphi)$ has a representation $\chi = \eta \circ \iota_X$ with $\eta \in \hat{X}$.*

Proof. For every $\chi \in \hat{G}$ with $m_G(\varphi\cdot\overline{\chi}) = \alpha \neq 0$ we can pick a continuous function $\psi^* : X \to \mathbb{C}$ such that $\|\psi^* - \varphi\|_1 < |\alpha|/2$. Then $\psi := \psi^* \circ \iota_X$ satisfies

$$|m_G(\varphi \cdot \overline{\chi}) - m_G(\psi \cdot \overline{\chi})| \leq m_G(|\varphi - \psi|) \leq \|\psi^* - \varphi^*\|_1 < |\alpha|/2.$$

In particular $m_G(\psi \cdot \overline{\chi}) \neq 0$. Applying Proposition 12 to the function $\psi \in AP(G)$ yields that the character χ can be realized on X. □

Thus for almost periodic functions φ the subgroup $\Gamma(\varphi)$ contains all the relevant information to reconstruct φ from its Fourier-data in a minimal way. It is not obvious how to obtain similar results for Hartman measurable functions that are not almost periodic. The following example illustrates how a straight forward approach may fail.

Example 14. Let $\varphi_n(k) := \prod_{j=1}^n \cos^2\left(2\pi\frac{k}{3^j}\right)$ on $G = \mathbb{Z}$. Each φ_n is a finite product of periodic (and hence almost periodic) functions. Since $AP(\mathbb{Z})$ is an algebra, φ is almost periodic. In [8] it is shown that $\varphi(k) := \lim_{n\to\infty}\varphi_n(k)$ exists and defines a non negative Hartman measurable function with $m_{\mathbb{Z}}(\varphi) = 0$. Since $\Gamma(\varphi_n) \cong \mathbb{Z}/3^n\mathbb{Z}$ we have (using obvious notation):

$$\lim_{n\to\infty}\Gamma(\varphi_n) = \bigcup_{n=1}^{\infty}\Gamma(\varphi_n) \cong \mathbb{Z}_3^{\infty},$$

the Prüfer 3-group (i.e. the subgroup of all complex 3^n-th roots of unity for $n \in \mathbb{N}$), but

$$\Gamma(\lim_{n\to\infty}\varphi_n) = \{0\}.$$

Proposition 15. *Let $\{K_n\}_{n=1}^{\infty}$ denote the family of Fejér kernels on $\mathbb{T}^k$*

$$K_n(\exp(it_1),\ldots,\exp(it_k)) = \frac{1}{k}\prod_{j=1}^{k}\left(\frac{\sin(\frac{1}{2}nt_j)}{\sin(\frac{1}{2}t_j)}\right)^2.$$

The linear convolution operators on $L^1(\mathbb{T}^k)$ defined by

$$\sigma_n : \varphi \mapsto K_n * \varphi$$

are non negative, their norm is uniformly bounded by $\|\sigma_n\| = 1$ and $\sigma_n\varphi(x) \to \varphi(x)$ a.e. for every $\varphi \in L^1(\mathbb{T}^k)$. Furthermore $\sigma_n\varphi \in span\,\Gamma(\varphi)$ for every $n \in \mathbb{N}$.

Proof. This is a reformulation of the results in section 44.51 in [7]. □

Let f be Riemann integrable on $X = \mathbb{T}^k$, w.l.o.g. real-valued, and $\varphi_i, \psi_i \in C(X)$ such that $\varphi_i \geq f \geq \psi_i$ and $\|\varphi_i - \psi_i\|_1 < \varepsilon_i$ for a sequence $\{\varepsilon_i\}_{i=1}^{\infty}$ of positive real numbers, tending monotonically to 0. We know that $\sigma_n f(x) \to f(x)$ for a.e. $x \in X$. Thus we have

$$\varphi_n^* := \sigma_n\varphi_n \geq \sigma_n f \geq \sigma_n\psi_n = \psi_n^*$$

and

$$\|\varphi_n^* - \psi_n^*\|_1 \leq \|\sigma_n(\varphi_n^* - \psi_n^*)\|_1 \leq \|\sigma_n\|\,\|\varphi_n - \psi_n\|_1 \leq \varepsilon_n.$$

