COUNTING THE NUMBER OF UNAVOIDABLE SETS OF A GIVEN SIZE IN A FINITE SET OF INTEGERS

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Abstract. In this paper, we introduce the U-function on $\mathbf{N}$, the set of natural numbers. $U(k,n)$, for positive integers k, n with $k \leq n$ and $n \geq 3$, counts the number of unavoidable sets of size k which are subsets of $\{1, 2, ..., n\}$, where the notion of avoidability is defined below. We use some straightforward observations as well as the results of Shan, Zhu, Dumitriu, and Develin to observe that $U(1,n) = U(2,n) = 0$ for all n, and to give a recursive formula for $U(3,n)$. Further, we provide nontrivial lower bounds for $U(k,n)$ for $k \geq 4$. We conclude by giving conditions under which r-step sequences, defined in the text, are avoidable.

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1. Introduction and Preliminary Results

Let $\mathbf{N}$ denote the set of natural numbers. A set $S \subset \mathbf{N}$ is said to be *avoidable* if there exists a partition of $\mathbf{N}$ into two (nonempty) disjoint sets A and B such that no element of S is the sum of two distinct elements of either A or B.

While avoidable sets in $\mathbf{N}$ have been studied for some time, not many families of such sets are known. To date, the Fibonacci [8] and Tribonacci [3] sequences have been categorized, as well as a family of sets [2, 5] that are similar in some sense to Beatty sequences. In this paper, we categorize certain subsets of $\mathbf{N}$ according to their avoidability.

First, we note the following.

Proposition 1.1. *A set S having four consecutive terms a, $a+1$, $a+2$, $a+3$ where $a > 1$ is even is unavoidable. Further, if S has three consecutive terms a, $a+1$, $a+2$ where $a > 1$ is odd, then S is unavoidable.*

Proof. When $a > 1$ is odd, then $\frac{a-1}{2} \in A$, say, while $\frac{a+1}{2} \in B$. But then neither A nor B can contain $\frac{a+3}{2}$. The first statement follows from the second. $\square$

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Observe, however, that there are infinitely many sets S with three consecutive terms that are avoidable. Specifically, apply the work of Dumitriu [4] and Develin [3] to Tribonacci sequences with initial terms a, $a+1$, $a+2$ where a is even.

Related to Proposition 1.1 is the following result.

Proposition 1.2. *Any avoidable set S with consecutive terms a, $a+1$ where $a \geq 3$ can contain $c < a$ only if $c = 1$ or 2, or $c = a-1$ with a odd.*

Proof. If S is avoidable with a even, then for each $1 \leq c \leq a$ we must have c, $a+1-c$ in opposite sets of the partition, specifically $\{1, 2, ..., a/2\} \in A$ and $\{1+(a/2), 2+(a/2), ..., a\} \in B$. Observe that this is the only way to partition $\{1, 2, ..., a\}$. If S is avoidable with a odd, then again for each $1 \leq c \leq a$ we must have c, $a+1-c$ in opposite sets of the partition, with the caveat that $c = \frac{a+1}{2}$ and a belong to the same set, specifically $\{1, 2, ..., (a-1)/2\} \in A$ and $\{(a+1)/2, (a+3)/2, ..., a\} \in B$. Again note that this is the only way to partition $\{1, 2, ..., a\}$. □

From this we immediately have the following.

Proposition 1.3. *If S is avoidable, then S has at most two nonoverlapping consecutive pairs, namely $\{1, 2\}$ and $\{a, a+1\}$ where $a \geq 3$.*

We recall the main result from [8]. For a, $b \in \mathbf{N}$ and k a nonnegative integer, let $c = a+b+k$, let $d = \gcd(a+k, b+k) = \gcd(c-a, c-b)$, and let e be the number of even integers among a, b, c. The sequence $U = \{u_m\}$ is defined recursively by $u_1 = a$, $u_2 = b$, and $u_m = u_{m-1} + u_{m-2} + k$ for $m \geq 3$. A pair of sets A, B is called an (a, b, k)-partition of $M \subset \mathbf{N}$ if $M = A \cup B$, $A \cap B = \emptyset$, and the sum of two distinct elements of A (respectively, B) is never in U.

Theorem 1.4 (Shan and Zhu, 1993)**.** *If $a > b+k$ and c is even then an (a, b, k)-partition of $\mathbf{N}$ does not exist. If $a \leq b+k$ or c is odd then there are 2^{g-1} different (a, b, k)-partitions of $\mathbf{N}$, where $g = \frac{k+e+1}{2}$ if $k+1 \geq d$ and $g = \lceil \frac{d+e-1}{2} \rceil$ if $k+2 \leq d$.*

As the set $\{a, b, c\}$ is certainly avoidable if each element of the set is odd (let A be the set of odd integers, and thus B is the set of evens), we can assume, in regards to the above theorem, that $e \geq 1$. Thus we have the following corollary.

Corollary 1.5. *Let a, b, $c \in \mathbf{N}$ be given with $a < b < c$.*

(1) *$\{a, b\}$ is avoidable.*
(2) *$\{a, b, c\}$ is avoidable if $a+b \leq c$.*

Proof. To see why any set $\{a, b\}$ is avoidable, apply Theorem 1.4 with a and b defined as in the theorem, and $d = a+b+k$ where d is chosen so that at least one of a, b, or d is even. Then $g \geq 1$, so that there exists an (a, b, k)-partition $\{A, B\}$ of $\mathbf{N}$, and thus $(A \cap \{1, 2, ..., b-1\}) \cup (B \cap \{1, 2, ..., b-1\})$ avoids $\{a, b\}$.

For the second part of the corollary, as $a < b$, $a \leq b+k$. If all of a, b and c are odd then $\{a, b, c\}$ is clearly avoidable, so we assume $e \geq 1$ in Theorem 1.4. Thus $g \geq 1$, and we are done. □

The above corollary provokes the following interesting question.

