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EXTREMAL PROBLEMS ABOUT ASYMPTOTIC BASES: A SURVEY

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Abstract

I give an account of results and open problems on asymptotic bases in general additive number theory, inspired and arisen from the paper "On bases with an exact order" by Paul Erdős and Ronald L. Graham, published in Acta Arith. 37 (1980), 201–207. I survey papers by Melvyn B. Nathanson, Xing-De Jia, John C. M. Nash, Alain Plagne, Julien Cassaigne, Bruno Deschamps and myself.

–To Ron Graham, with my warmest wishes for his 70th birthday.

1. The Original Erdős-Graham Question

I present here the original Erdős-Graham question and I discuss two observations that allow us to extend their setting.

Consider the set of positive squares $Q = \{1^2, 2^2, \ldots\}$. By Lagrange's theorem, every positive integer is the sum of *at most* 4 squares and 4 is minimal. It was proved by G. Pall [Pa] that every *sufficiently large* positive integer is the sum of *exactly* 5 positive squares and that 5 is minimal. Thus we have two distinct notions of "order", the *"at most order"*

$$\mathrm{ord}(Q) = 4,$$

and the *"exact order"*

$$\mathrm{ord}^*(Q) = 5.$$

Notice that in the definition of the "exact order" one cannot avoid the phrase "sufficiently large" [every *sufficiently large* positive integer]. The definition of the "at most order" in the Erdős-Graham paper is taken in the sense of "asymptotic bases": $\mathrm{ord}(A) = h$ means that every sufficiently large (positive) integer is the sum of at most h elements of A and h is minimal; $\mathrm{ord}^*(A) = k$ means that every sufficiently large (positive) integer is the sum of exactly k elements of A and k is minimal.

Erdős and Graham [EG] studied the following extremal problem: "Fix $h = \mathrm{ord}(A)$. When does $\mathrm{ord}^*(A)$ exist? Assuming that $\mathrm{ord}^*(A)$ exists, how big can $\mathrm{ord}^*(A)$ be in terms of h? (Of course, it is not less than h.)"

They proved that $\mathrm{ord}^*(A)$ exists if and only if the following, clearly necessary, condition is fulfilled:

$$\gcd\{a - a' \ ; \ a \in A, a' \in A\} = 1. \tag{1.1}$$

They also proved that, if $\mathrm{ord}^*(A)$ exists, then it is less than $(5/4)h^2 + O(h)$ [in fact their proof easily gives the bound $h^2 + O(h)$] and it may be greater than $(1/4)h^2 + O(h)$.

A first observation is that

$$\mathrm{ord}(A) = \mathrm{ord}^*(A \cup \{0\}). \tag{1.2}$$

The above relation suggests that, as a definition of "order", one can adopt in any case that of "exact order" and use (1.2) when the concept of "at most order" is needed. More precisely, let A be a set of integers. Let $B = A \cup \{0\}$. Then $\mathrm{ord}^*(B)$ is the "at most order" of A as in Lagrange's theorem. Let B be any set of integers. Then $\mathrm{ord}^*(B \setminus \{0\})$ is the "exact order" of $B \setminus \{0\}$ as in Pall's theorem.

A second observation is that the Erdős-Graham' results [EG] on the existence of $\mathrm{ord}^*(B \setminus \{0\})$ and on its magnitude in terms of $h = \mathrm{ord}(B)$, apply also when *any* element $b \in B$ is removed. This is so because the order ord^* and the property of being a basis (asymptotic basis) are invariant upon translation: translate by $-b$ [no matter if 0 belongs or not to the initial set], remove zero, apply Erdős-Graham' results, and finally re-translate in the opposite sense, *i.e.,* translate by $+b$.

2. A More General Setting

In this section I discuss the framework adopted in the present survey.

Let A be a set of integers, bounded from below. Thus we assume that A may contain a finite number of negative integers. This is convenient because, if this is the case, certain properties defined in the sequel (basicity, order) will be invariant upon arbitrary translations.

For two sets S, T the notation $S \sim T$ means that the symmetric difference $S \Delta T$ is finite. The relation $\sim$ is an equivalence relation.