Let $\varphi^* := \inf_{n\in\mathbb{N}}\varphi_n$ and $\psi^* := \sup_{n\in\mathbb{N}}\psi_n$. If we assume w.l.o.g. ψ_n to increase and φ_n to decrease as $n \to \infty$, the same will hold for ψ_n^* and φ_n^*. This implies that in the inequality

$$\varphi^*(x) = \lim_{n\to\infty}\varphi_n^*(x) \geq \limsup_{n\to\infty}\sigma_n f \geq \liminf_{n\to\infty}\sigma_n f \geq \lim_{n\to\infty}\psi_n^*(x) = \psi^*(x)$$

actually equality holds μ_X-a.e. on X. Thus we can apply Proposition 1 and conclude that any function f^* with $\varphi^* \geq f^* \geq \psi^*$ is Riemann integrable (and coincides μ_X-a.e. with f). In particular $f^\bullet := \limsup_{n\to\infty}\sigma_n f$ and $f_\bullet := \liminf_{n\to\infty}\sigma_n f$ are (lower resp. upper semicontinuous) Riemann integrable functions that coincide μ_X-a.e. with f.

Let us call a group compactification (ι_X, X) finite dimensional iff X is topologically isomorphic to a closed subgroup of $\mathbb{T}^n$ for some $n \in \mathbb{N}$. Note that if (ι_X, X) is finite dimensional, then every group compactification covered by (ι_X, X) is finite dimensional as well. A Hartman measurable function $\varphi \in \mathcal{H}(G)$ can be realized finite dimensionally iff there exists a realization of φ on some finite dimensional group compactification.

Proposition 16. *For a compact LCA group C the following assertions are equivalent:*

1. *C is finite dimensional,*
2. *$\hat{C}$ is finitely generated,*
3. *C is topological isomorphic to $\mathbb{T}^k \times F$ for $k \in \mathbb{N}$ and a finite group F of the form*

$$F = \prod_{i=1}^{N} (\mathbb{Z}/n_i\mathbb{Z})^{p_i}, \quad p_i \text{ prime.}$$

Proof. Folklore. □

Proposition 17. *Let $\varphi \in \mathcal{H}(G)$. If φ can be realized finite dimensionally, then there is an almost realization of φ on the (finite dimensional) compactification induced by $\Gamma := \Gamma(\varphi)$.*

Proof. Let φ be realized finite dimensionally on some group compactification (ι_X, X). Since there exists a group compactification covered by (ι_X, X), on which φ can be almost realized aperiodically (cf. Theorem 2), we can assume w.l.o.g. that φ can be almost realized aperiodically already on (ι_X, X). We have to show that (ι_X, X) and (ι_Γ, C_Γ) are equivalent.

Let ψ^* be an aperiodic realization of φ on $C_\Gamma \cong \mathbb{T}^k \times F$ with $k \in \mathbb{N}$ and F finite. Let us denote the elements of $\mathbb{T}^k \times F$ by tuples $(\vec{\alpha}, x)$. For every fixed $\vec{\alpha} \in \mathbb{T}^k$ define a mapping $\psi_{\vec{\alpha}} : F \to \mathbb{R}$ via

$$\psi^*_{\vec{\alpha}}(x) := \psi^*(\vec{\alpha}, x).$$

For each $\overline{\chi} \in \hat{F}$, the dual of the finite group F, define the F-Fourier coefficient of $\psi^*_{\vec{\alpha}}$ as

$$c_{\overline{\chi}}(\vec{\alpha}) := \int_F \psi^*_{\vec{\alpha}}(x)\overline{\chi}(x)dx = \frac{1}{\#F} \sum_{x \in F} \psi^*(\vec{\alpha}; x)\overline{\chi}(x) \in \mathbb{C}.$$