Question 1.6. *Given an avoidable set* $S = \{u_1, u_2, ..., u_m\} \subset \mathbf{N}$ *with* $u_i < u_j$ *for* $i < j$*, what is the smallest number* $b_m(S) > u_m$ *such that* $\{u_1, u_2, ..., u_m, b_m(S)\}$ *is also avoidable?*

Clearly, $b_m(S) \leq 2u_m - 3$. Beyond the trivial lower bound $b_m(S) \geq u_m + 1$, it is not clear what $b_m(S)$ should be in general. Observe that it is not necessary that $u_{m-1} + u_m \leq b_m(S)$, as $S_1 = \{6, 10, 13\}$ is avoidable by setting $A = \{1, 4, 7, 8, 10, 11\}$ and $B = \{2, 3, 5, 6, 9, 12\}$, while $S_2 = \{8, 12, 17\}$ is avoidable by setting $A = \{1, 4, 5, 6, 9, 10, 14, 15\}$ and $B = \{2, 3, 7, 8, 11, 12, 13, 16\}$. It is easy to deduce that $13 = b_2(S_1)$ and $17 = b_2(S_2)$.

It should be observed that S_2 is a subset of the set $S^* = \{3, 4, 8, 12, 17, 22, 43, 85, ...\}$ (each subsequent element is twice the previous element, minus one), an avoidable set put forward by Chow and Long [2] which is not equal to any avoidable set constructed by said authors. To explain what we mean by this assertion, it is helpful to recall the main results in Chow and Long's paper.

Theorem 1.7 (Chow and Long, 1999)**.** *Let* α *be an irrational number between 1 and 2, and define*

$$A_\alpha := \{n \in \mathbb{N} : \textit{the integer multiple of } \alpha \textit{ nearest } n \textit{ is greater than } n\},$$

$$B_\alpha := \{n \in \mathbb{N} : \textit{the integer multiple of } \alpha \textit{ nearest } n \textit{ is less than } n\}.$$

Let S_α *be the set of all positive integers avoided by the partition* $\{A_\alpha, B_\alpha\}$*. Then* S_α *contains all numerators of continued fraction convergents of* α*.*

Theorem 1.8 (Chow and Long, 1999)**.** *Let* α*,* A_α*,* B_α *and* S_α *be as in Theorem 1.7. Then every element of* S_α *is either the numerator of a convergent of* α*, the numerator of an intermediate fraction, or twice the numerator of a convergent.*

In the manner of [1], Chow and Long say that a set $S \subset \mathbb{N}$ is *saturated* if it is avoidable and it is maximal (with respect to set inclusion) among all avoidable sets. Saturated sets are not necessarily uniquely avoidable, and indeed Chow and Long prove that there exist saturated sets that are avoided by arbitrarily large numbers of partitions. It is in this overall context that the set S^* is presented for, while Chow and Long note that said set is uniquely avoidable, its saturation is not equal to any of the sets S_α presented in [2], and thus the sets S_α do not exhaust the class of all saturated uniquely avoidable sets.

2. A Function That Counts the Number of Unavoidable Sets of a Given Size

2.1. **Sets of Sizes Three and Four.** We now define the following arithmetic function. Given k, $n \geq 3$ with $n \geq k$, let $U(k, n)$ denote the number of sets $S \subset \{1, 2, ..., n\}$ with $|S| = k$ that are unavoidable. The reason that we restrict k and n to be greater than 2 is so that the quantity $U(k, n)$ will be nontrivial, as singleton sets are avoidable, while sets of size two, by Corollary 1.5, are also avoidable. By Proposition 1.1, sets S of the form $S = \{a, a+1, a+2\}$ with $a > 1$ odd are unavoidable. Of course, $U(k, n) \leq \binom{n}{k}$, and thus $U(k, n) = O(n^k)$ for any choice of k and n. Indeed, if $k \leq \lceil \frac{n}{2} \rceil = v$ then we have $U(k, n) \leq \binom{n}{k} - \binom{v}{k}$, as sets composed entirely of odd numbers are avoidable. As an unavoidable set of size k in $\{1,2,...,n\}$ remains unavoidable on $\{1,2,...,n+1\}$, we have $U(k,n) \leq U(k,n+1)$. Indeed, provided that $k < n$ we have $U(k,n) = U(k,n-1)+F$ where F denotes the number of unavoidable sets S of size k with $n \in S$. Defining $U_i(k,n)$ for $1 \leq i \leq n$ to be the number of unavoidable sets $S \subset \{1,2,...,n\}$ with $i \in S$, we have the following.

Proposition 2.1. *For all k, $n \geq 3$ with $k < n$ we have*

$$U(k,n) = U(k,n-1) + U_n(k,n).$$

From this we have

$$U(k,n) \leq U(k,n-1) + \binom{n-1}{k-1}.$$

When $k = 3$, it is easy to obtain linear lower and upper bounds for $U(3,n)$ in terms of $U(3,n-1)$. We address lower bounds first, via the following proposition.

Proposition 2.2. *For $n \geq 7$, $U(3,n) \geq U(3,n-1)+n-4$.*

Proof. It is straightforward to show that one can choose unavoidable sets of the form $\{i,n-1,n\}$ for $3 \leq i \leq n-2$ and n odd. Then, for n even, one does the additional work of noting that $\{n-4,n-2,n\}$ is also unavoidable, completing the proof. □

For upper bounds, we consider avoidable sets of the form $\{a,b,n\}$, using the bound $U_n(3,n) \leq \binom{n-1}{2}$ to obtain said upper bounds. Specifically, we observe, using the first part of Corollary 1.5, that all sets of either the form $\{1,j,n\}$, $1 < j < n$ and $n \geq 4$, or $\{2,j,n\}$, $2 < j < n$ and $n \geq 4$, are avoidable, thus $U_n(3,n) \leq \frac{1}{2}n^2 - \frac{7}{2}n + 6$. Further, for $n \geq 7$ odd we note that all sets of the form $\{j_1,j_2,n\}$ where $3 \leq j_1 < j_2 < n$ are distinct and odd, are avoidable, thus we can improve the upper bound for $U(3,n)$ here to $U_n(3,n) \leq \frac{3}{8}n^2 - \frac{5}{2}n + \frac{33}{8}$. As an alternate approach, the second part of Corollary 1.5 provides us with the following upper bounds for $U(3,n)$, depending on the parity of n. If n is even, $U_n(3,n) \leq \frac{(n-2)^2}{4}$. If n is odd, $U_n(3,n) \leq \frac{n^2-4n+3}{4}$.

More importantly, however, Corollary 1.5 allows us to get an exact count for $U(3,n)$. To do this, we rely on Corollary 1.5 as well as the following definition, lemmas, and theorem below (see [3]; proofs of the lemmas and theorem are due to Dumitriu [4]), and also the last two bounds for $U_n(3,n)$ given in the prior paragraph.

