ENUMERATION OF 3-LETTER PATTERNS IN COMPOSITIONS

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Abstract

Let A be any set of positive integers and $n \in \mathbb{N}$. A *composition of* n *with parts in* A is an ordered collection of one or more elements in A whose sum is n. We derive generating functions for the number of compositions of n with m parts in A that have r occurrences of 3-letter patterns formed by two (adjacent) instances of *levels*, *rises* and *drops*. We also derive asymptotics for the number of compositions of n that avoid a given pattern. Finally, we obtain the generating function for the number of k-ary words of length m which contain a prescribed number of occurrences of a given pattern as a special case of our results.

1. Introduction

A *composition* $\sigma = \sigma_1\sigma_2\ldots\sigma_m$ of $n \in \mathbb{N}$ is an ordered collection of one or more positive integers whose sum is n. The number of summands, namely m, is called the number of *parts* of the composition. We will look at *compositions of* n *with parts in* A, i.e., compositions whose parts are restricted to be from a set $A \subseteq \mathbb{N}$. Our aim is to count the number of compositions of n with parts in A which contain a 3-letter pattern τ exactly r times. This extends work on the statistics *rises* (a summand followed by a larger summand), *levels* (a summand followed by itself), and *drops* or *falls* (a summand followed by a smaller summand) in all compositions of n whose parts are in a given set A.

Several authors have studied compositions and the statistics rises, levels and falls from different viewpoints. Alladi and Hoggatt [1] studied compositions with parts in the set $A = \{1, 2\}$ in conjunction with the Fibonacci sequence. Chinn, Grimaldi and Heubach [12, 13, 14, 18, 19] have generalized to different sets A and have counted the number of compositions, rises, levels and drops, looking for connections to known sequences (which is only possible when considering a specific set A).

Carlitz and several co-authors ([6],[8],[9],[10],[11]) studied rises, levels and falls in compositions on the set $[n] = \{1, 2, \ldots, n\}$ as an extension of the study of these statistics or patterns in permutations. (Unlike Alladi and Hoggatt, Carlitz et al. included an additional rise at the beginning and an additional fall at the end of each composition, except in [10].) These authors extended enumeration questions for permutations to compositions by also considering the *specification* of a composition (a list of counts for the occurrences of each integer) as well as the statistic levels. More recently, Rawlings [25] enumerated compositions according to weak rises and falls (equality allowed) in connection with restricted words by adjacencies. He also introduced the notion of *ascent variation*, (the sum of the increases of the rises within a composition), which is motivated by a connection to the perimeter of directed vertically convex polyominoes. Furthermore, Heubach and Mansour [21] developed a general framework which gives the generating function for the number of rises, levels and falls for any ordered subset A of $\mathbb{N}$.

"Closest" to permutations are those compositions that do not have levels, which were called *waves* by Carlitz and his co-authors. These compositions have also been referred to as *Smirnov sequences* (see for example [17]) and *Carlitz compositions* [23]. One special case of Carlitz compositions are up-down sequences, in which rises and falls alternate, which were studied by Carlitz and Scoville [9] and Carlitz [6]. In addition, the problem of enumerating Carlitz compositions according to rises and falls reduces to the Simon Newcomb problem ([5], [15]) when the number of falls is disregarded. Another extension of questions first studied for permutations are the generating functions for the number of compositions according to specification, rises, falls and maxima by Carlitz and Vaughan [11], and enumeration of pairs of sequences according to rises, levels and falls [10]. Recently, Gho, Hitczenko, Louchard and Prodinger have studied distinctness and other characteristics of compositions and Carlitz compositions using a probabilistic approach [16, 22, 24].

A widely studied topic for permutations is pattern avoidance (see [2] and references therein for an overview). We follow the motivation of extending enumeration questions from permutations to compositions by expressing rises, levels and drops as 2-letter patterns. For example, the pattern 11 corresponds to any occurrence of a_ia_i with $a_i \in A$, and thus any occurrence of the pattern 11 corresponds to the occurrence of a level. Likewise, the pattern 12 corresponds to a_ia_j with $a_i < a_j$, i.e., a rise, and the pattern 21 corresponds to a fall or drop.

In this paper, we generalize to 3-letter patterns. For example, the pattern 111 corresponds to any occurrence of $a_ia_ia_i$ with $a_i \in A$, and therefore corresponds to a level followed by a level; we will refer to this statistic by the shorthand level+level. Likewise, we can define patterns for all the combinations of rises, levels and drops. To illustrate this idea, we look at the composition 1413364. In terms of rises, levels and drops, this composition is represented by rise+drop+rise+level+rise+drop. It contains the patterns (from left to right) 121 (rise+drop), 312 (drop+rise), 122 (rise+level), 112 (level+rise) and 132 (rise+drop).