We want to show that $c_{\overline{\chi}} : \mathbb{T}^k \to \mathbb{C}$ is a Riemann integrable function: The mapping $\gamma_x : \mathbb{T}^k \to X$ defined via $\vec{\alpha} \mapsto (\vec{\alpha}; x)$ is continuous and measure-preserving for every $x \in F$. ψ^* is by definition Riemann integrable. Thus the mapping $\psi^* \circ \gamma_x : \mathbb{T}^k \to \mathbb{C}$ is Riemann integrable for each $x \in F$. Note that

$$c_{\overline{\chi}}(\vec{\alpha}) = \sum_{x \in F} (\psi^* \circ \gamma_x)(\vec{\alpha})\overline{\chi}(x).$$

Hence, for each fixed character $\chi \in \hat{F}$, the mapping $c_{\overline{\chi}} : \mathbb{T}^k \to \mathbb{C}$ defined via $\vec{\alpha} \mapsto \sum_{x \in F} (\psi^* \circ \gamma_x)(\vec{\alpha})\overline{\chi}(x)$ is Riemann integrable on $\mathbb{T}^k$.

Thus Proposition 15 implies $\sigma_n c_\chi(\vec{\alpha}) \to c_\chi(\vec{\alpha})$ a.e. on $\mathbb{T}^k$. Taking into account that the Haar measure on F is the normalized counting measure, we get

$$\psi_n^*(\vec{\alpha};x) := \sum_{\overline{\chi}\in\hat{F}} (\sigma_n c_{\overline{\chi}}(\vec{\alpha}))\,\overline{\chi}(x) \to \sum_{\overline{\chi}\in\hat{F}} c_{\overline{\chi}}(\vec{\alpha})\overline{\chi}(x) = \psi_{\vec{\alpha}}^*(x) = \psi^*(\vec{\alpha};x) \tag{3}$$

for almost every $\vec{\alpha} \in \mathbb{T}^k$ and every $x \in F$, as $n \to \infty$. Since Haar measure μ_C on C is the product measure of the Haar measures on the groups $\mathbb{T}^k$ and F, the relation (3) holds μ_C-a.e. on C. We conclude that any function majorizing $\liminf_{n\to\infty} \psi_n^*$ and minorizing $\limsup_{n\to\infty} \psi_n^*$ is an almost realization of φ. Note that according to the properties of the Fejér kernels on $\mathbb{T}^k$ (see 44.51 in [7]) for each character $(\eta \times \chi)(\vec{\alpha};x) := \eta(\vec{\alpha})\chi(x)$, $\eta \in \hat{\mathbb{T}}^k$ and $\chi \in \hat{F}$, there exists an $n_0 \in N$ such that for $n \geq n_0$ in the Fourier expansion of ψ_n^* the Fourier coefficient (computed in C) associated with the character does not vanish iff the $\mathbb{T}^k$-Fourier coefficient of c_χ

$$c_\eta(c_\chi) = \int_{\mathbb{T}^k} c_\chi(\vec{\alpha})\overline{\eta}(\vec{\alpha})d\vec{\alpha}$$

does not vanish. A simple computation shows that the Fourier coefficients of ψ^* are given by

$$\begin{aligned} c_{\eta\times\chi}(\psi^*) &= \int_{\mathbb{T}^k}\int_F \psi^*(\vec{\alpha},x)\overline{\eta}(\vec{\alpha})\overline{\chi}(x)d\vec{\alpha}dx \\ &= \int_{\mathbb{T}^k} c_\chi(\vec{\alpha})\overline{\eta}(\vec{\alpha})d\vec{\alpha} = c_\eta(c_\chi) \end{aligned}$$

So the character $\eta\times\chi$ contributes to the Fourier expansion of ψ^* if and only if $c_{\eta\times\chi}(\psi) \neq 0$. Thus $\psi_n^* \in \operatorname{span}\Gamma(\varphi)$ for every $n \in \mathbb{N}$, implying that there exist almost realizations of φ on the group compactification induced by $\Gamma(\varphi)$, e.g. $\liminf_{n\to\infty}\psi_n^*$ or $\limsup_{n\to\infty}\psi_n^*$. □

Combining this result with the results of the previous section we obtain

Theorem 3. *Let $\varphi \in \mathcal{H}(G)$ and $\Gamma = \Gamma(\varphi) \leq \hat{G}$. The following assertions hold:*