Definition 2.3. *The symmetric Boolean variable $P(a,b,c)$ is defined to be equal to* 1 *if there exists a partition of $\{1,...,\max(a,b,c)-1\}$ avoiding $\{a,b,c,a+b+c\}$, and* 0 *otherwise.*

Lemma 2.4. *Let a, b, $c \in \mathbf{N}$ be given, with $a < b < c < a+b$. If $a+b+c$ is even then a partition of $\{1,2,...,c-1\}$ avoiding $\{a,b,c\}$ does not exist.*

Lemma 2.5. *Let T denote the Tribonacci sequence defined by $t_m = t_{m-1} + t_{m-2} + t_{m-3}$ for $m \geq 4$ with initial terms $t_1 = a$, $t_2 = b$, and $t_3 = c$. Then T is avoidable if and only if there is a partition of $\{1,2,...,c-1\}$ avoiding a, b, c, and $a+b+c$. When $a < b < c < a+b$ it suffices to avoid a, b, and c.*

Theorem 2.6. *Say $t_1 = a$, $t_2 = b$, and $n = c$ with a, b, and c not all odd. If $a < b < c < a + b$ then $P(a, b, c) = 1$ if and only if either:*

(1) *a and b are even, c is odd, and $c \geq b + \frac{a}{2}$; or*
(2) *b and c are even, a is odd, and $a \leq \frac{c}{2}$; or*
(3) *a and c are even, b is odd, and either $2b = a + c$, or $c \geq b + \frac{a}{2}$ and $a \leq \frac{c}{2}$.*

It should be observed that, in regards to the first condition of Theorem 2.6, there is a typographical error in both [3] and [4]. Namely, the condition for both a and b even is incorrectly stated, as the statement in both is $c \geq a + \frac{b}{2}$. (To see why we need the above condition instead, try $a = 4$, $b = 10$, $c = 11$; the resulting set $\{4, 10, 11\}$ is unavoidable.) However, Dumitriu's proofs are correct, and so we proceed.

Corollary 1.5 shows that we want to count the number of unavoidable sets of the form $S = \{t_1, t_2, n\}$ where $t_1 < t_2 < n < t_1 + t_2$ and at least one of the members of S is even; the inequality condition conforms to the hypothesis of Theorem 2.6. Further, according to Lemma 2.5 it suffices to avoid t_1, t_2, and n, which are a, b, and c for us respectively.

Letting $D(c)$ denote the number of triplets (a, b, c) for which $P(a, b, c) = 1$, $c < a + b$ and at least one of a, b, or c is even, it follows that

$$U_n(3, n) = \frac{(n-2)^2}{4} - D(n)$$

if n is even, while if n is odd, we have

$$U_n(3, n) = \frac{n^2 - 4n + 3}{4} - D(n) - \delta(n),$$

where $\delta(n)$ represents the number of triplets (a, b, n) with $n < a + b$ where a and b are odd. Observe that $\delta(5) = 0$ and $\delta(7) = 1$. For $n \geq 9$, it is an easy matter to check that

$$\delta(n) = 2\sum_{j=1}^{\frac{n-5}{4}} j \tag{1}$$

if $n \equiv 1 \pmod 4$, while if $n \equiv 3 \pmod 4$ we have

$$\delta(n) = -\frac{n-3}{4} + 2\sum_{j=1}^{\frac{n-3}{4}} j. \tag{2}$$

Now we evaluate $D(n)$. While the counts we perform here and later (when n is even) may seem both tedious and redundant (in light of Propositions 2.8 and 2.10), we feel it is important nonetheless to include these arguments because

- The following arguments place the notion of evaluating $D(n)$ in a geometric context, and is thus appealing to those readers who prefer visual presentations.

- The geometric proofs used here may take on greater meaning, should the work in this paper be generalized to higher dimensions, that is, the calculation of $U(k, n)$ for $k > 3$. (See the end of Section 2, specifically, Problem 2.19 for a discussion of this.)

Suppose first that n is odd, $n \geq 5$. As $D(5) = D(7) = 1$ (the sets $\{2, 4, 5\}$ and $\{2, 6, 7\}$ are avoidable by Theorem 2.6), we focus our attention on $n \geq 9$. We count the number of avoidable sets satisfying part one of Theorem 2.6, namely $a < b < n < a + b$, a and b even, n odd, and $n \geq b + \frac{a}{2}$. Treat n as a constant, a as the independent variable and b as the dependent variable, and graph the lines $a = b$, $b = n - \frac{a}{2}$, and $b = n - a$ on the ab-plane. (We urge the reader to note that the reference to a as "the independent variable" and b as "the dependent variable" is done in the context of graphing, that is, the a-axis will be for us the horizontal axis while the b-axis is the vertical axis. However, our approach will be to adjust values of b, then ask what happens to the values of a accordingly.) We need to count all the ordered pairs (a, b) in this feasible region that have even coordinates. We first focus on the values of b. If $b = n - 1$ then $a = 2$. Decreasing b to $b = n - 3$ gives us two choices for a, namely $a = 4, 6$. Each time we decrease the value of b by 2, it is clear that we pick up an additional choice for a, that is, when $b = n - 5$ we have three choices for a, and so forth. Continue this count while $b > \frac{2n}{3}$. To formulate the count thus far one only need to evaluate the sum $\sum_{i=1}^{\lfloor \frac{n+1}{6} \rfloor} i = \frac{1}{2}(\lfloor \frac{n+1}{6} \rfloor^2 + \lfloor \frac{n+1}{6} \rfloor)$.

Now consider the values for b where $\frac{n}{2} < b \leq \frac{2n}{3}$. Recalling the feasible region we constructed above, note that if we have a value for b and $b - 2$ also intersects the feasible region, we will lose two choices for the possible values of a. That is, when we are in this part of the feasible region decreasing the value of b by 2 decreases the choices for a by 2. The count of the possible ordered pairs (a, b) with even coordinates will depend on two factors: the parity of $\lfloor \frac{n+1}{6} \rfloor$, and the value of $\frac{2n}{3}$. We argue according to n's residue modulo 6.