Statistic	Pattern	Statistic	Pattern
level+level	111	rise+rise	123
level+rise	112	rise+drop=peak	121+132+231
level+drop	221	drop+rise=valley	212+213+312

TABLE 1. Statistics and their associated patterns

Due to symmetry, each rise is matched by a drop (for each composition that is not symmetric, there is a composition whose parts are in reverse order, and for symmetric compositions, the rises and drops are matched within the composition). Thus, the statistic rise+level, 122, occurs as often as the statistic level+drop, 221. Table 1 lists the statistics to be considered, and their corresponding patterns. Note that the statistic rise+drop is represented by three different patterns, which take into account the relative size of the actual summands. For example, the pattern 121 indicates the occurrence of $a_1a_2a_1$ where $a_1 < a_2$, whereas 132 indicates the occurrence of $a_1a_2a_3$, where $a_1 < a_3 < a_2$.

In Section 2 we derive the generating functions for the number of compositions of n, with m parts in A, which contain a given pattern τ exactly r times, for each of the patterns listed above. In Section 3 we use tools from complex analysis to derive the exact asymptotics for the number of compositions of n that avoid a given pattern τ. Finally, in Section 4, we apply our results to words, i.e., elements of $[k]^n$, where $[k]$ is a (totally ordered) alphabet on k letters. We obtain previous results in [3, 4], and obtain new results for the patterns peak and valley. This application shows that compositions form a larger class of combinatorial objects, containing words as a subclass.

2. Compositions of n with parts in A

Let $A = \{a_1, a_2, a_3, \ldots, a_d\}$ or $A = \{a_1, a_2, a_3, \ldots\}$, where $a_1 < a_2 < \ldots$. are positive integers. We will refer to such a set as an *ordered subset* of $\mathbb{N}$. In the theorems and proofs, we will treat the two cases together if possible, and will note if the case $|A| = \infty$ requires additional steps.

Let $C_\tau(n, r)$ (respectively $C_\tau(j; n, r)$) denote the number of compositions of n with parts in A (respectively with j parts in A) which contain the pattern τ exactly r times. The corresponding generating functions are given by

$$C_\tau(x, y) = \sum_{n,r\geq 0} C_\tau(n, r)x^n y^r$$

and

$$C_\tau(x, y, z) = \sum_{n,r,j\geq 0} C_\tau(j; n, r)x^n y^r z^j = \sum_{j\geq 0} C_\tau(j; x, y)z^j.$$

More generally, let $C_\tau(\sigma_1\ldots\sigma_\ell|n,r)$ (respectively $C_\tau(\sigma_1\ldots\sigma_\ell|j;n,r)$) be the number of compositions of n with parts in A (respectively with j parts in A) which contain τ exactly r times and whose first ℓ parts are $\sigma_1,\ldots,\sigma_\ell$. The corresponding generating functions are given by

$$C_\tau(\sigma_1\ldots\sigma_\ell|x,y)=\sum_{n,r\geq 0}C_\tau(\sigma_1\ldots\sigma_\ell|n,r)x^ny^r$$

and

$$C_\tau(\sigma_1\ldots\sigma_\ell|x,y,z)=\sum_{n,r,j\geq 0}C_\tau(\sigma_1\ldots\sigma_\ell|j;n,r)x^ny^rz^j=\sum_{j\geq 0}C_\tau(\sigma_1\ldots\sigma_\ell|j;x,y)z^j.$$

The initial conditions are $C_\tau(j;x,y)=0$ for $j<0$, $C_\tau(0;x,y)=1$, and $C_\tau(\sigma_1,\ldots\sigma_l|j;x,y)=0$ for $j\leq l-1$. In addition,

$$C_\tau(x,y,z)=1+\sum_{a\in A}C_\tau(a|x,y,z). \tag{1}$$

In this section we study the generating functions $C_\tau(x,y,z)$ for different patterns $\tau=\tau_1\tau_2\tau_3$. To find an explicit expression for $C_\tau(x,y,z)$, we derive recursive equations using a variety of strategies for the different patterns. For patterns that contain levels, namely 111, 112, and 221, the recursion is in terms of $C_\tau(\sigma_1|j;x,y)$, $C_\tau(\sigma_1\sigma_2|j;x,y)$, $C_\tau(\sigma_1\sigma_2\sigma_3|j;x,y)$ and $C_\tau(x,y,z)$, which is usually straightforward. However, solving the resulting system of equations can be difficult, as for example in the case of the pattern 112. We will describe the derivation of the recursive equations in detail for the pattern 111.