1. *$(\iota_\Gamma, C_\Gamma) \leq (\iota_X, X)$ for every compactification (ι_X, X) on which φ can be realized. In particular $\mathcal{F}(\varphi) \subseteq \mathfrak{U}_{(\iota_\Gamma, C_\Gamma)}$.*
2. *Assume that $\varphi \in AP(G)$ or that φ can be realized finite dimensionally. Then φ can be realized aperiodically on C_Γ. In particular $\mathcal{F}(\varphi) = \mathfrak{U}_{(\iota_\Gamma, C_\Gamma)}$.*

We strongly conjecture that the second assertion in Theorem 3 holds for *any* Hartman measurable function, at least on LCA groups G with separable dual $\hat{G}$. A proof of this might utilize more general summation methods (in the flavour of Theorems 44.43 and 44.47 in [7]) than the Fejér summation presented here.

In [12] it is shown that for any Hartman measurable set $M \subseteq G = \mathbb{Z}$ and the induced filter $\mathcal{F} = \mathcal{F}(M)$ there is an aperiodic realization of $\varphi_M = 1\!\!1_M$ on the compactification determined by the subgroup $\mathrm{Sub}(M) = \{\alpha : \mathcal{F}\text{–}\lim_{n\in\mathbb{Z}} \lfloor n\alpha \rfloor = 0\}$ or, equivalently, $\mathrm{Sub}(M) = \{\alpha : \mathcal{F}\text{–}\lim_{n\in\mathbb{Z}} e^{2\pi i n\alpha} = 1\}$.

Together with Theorem 3 this implies that for Hartman sets M with finite dimensional realization both the group compactifications of $\mathbb{Z}$ induced by the subgroups $\Gamma(\varphi_M)$ and $\mathrm{Sub}(M)$ admit aperiodic realizations of φ_M. Hence uniqueness of the minimal compactification with aperiodic realization (Corollary 7) implies that in this special case $\Gamma(\varphi_M) = \mathrm{Sub}(M)$. In the general situation we can prove up to now far only the following

Proposition 18. *For a Hartman measurable function $\varphi \in \mathcal{H}(G)$ let $\mathcal{F} = \mathcal{F}(\varphi)$, $\Gamma = \Gamma(\varphi)$ and Sub$(\varphi) = \{\chi \in \hat{G} : \mathcal{F}\text{–}\lim_{g\in G} \chi(g) = 1_{\mathbb{C}}\}$. Then $\Gamma(\varphi) \leq$ Sub(φ).*

Proof. Suppose $\chi \in \Gamma(\varphi)$. To prove $\mathcal{F}\text{–}\lim_{g\in G} \chi(g) = 1_{\mathbb{C}}$ (unit element of the multiplicative group of complex numbers) we have to show that for every $\varepsilon > 0$ the set $\{g \in G : |1-\chi(g)| < \varepsilon\}$ belongs to the filter $\mathcal{F}(\varphi)$, i.e. that there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\{g \in G : m_G(|\tau_g\varphi - \varphi|) < \delta\} \subseteq \{g \in G : |1 - \chi(g)| < \varepsilon\} \in \mathcal{F}(\varphi). \qquad (4)$$

Using the fact that m_G is an invariant mean and that χ is a homomorphism, we have

$$\chi(g)\, m_G(\varphi \cdot \overline{\chi}) = m_G(\tau_g\varphi \cdot \overline{\chi}) = m_G((\tau_g\varphi - \varphi) \cdot \overline{\chi}) + m_G(\varphi \cdot \overline{\chi}).$$

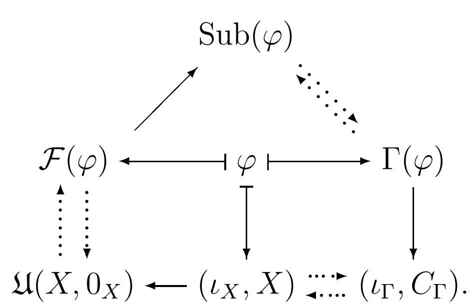
Using $\|\chi\|_\infty = 1$ this implies

$$|1 - \chi(g)| \cdot |m_G(\varphi \cdot \overline{\chi})| = |m_G((\tau_g\varphi - \varphi) \cdot \overline{\chi})| \leq m_G(|\tau_g\varphi - \varphi|).$$