Let h be a positive integer and A be as above. We denote by hA the $h-$fold sum of A

$$hA = \{x_1 + \cdots + x_h \ ; \ x_j \in A, 1 \leq j \leq h\}.$$

We say that A is an *asymptotic basis* if there is an integer $h \geq 1$ such that $hA \sim \mathbb{N} := \{1, 2, \ldots\}$. If this is the case, the smallest integer h verifying this property is called *the order* of the asymptotic basis A and will be denoted by $G(A)$, notation borrowed from the Waring's problem. For instance, if $C = \{0^3, 1^3, 2^3, \ldots\}$ is the set of cubes of the non-negative integers, we know, after Linnik, that

$4 \le G(C) \le 7$. $G(C)$ is noted $G(3)$ in Waring's problem. So G is the ord* of the preceding section. For the reasons presented in section 1, the word "exact" in the term "exact order" [EG], [Pl1], [Pl2] is not essential.

Notice the equivalence between the relations $hA \sim \mathbb{N}$ and $G(A) \le h$.

3. The Erdős-Graham' results in the new setting

(3.1) **Theorem EG1.** Let A be an asymptotic basis and $a \in A$. The set $A \setminus \{a\}$ is an asymptotic basis if and only if the following, clearly necessary, condition is satisfied:

(3.2) $$\gcd\{b - b' \ ; \ b \in A \setminus \{a\}, b' \in A \setminus \{a\}\} = 1.$$

The proof is in [EG]. The theorem was stated in the above form in [G1].

Let A be an asymptotic basis. Put

$$A^* = \{a \in A; \ (3.2) \text{ holds } \}.$$

For $h = 1, 2, \ldots$, let

$$\mathbf{X}(h) = \max_{hA \sim \mathbb{N}} \mathbf{x}(A) \quad \text{where} \quad \mathbf{x}(A) = \max_{a \in A^*} G(A \setminus \{a\}).$$

From [EG], it follows the

(3.3) **Theorem EG2.** For any $h \ge 1$, we have

$$\frac{1}{4}h^2 + O(h) \le \mathbf{X}(h) \le \frac{5}{4}h^2 + O(h).$$

As already mentioned, the method used in [EG] gives easily $h^2 + O(h)$ instead of $(5/4)h^2 + O(h)$ in the last inequality.

4. Further Results on the Function X

Notice [EG] that $A \setminus A^*$ is finite and more precisely [G1] that $|A \setminus A^*| < h$, where $h = G(A)$. In [DG] we improve this result to $|A \setminus A^*| \le C_1 \sqrt{\frac{h}{\ln h}}$, for some absolute constant C_1. In the same paper, we prove that for each one of infinitely many h, there is an asymptotic basis A with $G(A) = h$ and such that $|A \setminus A^*| \ge C_2 \sqrt{\frac{h}{\ln h}}$, where $C_2 < C_1$ is another absolute constant.

I proved [G1, G2] that

(4.1) $$\frac{1}{3}h^2 - \frac{2h}{3} \le \mathbf{X}(h) \le h^2 + h.$$

For the first inequality, I used the following method: I considered a small subinterval $I = [\alpha, \beta]$ of [0,1] such that $h(I \cup \{0\}) = [0, 1]$, modulo 1. The notation hJ means

$$hJ = \{x_1 + \cdots + x_h \ ; \ x_j \in J, 1 \le j \le h\}.$$

The set A such as $G(A) = h$ and $G(A \setminus \{0\}) \geq \frac{1}{3}h^2 - \frac{2h}{3}$ is the union of $\{0\}$ and all classes modulo a big integer N whose representatives, when divided by N, belong to I modulo 1.

For the second inequality in (4.1), I used Kneser's theorem [K, HR]. Since then, this idea found many applications. See [J, Ns, Nt] for finding upper bounds of $\mathbf{X}$ and the related function $\mathbf{X}_k$ (defined below in this section). For further applications in other problems we refer to the survey in [G3].

Using similar ideas with much more sophisticated calculations, J.C.M. Nash [Ns] proved that

$$\mathbf{X}(h) \leq \frac{1}{2}h^2 + \frac{3h}{2}.$$

Recently A. Plagne [Pl1] using the isoperimetric method of Y. O. Hamidoune and the theory of critical pairs in Kneser's theorem, established the inequality

$$\lfloor \frac{h(h+4)}{3} \rfloor \leq X(h) \leq \frac{h(h+1)}{2} + \lceil \frac{h-1}{3} \rceil.$$

As for specific values of the function $\mathbf{X}$, it is known that $\mathbf{X}(2) = 4$ [EG], $\mathbf{X}(3) = 7$ [Ns], $\mathbf{X}(4) = 10$ or 11, $\mathbf{X}(5) = 15$ or 16, and $20 \leq \mathbf{X}(6) \leq 23$ [Pl1].