Suppose first that $n \equiv 5 \pmod 6$. Then when $b = \frac{2n+2}{3}$, a can take on any even number between $\frac{n+1}{3}$ and $\frac{2n-4}{3}$ inclusive. There are $\frac{n+1}{6}$ such choices for a. Now reduce b to $b = \frac{2n-4}{3}$, and note now that we have only $\frac{n+1}{6} - 2$ choices for a, namely the even integers between $\frac{n+7}{3}$ and $\frac{2n-10}{3}$. As b is now less than $\frac{2n}{3}$, each reduction of b's value by 2 reduces the number of choices for a by 2. Thus we are adding all the positive integers less than or equal to $\lfloor \frac{n+1}{6} \rfloor - 2$ that have the same parity as $\lfloor \frac{n+1}{6} \rfloor$.

Now suppose that $n \equiv 1 \pmod 6$. When $b = \frac{2n+4}{3}$, a can take on any even number between $\frac{n-1}{3}$ and $\frac{2n-8}{3}$ inclusive. Note that a cannot equal $\frac{2n-2}{3}$, as this violates the condition $n \geq b + \frac{a}{2}$. Thus the number of choices for a is $\lfloor \frac{n+1}{6} \rfloor$. Now reduce b to $\frac{2n-2}{3}$, and observe that we only lose one choice for a, namely $a = \frac{n-1}{3}$. Now that $b < \frac{2n}{3}$, each reduction of b's value by 2 reduces the number of choices for a by 2. Hence we are adding all the positive integers less than or equal to $\lfloor \frac{n+1}{6} \rfloor - 1$ that have parity opposite that of $\lfloor \frac{n+1}{6} \rfloor$.

Finally, suppose that $n \equiv 3 \pmod 6$. When $b = \frac{2n+6}{3}$, a can take on any even number between $\frac{n-3}{3}$ and $\frac{2n-12}{3}$ inclusive, for a total of $\lfloor \frac{n+1}{6} \rfloor$ choices for a. When we reduce b to $b = \frac{2n}{3}$, we still have $\lfloor \frac{n+1}{6} \rfloor$ choices for a, namely the even integers from $\frac{n+3}{3}$ and $\frac{2n-6}{3}$ inclusive. Now that $b \leq \frac{2n}{3}$, each reduction of b's value by 2 reduces the number of choices for a by 2. Hence we are adding the positive integers with the same parity as $\lfloor \frac{n+1}{6} \rfloor$ in the interval $[\lfloor \frac{n+1}{6} \rfloor, 1]$.

We summarize as follows:

If $n \equiv 5 \pmod 6$ then $\mathcal{I} = \{i \in [\lfloor \frac{n+1}{6} \rfloor - 2, 1] : i\ ,\ \lfloor \frac{n+1}{6} \rfloor$ have the same parity$\}$.
If $n \equiv 1 \pmod 6$ then $\mathcal{I} = \{i \in [\lfloor \frac{n+1}{6} \rfloor - 1, 1] : i\ ,\ \lfloor \frac{n+1}{6} \rfloor$ have opposite parity$\}$.
If $n \equiv 3 \pmod 6$ then $\mathcal{I} = \{i \in [\lfloor \frac{n+1}{6} \rfloor, 1] : i\ ,\ \lfloor \frac{n+1}{6} \rfloor$ have the same parity$\}$.

Hence for n odd

$$D(n) = \frac{1}{2}\left(\lfloor \frac{n+1}{6} \rfloor^2 + \lfloor \frac{n+1}{6} \rfloor\right) + \sum_{i \in \mathcal{I}} i. \tag{3}$$

Thus we have the following theorem.

Theorem 2.7. *For $n \geq 9$ odd, we have*

$$U(3,n) - U(3,n-1) \quad = \quad \frac{n^2 - 4n + 3}{4} - \delta(n) - D(n),$$

where $\delta(n)$ is given by Equation (1) or (2), as appropriate, and $D(n)$ is given by (3).

Interestingly, Equation 3 can be connected to partitions in the following way, where by "partitions" we mean additive partitions in the "classical" sense of the term (see Section 2.5 of [6], for example).[1]

Proposition 2.8. *For $n \geq 5$ odd, $D(n) = p_3\left(\frac{n-5}{2}\right)$, where for nonnegative integer x, $p_3(x) = \left[\frac{(x+3)^2}{12}\right]$ is the number of additive partitions of x into at most three parts (not necessarily distinct), and $[y]$ represents the nearest integer to y.*

Proof. Observe that with $x = \frac{n-5}{2}$, $p_3(x) = \left[\frac{(n+1)^2}{48}\right]$. We will do our work modulo 12, and prove the case $n \equiv 3 \pmod{12}$, noting that the proofs of the other cases proceed similarly. For $n \equiv 3 \pmod{12}$, $\lfloor \frac{n+1}{6} \rfloor = \frac{n-3}{6}$ and thus

$$D(n) = \frac{n^2 - 9}{72} + \sum_{i \in \mathcal{I}} i,$$

where $\mathcal{I} = \{2, 4, ..., (n-3)/6\}$. Thus, a little arithmetic shows that

$$D(n) = \frac{n^2 - 9}{72} + \frac{n^2 + 6n - 27}{144} = \frac{n^2 + 2n - 15}{48} = \left[\frac{(n+1)^2}{48}\right].$$

□

[1]The authors thank James Sellers of The Pennsylvania State University and Timothy Flowers of Clemson University for pointing this out. The first author would also like to thank Florian Luca, Pantelimon Stanica, and others who attended the Integers Conference at the State University of West Georgia in November 2003 for helpful discussions.

For n even, we note first that $D(4) = 1$. In reference to Theorem 2.6, we can count explicitly the number of sets of the form $\{a, b, n\}$, with $a < b < n < a+b$ and $P(a, b, n) = 1$. Fix n.

Count for part two of Theorem 2.6, $a < b < n < a+b$ with a odd, b and n even and $a \leq \frac{n}{2}$. Observe that if $n \equiv 2 \pmod 4$ then $\frac{n}{2}$ is odd and thus is a candidate for a value of a. Letting $a = \frac{n}{2}$, the choices for b are the even integers in the interval $[\frac{n}{2}+1, n-2]$. Decreasing the value of a by 2, the subsequent choices for b are restricted to the even integers in the interval $[\frac{n}{2}+3, n-2]$, and so on. One terminates the count when $a = 3$ and $b = n-2$.