Since $m_G(\varphi \cdot \overline{\chi}) \neq 0$ we can define $\delta := \varepsilon \cdot \frac{m_G(|\tau_g\varphi - \varphi|)}{|m_G(\varphi\cdot\overline{\chi})|} > 0$. With this choice of δ indeed $m_G(|\tau_g\varphi - \varphi|) < \delta$ implies $|1 - \chi(g)| < \varepsilon$, i.e. the inclusion (4) holds. □

5 Summary

The content of the present paper essentially deals with the definition and properties of the objects occurring in the diagram below. Abusing the terminus technicus of *commutative diagrams* in a kind of sloppy way, the theorems of this paper circle around the question under which assumptions this diagram is commutative:



Section 3 deals with the left half of this diagram: to every Hartman measurable function φ a filter $\mathcal{F}(\varphi)$ is associated and to every group compactification (ι_X, X) on which φ can be realized a filter $\mathfrak{U}(X, 0_X)$ is associated. In general $\mathfrak{U}(X, 0_X) \supseteq \mathcal{F}(\varphi)$, and there always exists compactifications such that equality holds (indicated by $\uparrow\downarrow$).

Section 4 deals with the right half of the diagram: to every Hartman measurable function φ a subgroup $\Gamma(\varphi)$ of the dual is associated, which in turn induces a group compactification (ι_Γ, C_Γ). In general $(\iota_\Gamma, C_\Gamma) \leq (\iota_X, X)$ for every group compactification (ι_X, X) on which φ can be realized. If φ is either almost periodic or can be realized finite dimensionally then (ι_A, X_A) is itself a group compactification on which φ can be realized (indicated by $\rightleftarrows$) and the filter $\mathfrak{U}(X_A, 0_{X_A})$ associated with this particular compactification coincides with $\mathcal{F}(\varphi)$. The filter $\mathcal{F}(\varphi)$ in turn defines a subgroup $\mathrm{Sub}(\varphi)$ of the dual $\hat{G}$. While it can be shown that in general $\Gamma(\varphi) \leq \mathrm{Sub}(\varphi)$ it is and open problem whether this inclusion can be reversed.

References

[1] R. Burckel, *Weakly Almost Periodic Functions on Semigroups*, Gordon and Breach, New York, 1970.

[2] M. Beiglböck, C. Steineder, R. Winkler *Sequences and filters of characters characterizing subgroups of compact abelian groups*, to appear in Top. Appl.

[3] J. Elstrodt, *Maß- und Integrationstheorie*, Springer-Verlag Berlin Heidelberg New York, 1999.

[4] E. Følner, *A Proof of the Main Theorem for Almost Periodic Functions in an Abelian Group*, Ann. of Math., 50/5 (1949), 559-569.

[5] S. Frisch, M. Pašteka, R. Tichy, R. Winkler, *Finitely additive measures on groups and rings*, Rend. Circ. Mat. Palermo, Series II, 48 (1999), 323-340.

[6] S. Hartman, *Remarks on equidistribution on non-compact groups*, Compositio Math., 16 (1964), 66-71.

[7] E. Hewitt and K. Ross, *Abstract Harmonic Analysis I,II*, Springer-Verlag Berlin-Heidelberg-New York, 1963.

[8] G. Maresch and R. Winkler, *Hartman measurable functions and related function spaces*, E-print, 2005, available at *www.dmg.tuwien.ac.at/maresch*

[9] J. Schmeling, E. Szabò, R. Winkler, *Hartman and Beatty bisequences*, Algebraic Number Theory and Diophantine analysis, 405-421, Walter de Gruyter, Berlin New York, 2000.

[10] C. Steineder and R. Winkler, *Complexity of Hartman sequences*, to appear in Journal de Théorie des Nombres de Bordeaux.

[11] M. Talagrand, *Closed convex hull of measurable functions, Riemann measurable functions and measurability of translations*, Ann. Inst. Fourier (Grenoble), 32/1 (1982), 39-69.

[12] R. Winkler, *Ergodic Group Rotations, Hartman Sets and Kronecker Sequences*, Monatsh. Math., 135 (2002), 333-343.