In a series of papers (see [J], [Ns] and [Nt] for further references) X.-D. Jia, J. C. M. Nash and M. B. Nathanson, generalizing (4.1), studied the function $\mathbf{X}_k$ defined as $\mathbf{X}$, but taking away from the asymptotic basis a finite set of cardinality k instead of a single element:

$$\mathbf{X}_k(h) = \max_{hA \sim \mathbb{N}} \Big(\max_{|F|=k, A \setminus F \text{ basis}} G(A \setminus F) \Big).$$

Here it is not sufficient to take $F \subset A^*$ as the following example shows: $A = \{1, 3\} \cup \overline{0}_2$, where

$$\overline{k}_t = \{x \in \mathbb{N}; x \equiv k (\text{mod } t)\}.$$

We have $A^* = A$ but $A \setminus \{1, 3\}$ is not an asymptotic basis. So one must consider

$$A^*, (A^* \setminus \{a\})^*, \ldots \ [k \text{ times}].$$

X.-D. Jia, J. C. M. Nash and M. B. Nathanson proved that

$$\frac{1}{(k+1)^{k+1}} h^{k+1} + O(h^k) \leq \mathbf{X}_k(h) \leq (k+2)^k h^{k+1} + O(h^k),$$

the implicit constants in the $O(h^k)$ depending on k.

5. The Function S

Known examples suggest that $G(A \setminus \{a\})$ is "very big" only for "some" (few) elements a of A. In order to study this aspect, I introduced [G4] the quantity

$$\mathbf{S}(h) = \max_{hA \sim \mathbb{N}} \mathbf{s}(A) \quad \text{where} \quad \mathbf{s}(A) = \limsup_{a \to +\infty, a \in A} G(A \setminus \{a\}).$$

Since $G(A \setminus \{a\})$ takes only finitely many values for a running through A^*, the superior limit $\mathbf{s}(A)$ is the biggest one among the values taken infinitely often. Thus the introduction of $\mathbf{s}$ and $\mathbf{S}$ allows us to exclude a finite number of exceptions. I proved [G4] that $\mathbf{S}(3) \leq 6$ (while $\mathbf{X}(3) = 7$ [Ns]) and I conjectured that $\mathbf{S}(h) < \mathbf{X}(h)$ for any h. In [G4] I asserted that $\mathbf{S}(2) = 3$ (while $\mathbf{X}(2) = 4$ [EG]), but the reference to a previous work by myself was not adequate.

A. Plagne proved in [Pl2] that $\mathbf{S}(h) < \mathbf{X}(h)$ for $h \geq 64$. Later, J. Cassaigne and A. Plagne [CP] proved that

$$\mathbf{S}(h) \leq 2h$$

for any h and that $\mathbf{S}(2) = 3$.

6. Some Open Questions

We may introduce two more functions:

$$\mathbf{I}(h) = \max_{hA\sim\mathbb{N}} \mathbf{i}(A) \quad \text{where} \quad \mathbf{i}(A) = \liminf_{a\to+\infty, a\in A} G(A \setminus \{a\}).$$

$$\mathbf{N}(h) = \max_{hA\sim\mathbb{N}} \mathbf{n}(A) \quad \text{where} \quad \mathbf{n}(A) = \min_{a\in A^*} G(A \setminus \{a\}).$$

We have

$$h + 1 \leq \mathbf{N}(h) \leq \mathbf{I}(h) \leq \mathbf{S}(h) \leq 2h.$$

The first inequality is due to E. Härtter [H].

I do not know whether the functions $\mathbf{N}$ and $\mathbf{I}$ are interesting. The only known value is $\mathbf{N}(2) = \mathbf{I}(2) = \mathbf{S}(2) = 3$. In [G5] I proved a little more than $\mathbf{I}(h) \leq 2h$:

(6.1) **Proposition.** If $hA \sim \mathbb{N}$, then for any $k \in \mathbb{N}$, there are infinitely many mutually disjoint subsets F of A with $|F| = k$ and $G(A \setminus F) \leq 2h$.

A long standing problem is to prove that the limit of $\frac{\mathbf{X}(h)}{h^2}$, for h tending to infinity, exists; and, if this is the case, to determine its value.

A concrete open problem is to determine $\mathbf{S}(3)$. We only know that $4 \leq \mathbf{S}(3) \leq 6$.

Finally, we would like to improve the upper and lower bounds for $\mathbf{X}_k(h)$.

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