If $n \equiv 0 \pmod 4$ then $\frac{n}{2}$ is even and cannot be a candidate for a. Letting $a = \frac{n}{2} - 1$, then b can be any even integer in the interval $[\frac{n}{2}+2, n-2]$. Again decreasing the value of a by 2 and adjusting the interval for b accordingly, we terminate the count when $a = 3$ and $b = n-2$.

Thus the number of avoidable sets corresponding to part two of Theorem 2.6 is

$$D_2(n) = \frac{\lfloor \frac{n-2}{4} \rfloor^2 + \lfloor \frac{n-2}{4} \rfloor}{2}, \quad n > 4. \tag{4}$$

Now consider the avoidable sets which satisfy the first condition of part three of Theorem 2.6, namely $a < b < n < a+b$ with b odd, a and n even and $2b = a+n$. To find the number of avoidable sets satisfying this condition, fix n and approach the matter geometrically by considering a as one would the variable x, b as the variable y, and graphing lines on the ab-plane. Specifically, find the intersection of the lines $2b = a+n$ and $n = a+b$, as well as the intersection of the lines $2b = a+n$ and $a = b$. Note from the graph that the number of avoidable sets is precisely the number of odd integers in the interval $(\frac{2n}{3}, n)$. Considering n modulo 6, one concludes that the number of avoidable sets satisfying the stated condition is

$$D_{3,1}(n) = \lfloor \frac{n+2}{6} \rfloor, \quad n \geq 4. \tag{5}$$

For the second condition of part three of Theorem 2.6, namely $a < b < n < a+b$ with b odd, a and n even and $n \geq b + \frac{a}{2}$ and $a \leq \frac{n}{2}$, we argue as follows.

If $n \equiv 0 \pmod 8$, then $\frac{n}{2}$ is even and can be a choice for a. If $b = n-1$, then for a, any even integer in the interval $[2, \frac{n}{2}]$ would satisfy $n < a+b$; however, 2 is the only element that also satisfies $n \geq b + \frac{a}{2}$. Now decrease b by 2 to obtain $b = n-3$. Any even integer a in $[4, \frac{n}{2}]$ would satisfy $n < a+b$, but only 4 and 6 would satisfy $n \geq b + \frac{a}{2}$. Thus, each decrease of b by 2 yields one more choice for a than the previous step; specifically, if $b = n-j$ for j an odd integer, then the number of choices for a is $\frac{j+1}{2}$. Continue this process so long as $b > \frac{3n}{4}$. For the remaining possible values for b, specifically, the odd integers in the interval $[\frac{n}{2}, \frac{3n}{4}]$ the inequality $n \geq b + \frac{a}{2}$ is no longer applicable, all one needs to be concerned with is the inequality $n < a+b$. Let $b = \frac{3n}{4} - 1$, then any even integer in the interval $[\frac{n}{4}+2, \frac{n}{2}]$ would suffice. Decreasing b by 2 restricts the choices for a to the even integers in the interval $[\frac{n}{4}+4, \frac{n}{2}]$. Continue in this manner until $b = \frac{n}{2}+1$. Note that $a = \frac{n}{2}$ and $b = \frac{n}{2}+1$ satisfy $n < a+b$. Also note that when $b = \frac{3n}{4} - 1$ and

$b = \frac{3n}{4} + 1 = n - (\frac{n}{4} - 1)$ we have the same number of possible values for a, namely $\frac{n}{8}$.

If $n \equiv 2 \pmod 8$, replace $\frac{n}{2}$ with $\frac{n}{2} - 1$, $\frac{n}{4}$ with $\lfloor \frac{n}{4} \rfloor$, and $\frac{3n}{4}$ with $\lceil \frac{3n}{4} \rceil$, then proceed as above.

If $n \equiv 4 \pmod 8$, then $\frac{n}{2}$ is even and can be a choice for a. Note that both $\frac{n}{4}$ and $\frac{3n}{4}$ are both odd integers. Proceeding as we did in the $n \equiv 0 \pmod 8$ case, we replace $b = \frac{3n}{4} - 1$ with $b = \frac{3n}{4}$, and thus $[\frac{n}{4} + 2, \frac{n}{2}]$ with $[\frac{n}{4} + 1, \frac{n}{2}]$, and so forth. In this case note that when $b = \frac{3n}{4}$ there is one more possible value for a, as compared to when $b = \frac{3n}{4} + 2$.

Finally, let $n \equiv 6 \pmod 8$. We proceed as we did in the case where $n \equiv 4 \pmod 8$, except we replace $\frac{n}{2}$ with $\frac{n}{2} - 1$, $\frac{3n}{4}$ with $\lceil \frac{3n}{4} \rceil$, and $\frac{n}{4}$ with $\lfloor \frac{n}{4} \rfloor$.

Thus the number of avoidable sets satisfying this condition is:

$$(6) \qquad D_{3,2}(n) = \left\{ \begin{array}{ccc} \lfloor \frac{n}{8} \rfloor^2 + \lfloor \frac{n}{8} \rfloor & \text{if} \quad n \equiv 0, 2 & \pmod 8 \\ \lfloor \frac{n+4}{8} \rfloor^2 & \text{if} \quad n \equiv 4, 6 & \pmod 8 \end{array} \right\} - \varepsilon(n), \quad n \geq 4$$

where $\varepsilon(n)$ is the number of triples $\{a, b, n\}$ that were counted in (5). Thus $D(n) = D_2(n) + D_{3,1}(n) + D_{3,2}(n)$.