For the pattern 123, we break the composition of n into pieces, some of which have parts in A, and others that contain only parts larger than the part under consideration. We then define a second generating function which will play a major role in the recursive equation for $C_\tau(x,y,z)$. In the case of peaks (valleys), we split the composition into parts according to where the largest (smallest) part occurs, and derive a recursion that will lead to a continued fraction expansion for the generating function $C_\tau(x,y,z)$.

2.1 The pattern 111 (the statistic level+level) In the following theorem we present the generating function for the number of compositions of n with j parts in A that contain the pattern 111 exactly r times.

Theorem 2.1. *Let A be any ordered subset of $\mathbb{N}$. Then*

$$C_{111}(x,y,z)=\frac{1}{1-\sum_{a\in A}\frac{x^az(1+(1-y)x^az)}{1+x^az(1+x^az)(1-y)}}.$$

Proof. The pattern 111 occurs when $a\in A$ occurs three times in a row. Thus, for fixed $a\in A$ and $j\geq 2$

$$\begin{aligned}C_{111}(a|j;x,y)&=C_{111}(aa|j;x,y)+\textstyle\sum_{b\in A,b\neq a}C_{111}(ab|j;x,y)\\&=C_{111}(aa|j;x,y)+x^aC_{111}(j-1;x,y)-x^aC_{111}(a|j-1;x,y).\end{aligned}\tag{2}$$

Note that the factor of x^a reflects the fact that we are looking at compositions of $n-a$. We now apply a similar argument to $C_{111}(aa|j;x,y)$ to obtain for $j \geq 3$

$$\begin{aligned} C_{111}(aa|j;x,y) &= C_{111}(aaa|j;x,y) + \textstyle\sum_{b\in A, b\neq a} C_{111}(aab|j;x,y) \\ &= x^a\, yC_{111}(aa|j-1;x,y) + x^{2a}\textstyle\sum_{b\in A, b\neq a} C_{111}(b|j-2;x,y) \\ &= x^a\, yC_{111}(aa|j-1;x,y) + x^{2a}C_{111}(j-2;x,y) - x^{2a}C_{111}(a|j-2;x,y). \end{aligned} \tag{3}$$

Multiplying (2) and (3) by z^j, summing over all $j \geq 1$, taking into account that the recurrences hold for $j \geq 2$ and $j \geq 3$, and solving the resulting system of two equations for $C_{111}(a|x,y,z)$, we get that

$$C_{111}(a|x,y,z) = \frac{x^a z(1+(1-y)x^a z)}{1+x^a z(1+x^a z)(1-y)} C_{111}(x,y,z).$$

Summing over all $a \in A$ and using Eq. (1) for $\tau = 111$, we get the desired result. □

Applying Theorem 2.1 to $A = \mathbb{N}$ with $a_i = i$ for $i \geq 1$, we get that the generating function for the number of compositions of n with parts in $\mathbb{N}$ which avoid the pattern 111 is given by

$$C_{111}(x,0,1) = \frac{1}{1 - \sum_{i\geq 1} \frac{x^i(1+x^i)}{1+x^i(1+x^i)}},$$

and the values of the corresponding sequence are 1, 1, 2, 3, 7, 13, 24, 46, 89, 170, 324, 618, 1183, 2260, 4318, 8249, 15765, 30123, 57556, 109973, 210137, 401525, 767216, 1465963, 2801115, 5352275 for $n = 0, 1, \ldots, 25$. Note that compositions that avoid the pattern 111 have only isolated levels.

Remark: We note that *Carlitz compositions* of n, introduced in [7], are those compositions of n in which no adjacent parts are the same. Thus, Carlitz compositions are precisely those compositions that avoid levels, or equivalently, avoid the (2-letter) pattern 11. One possible generalization of Carlitz compositions is to define *ℓ-Carlitz compositions of n* to be those compositions of n that avoid ℓ consecutive levels, or in terms of pattern avoidance, avoid the pattern $11\ldots11$ consisting of $\ell+1$ 1's.

2.2 The patterns 112 and 221 (the statistics level+rise and level+drop) In the following theorem we present the generating functions for the number of compositions of n with j parts in A which contain the patterns 112 and 221, respectively, exactly r times.