To calculate $\varepsilon(n)$, we need to count the number of $\{a, b, n\}$ that satisfy the requirements for both (5) and (6). That is, a and n are even, b is odd, $n < a+b$, $a \leq \frac{n}{2}$, $n \geq b + \frac{a}{2}$, and $2b = a + n$. Treat n as a constant, a as the independent variable, and b as the dependent variable. Solving this system yields the inequality $\frac{2n}{3} < b \leq \frac{3n}{4}$. Since b must be an odd integer we have $\varepsilon(n) = \#\{b \in \mathbf{N} : \frac{2n}{3} < b \leq \frac{3n}{4} \text{ and } b \text{ is odd}\}$. One can thus determine a formula for $\varepsilon(n)$ by working modulo $2\,\mathrm{lcm}(3, 4) = 24$. For the sake of brevity, we will only discuss the case $n \equiv 0 \pmod{24}$, as the other cases are similarly handled. For this case, both $\frac{2n}{3}$ and $\frac{3n}{4}$ are even, thus the number of odd integers in the interval $[\frac{2n+3}{3}, \frac{3n-4}{4}]$ is $\frac{n}{24}$. We have

$$(7) \qquad \varepsilon(n) = \left\{ \begin{array}{ll} \lceil \frac{n}{24} \rceil & \text{if} \quad n \equiv 4, 10, 12, 16, 18, 20, 22 \pmod{24} \\ \lfloor \frac{n}{24} \rfloor & \text{if} \quad n \equiv 0, 2, 6, 8, 14 \pmod{24} \end{array} \right\}, \quad n \geq 4.$$

Theorem 2.9. *For $n \geq 6$ even, we have $U(3, n) - U(3, n-1) = \frac{(n-2)^2}{4} - D(n)$ where $D(n)$ is the sum of the expressions given in (4), (5), and (6).*

As it turns out, we can also discern an intimate connection between the functions $D(n)$ and $p_3(n)$ for n even.[2]

Proposition 2.10. *For $n \geq 6$ even, we have*

$$D(n) = \left\{ \begin{array}{ccc} p_3\left(\frac{3n-8}{4}\right) & \text{if} \quad n \equiv 0, 4 & \pmod 8 \\ p_3\left(\frac{3(n-2)}{4}\right) - \tau(n) & \text{if} \quad n \equiv \pm 2 & \pmod 8 \end{array} \right\},$$

where $\tau(n) = 0$ or 1 according to whether $n \equiv -2$ or $2 \pmod 8$, respectively.

[2] Again, the authors thank James Sellers and Timothy Flowers for drawing our attention to this.

Proof. The proof is accomplished in the same manner as the proof of Proposition 2.8, and is entirely straightforward. One finds that by working modulo 24 (in accordance with Equation 7), one obtains a uniform expression for $D(n)$ in each case, regardless of the residue of n modulo 24. $\square$

We now turn our attention to the case $k = 4$. For this case we can use Propositions 1.1, 1.2 and 1.3 to say the following.

Theorem 2.11. *For n sufficiently large,*

$$U(4, n) \;\; \geq \;\; (n-4) + \frac{(n-6)(n-5)}{2} + \sum_m m + \sum_j \frac{j(j+1)}{2}, \tag{8}$$

where the first sum is taken over all even m from 2 to $n-6$ if n is even, and is taken over all odd m from 1 to $n-6$ if n is odd, while the second sum is taken over all odd j from 1 to $n-7$ if n is even, and is taken over all even j from 2 to $n-7$ if n is odd.

Proof. By Proposition 1.1, the sets $\{2,3,4,5\}$, $\{3,4,5,6\}$, ..., $\{n-3,n-2,n-1,n\}$ are unavoidable. There are $n-4$ such sets. By Propositions 1.2 and 1.3, the sets $\{3,4\} \cup \{6,7\}$, $\{3,4\} \cup \{7,8\}$, ..., $\{3,4\} \cup \{n-1,n\}$ are unavoidable, as are the sets $\{4,5\} \cup \{7,8\}$, $\{4,5\} \cup \{8,9\}$, ..., $\{4,5\} \cup \{n-1,n\}$, and so forth, on up to $\{n-4,n-3\} \cup \{n-1,n\}$, for a total of $\frac{(n-6)(n-5)}{2}$ sets. Noting that $\{a,a+1,a+2\}$ is unavoidable for odd $a \geq 3$, we conclude that for even n the sets $\{a,a+1,a+2\} \cup \{i\}$ for odd a from 3 to $n-5$ and for i from $a+4$ to n are unavoidable, while for odd n the sets $\{a,a+1,a+2\} \cup \{i\}$ for odd a from 3 to $n-4$ and for i from $a+4$ to n are unavoidable. There are a total of $\sum_m m$ such sets, where the sum is taken over all even m from 2 to $n-6$ if n is even, and is taken over all odd m from 1 to $n-6$ if n is odd.

Finally, using Propositions 1.2 and 1.3 we observe that for each i from 3 to $\lfloor \frac{n-2}{2} \rfloor$, the sets $\{i\} \cup \{2i, 2i+1, 2i+2\}$, $\{i\} \cup \{2i, 2i+1, 2i+3\}$, ..., $\{i\} \cup \{2i, 2i+1, n\}$, $\{i\} \cup \{2i+1, 2i+2, 2i+3\}$, $\{i\} \cup \{2i+1, 2i+2, 2i+4\}$, ..., $\{i\} \cup \{n-2, n-1, n\}$ are unavoidable. The total number of such sets is $\sum_j \frac{j(j+1)}{2}$ where the sum is taken over all odd j from 1 to $n-7$ if n is even, and is taken over all even j from 2 to $n-7$ if n is odd. This completes the proof. $\square$

Corollary 2.12. $U(4, n) = \Omega(n^3)$.

Proof. This is an immediate consequence of the well-known "sum-of-squares" identity $\sum_{j=1}^{t} j^2 = \frac{t(t+1)(2t+1)}{6}$. $\square$

We close this section by using the formulas given above to determine the values of $U(3, n)$ and $U(4, n)$ for $3 \leq n \leq 12$ and $4 \leq n \leq 12$ in Table 1. As a somewhat interesting aside, we note that for $6 \leq n \leq 10$, $U(3, n) = a(n-5)$ where for $m \geq 1$, $a(m) = T(m+1, m)$, $T(i, j) = b(i+1) - b(i+1-j)$ for $j = 1, 2, ..., i$ denoting the (i, j)th entry of the triangular array described in the On-Line Encyclopedia of Integer Sequences at

http://www.research.att.com/~njas/sequences/index.html
(sequence A048201), where for $m \geq 3$, $b(m)$ equals the least number s such that $s - b(m-1) \neq b(j) - b(k)$ for all j, $k < s$, with initial conditions $b(1) = 1$ and $b(2) = 2$. However, this pattern begins to diverge at $n = 11$ ($U(3, 11) = 42$ while $a(6) = 43$, and $U(3, 12) = 59$ while $a(7) = 58$, et cetera).

n	$U(3,n)$	$U(4,n)$
3	0	n/a
4	0	0
5	1	2
6	2	5
7	6	17
8	11	37
9	19	71
10	29	122
11	42	196
12	59	304

TABLE 1. Values for $U(3, n)$ and $U(4, n)$

2.2. **The General Case.** In order to obtain a lower bound for $U(k, n)$ for all $k \geq 4$, we define the following function. For j, l, $m \geq 1$ with $l \leq m$, let $G_{m,l} := G(m, l)$ denote the number of ways to choose l numbers from $\{j, j+1, ..., j+m-1\}$ so that the resulting set has at least one pair a, b with $b = a+1$, that is, at least one consecutive pair of integers. Observe that $G_{m,l} = \binom{m}{l} - F(m, l)$ where $F_{m,l} := F(m, l)$ denotes the number of ways to choose l numbers from $\{j, j + 1, ..., j + m - 1\}$ such that the resulting set has no consecutive pair. For $m \geq 4$, we have

$$F_{m,l} = F_{m-2,l} + F_{m-2,l-1} + F_{m-3,l-1}. \tag{9}$$

This is seen by counting the number of sets with no consecutive pairs that contain at least one of j or $j+1$. Clearly, no such set contains both. The number of such sets containing j is $F_{m-2,l-1}$, the number containing $j + 1$ is $F_{m-3,l-1}$, and the number containing neither is $F_{m-2,l}$.

Equation (9) is a recurrence relation with two indices, l and m, and can be determined straightforwardly via generating functions. To begin, observe that $F_{m,0} = 1$ for each $m \geq 1$, $F_{m,1} = m$ for each positive m, and $F_{m,l} = 0$ for $0 \leq m < l$. We set $F_{0,0} = 1$. For a given m, we multiply both sides of (9) by x^l and sum from $l = 1$ to infinity to obtain, for $m \geq 4$,

$$F_m(x) = (1 + x)F_{m-2}(x) + xF_{m-3}(x) \tag{10}$$

where $F_m(x) = \sum_{l=0}^{\infty} F_{m,l} x^l$. To solve (10), we note the boundary conditions $F_0(x) = 1$, $F_1(x) = 1 + x$, $F_2(x) = 1 + 2x$, and $F_3(x) = 1 + 3x + x^2$. Multiplying both sides of (10) by y^m and summing from $m = 4$ to infinity yields

$$\mathcal{F}(x,y) = \frac{xy^2 + (x+1)y + 1}{1 - (x+1)y^2 - xy^3} \tag{11}$$

where $\mathcal{F}(x,y) = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} F_{m,l} x^l y^m$.

We are now ready to provide the desired lower bound.

Theorem 2.13. *For all $k \geq 4$ and n sufficiently large,*

$$U(k,n) \geq \sum_{i=3}^{\lfloor \frac{n-k+2}{2} \rfloor} G_{n-2i+1,k-1}. \tag{12}$$

Proof. We use the same reasoning as in the proof of Theorem 2.11. Specifically, the above bound is an immediate consequence of the fact that all sets of the form $\{i\} \cup T$, $3 \leq i \leq \lfloor \frac{n-k+2}{2} \rfloor$, $T \subset \{2i, 2i+1, ..., n\}$ with $|T| = k-1$ and T containing at least one consecutive pair, are unavoidable. $\square$

Corollary 2.14. *For all $k \geq 4$, $U(k,n) = \Omega(n^{k-1})$.*

Proof. We proceed via induction on k. Specifically, we show that for each $k \geq 4$, and for sufficiently large n, we have

$$\sum_{i=3}^{\lfloor \frac{n-k+2}{2} \rfloor} G_{n-2i+1,k-1} \geq c_{k-1} n^{k-1}$$

for some positive constant c_{k-1} which depends upon k. This, coupled with Theorem 2.13, will establish the corollary.

For $k = 4$, one straightforwardly determines that

$$F_{m,3} = \sum_{j=1}^{m-4} \sum_{i=1}^{j} i = \frac{m^3 - 9m^2 + 26m - 24}{6},$$

thus $G_{m,3} = (m-2)^2$, and so

$$\sum_{i=3}^{\lfloor \frac{n-2}{2} \rfloor} G_{n-2i+1,3}$$
$$= \tfrac{4}{3} \lfloor \tfrac{n}{2} \rfloor^3 + n^2 \lfloor \tfrac{n}{2} \rfloor - 2n \lfloor \tfrac{n}{2} \rfloor^2 - \tfrac{1}{3} \lfloor \tfrac{n}{2} \rfloor - 3n^2 + 18n - 35,$$

establishing the claim for $k = 4$. (Of course, one can also appeal to Corollary 2.12.)

For the inductive step, we first observe that, for $l \geq 4$,

$$F_{m,l} = \sum_{j=2l-3}^{m-2} F_{j,l-1}.$$

That this is true is seen by noting that the number of sets included in the count for $F_{m,l}$ with starting element 1 is $F_{m-2,l-1}$, while the number of such sets with smallest element 2 is $F_{m-3,l-1}$, and so forth, on down to the set $\{m-(2l-2), m-(2l-4), ..., m\}$, with $1 = F_{2l-3,l-1}$. In particular,

$$F_{m,k-1} = \sum_{j=2k-5}^{m-2} F_{j,k-2}.$$

Thus,

$$\begin{aligned} G_{m,k-1} &= \binom{m}{k-1} - \sum_{j=2k-5}^{m-2} F_{j,k-2} \\ &= \sum_{j=2k-5}^{m-2} G_{j,k-2} + \left[\binom{m}{k-1} - \sum_{j=2k-5}^{m-2} \binom{j}{k-2}\right]. \end{aligned}$$

Using the combinatorial identity (see page 52 of [6])

$$\binom{m}{r+1} + \sum_{j=0}^{n-1} \binom{m+j}{r} = \binom{m+n}{r+1}, \tag{13}$$

and the well-known identity $\binom{j}{i} + \binom{j}{i+1} = \binom{j+1}{i+1}$, we have

$$\sum_{i=3}^{\lfloor \frac{n-k+2}{2} \rfloor} G_{n-2i+1,k-1} = \sum_{i=3}^{\lfloor \frac{n-k+2}{2} \rfloor} \sum_{j=2k-5}^{n-2i-1} G_{j,k-2} + \sum_{i=3}^{\lfloor \frac{n-k+2}{2} \rfloor} \binom{n-2i}{k-2} + \mathcal{S}(n,k), \tag{14}$$

where $\mathcal{S}(n,k) = \binom{2k-5}{k-1}\left[-2 + \lfloor \frac{n-k+2}{2} \rfloor\right]$. As

$$\sum_{i=3}^{\lfloor \frac{n-k+2}{2} \rfloor} \binom{n-2i}{k-2} \geq \frac{1}{2} \sum_{i=0}^{n-k-4} \binom{k-2+i}{k-2},$$

we conclude that

$$\sum_{i=3}^{\lfloor \frac{n-k+2}{2} \rfloor} G_{n-2i+1,k-1} \geq \sum_{i=3}^{\lfloor \frac{n-k+2}{2} \rfloor} \sum_{j=2k-5}^{n-2i-1} G_{j,k-2} + \mathcal{S}(n,k) + \frac{1}{2}\binom{n-5}{k-1}, \tag{15}$$

where the last term on the right is a polynomial in n of degree $k-1$, with positive leading coefficient. As

$$\begin{aligned}\sum_{i=3}^{\lfloor \frac{n-k+2}{2} \rfloor} \sum_{j=2k-5}^{n-2i-1} G_{j,k-2} &\geq \sum_{i=3}^{\lfloor \frac{n-k+2}{2} \rfloor} G_{(n-2)-2i+1,k-2} \\ &\geq \sum_{i=3}^{\lfloor \frac{n-k}{2} \rfloor} G_{(n-2)-2i+1,k-2} \\ &\geq c_{k-2} n^{k-2} \qquad (16)\end{aligned}$$

by the inductive hypothesis, we infer that

$$\sum_{i=3}^{\lfloor \frac{n-k+2}{2} \rfloor} G_{n-2i+1,k-1} \geq c_{k-1} n^{k-1}$$

for some positive constant c_{k-1}. This, coupled with Theorem 2.13, establishes the corollary. □

Corollary 2.15. *For all $k \geq 4$, $U(k,n) = \Theta(n^{k-1} f(n))$ where $f(n) = n$ or $f(n) = o(n)$.*

Problem 2.16. *Precisely determine f.*

We conclude this section by offering the following open problems.

Problem 2.17. *Determine the behavior of $U_i(k,n)$ for $3 \leq i, k \leq n$.*

Of course we have resolved the case $i = n$ and $k = 3$.

Problem 2.18. *A close connection has been observed between $D(n)$ and $p_3(n)$ for $k = 3$ (see Propositions 2.8 and 2.10). Are such relations between $D(n)$ and $p_k(n)$ present for higher values of k?*

Problem 2.19. *Obtain formulas for $U(k,n)$ in higher dimensions by counting points in regions bounded by hyperplanes of dimension $k-2$.*

This last problem is given in accordance with the method used to obtain the results in Theorems 2.7 and 2.9, and, as the regions we considered were triangular and thus convex, is closely related to the famous problem, first enunciated by Minkowski, of counting the number of lattice points in a convex region in m dimensions (see [7] for a survey of this well-studied problem).

3. Avoidability of (Unions of) Arithmetic Progressions

Finally, we consider sequences $S = \{s_n\}$ that we shall refer to as *r-step sequences*. Specifically, r-step sequences are unions of r arithmetic progressions having the same difference. Thus, the sequences in which we are interested are those in which $s_0 = c$ and $s_{jr+w} = s_{jr+w-1} + v_{w-1}$ for $j \geq 0$ and $1 \leq w \leq r$, with $s_{jr} = s_{(j-1)r+r-1} + v_{r-1}$ for each positive j. The vector $(c, v_0, v_1, ..., v_{r-1}) \in \mathbf{N}^{r+1}$ is given. When $r = 1$ we have an arithmetic sequence with $v_0 = a$ and $s_n = s_{n-1} + a = s_0 + na$ for all natural numbers n. The characterization of r-step sequences turns out to be fairly easy.

Indeed, the following theorem shows that the only avoidable progressions are ones that have only odd numbers.

Theorem 3.1. *Let $S = \{s_n\}$ be given, where $s_0 = c$ and $s_{jr+w} = s_{jr+w-1} + v_{w-1}$ for $j \geq 0$ and $1 \leq w \leq r$, with $s_{jr} = s_{(j-1)r+r-1} + v_{r-1}$ for each positive j, and with vector $(c, v_0, v_1, ..., v_{r-1}) \in \mathbf{N}^{r+1}$. Set $d = \sum_{i=0}^{r-1} v_i$, let $0 \leq \bar{c}_{-1} \leq d-1$ denote the reduction of c modulo d, and let $\bar{c}_j$ denote the reduction of $C_j = c + \sum_{i=0}^{j} v_i$ modulo d for j from -1 to $r-1$. Then S is avoidable precisely when none of $2x = \bar{c}_{-1} \pmod{d}$, $2x = \bar{c}_0 \pmod{d}$, ..., $2x = \bar{c}_{r-1} \pmod{d}$ has a solution in $\mathbf{Z}_d$, the integers modulo d. That is, S is avoidable precisely when c is odd, each C_i is odd, and d is even.*

Proof. To go in one direction, suppose $2x = \bar{c}_j \pmod{d}$ has a solution $x_0 \in \mathbf{Z}_d$ for some j between -1 and $r-1$, and write $C_j = dq + \bar{c}_j$. Then $2x_0 = C_j + d(t-q)$ for some t. If S is avoidable, then write $\mathbf{N} = A \cup B$ where A, $B \neq \emptyset$ and $A \cap B = \emptyset$, suppose x_0 lies in A, and let $l > 0$ represent the smallest integer such that $t-q+l > 0$. Then $x_0+ld \in B$, else $x_0+(x_0+ld) = C_j+d(t-q+l) = s_{(t-q+l)r+j+1}$, a contradiction. But then $x_0 + (l+1)d$ cannot appear in either A or B, using similar reasoning, and we conclude that S is unavoidable.

For the other direction, suppose none of $2x = \bar{c}_{-1} \pmod{d}$, $2x = \bar{c}_0 \pmod{d}$, ..., $2x = \bar{c}_{r-1} \pmod{d}$ has a solution $x_0 \in \mathbf{Z}_d$. This happens precisely when c is odd, each C_i is odd, and d is even. Then it is easy to see that S, as it is a subset of the odd positive integers, can be avoided by letting A be the set of odd positive integers. $\square$

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