Theorem 2.2. *Let A be any ordered subset of $\mathbb{N}$. Then*

$$C_{112}(x,y,z) = \frac{1}{1 - \sum_{j=1}^{d}\left(x^{a_j} z \prod_{i=1}^{j-1}(1-(1-y)x^{2a_i}z^2)\right)}$$

and

$$C_{221}(x,y,z) = \frac{1}{1 - \sum_{j=1}^{d}\left(x^{a_j} z \prod_{i=j+1}^{d}(1-(1-y)x^{2a_i}z^2)\right)}.$$

Proof. To derive the generating function $C_{112}(x, y, z)$ we use arguments similar to those in the proof of Theorem 2.1 to obtain for every $a \in A$

$$\begin{aligned} C_{112}(a|x,y,z) &= \frac{x^{2a}z^2}{1-x^{2a}z^2} + \frac{x^{2a}z^2}{1-x^{2a}z^2} \sum_{b\in A, b<a} C_{112}(b|x,y,x) \\ &\quad + \frac{x^{2a}z^2y}{1-x^{2a}z^2} \sum_{b\in A, b>a} C_{112}(b|x,y,z) + \frac{x^a z}{1+x^a z} C_{112}(x,y,z). \end{aligned} \tag{4}$$

Let's now assume that A is finite, i.e. $A = \{a_1, \ldots, a_d\}$. Setting $x_0 = C_{112}(x, y, z)$, $x_i = C_{112}(a_i|x, y, z)$, $\alpha_i = \frac{x^{2a_i}z^2}{1-x^{2a_i}z^2}$, and $\beta_i = \frac{x^{a_i}z}{1+x^{a_i}z}$, the above equation is of the form

$$x_i - \alpha_i \sum_{j<i} x_j - \alpha_i y \sum_{j>i} x_j - \beta_i x_0 = \alpha_i \quad \text{for} \quad i = 1, \ldots, d.$$

Together with Eq. (1) for $\tau = 112$ this results in the following system of equations:

$$\begin{pmatrix} -\beta_1 & 1 & -\alpha_1 y & -\alpha_1 y & \cdots & -\alpha_1 y & -\alpha_1 y \\ -\beta_2 & -\alpha_2 & 1 & -\alpha_2 y & \cdots & -\alpha_2 y & -\alpha_2 y \\ -\beta_3 & -\alpha_3 & -\alpha_3 & 1 & \cdots & -\alpha_3 y & -\alpha_3 y \\ -\beta_4 & -\alpha_4 & -\alpha_4 & -\alpha_4 & \cdots & -\alpha_4 y & -\alpha_4 y \\ \vdots & & & \vdots & & & \vdots \\ -\beta_{d-1} & -\alpha_{d-1} & -\alpha_{d-1} & -\alpha_{d-1} & \cdots & 1 & -\alpha_{d-1} y \\ -\beta_d & -\alpha_d & -\alpha_d & -\alpha_d & \cdots & -\alpha_d & 1 \\ 1 & -1 & -1 & -1 & \cdots & -1 & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{d-2} \\ x_{d-1} \\ x_d \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \vdots \\ \alpha_{d-1} \\ \alpha_d \\ 1 \end{pmatrix}.$$

Let M_d be the $(d+1)\times(d+1)$ matrix of the system of equations and let N_d be the matrix that results from replacing the first column of M_d with the right-hand side of the system of equations. Then $C_{112}(x, y, z) = \frac{\det(N_d)}{\det(M_d)}$. We start by computing $\det(M_d)$. If we subtract α_j times the last row from row j for $j = 1, 2, \ldots, d$, (the elementary operation is $R_j - \alpha_j R_{d+1} \to R_j$), and then subtract the jth column from the $(j + 1)$st column ($C_{j+1} - C_j \to C_{j+1}$) for $j = 2, 3, \ldots, d$, and denote the resulting matrix by M'_d, then we get that $\det(M_d) = \det(M'_d)$. Let A_d be the matrix M'_d without the first column and the last row, and let B_d be the matrix M'_d without the second column and the last row. Thus, $\det(M_d) = (-1)^d(\det(A_d)+\det(B_d))$. It is easy to see that $\det(A_d) = \prod_{j=1}^{d}(1 + \alpha_j)$, and (by expanding along the last row of B_d) that $\det(B_d) = (1 + \alpha_d)\det(B_{d-1}) - (\alpha_d + \beta_d)\prod_{j=1}^{d-1}(1 + \alpha_j y)$. Using induction on d with $\det(B_1) = -(\alpha_1 + \beta_1)$ we get that

$$\det(B_d) = -\sum_{j=1}^{d}(\alpha_j + \beta_j)\prod_{i=j+1}^{d}(1 + \alpha_i)\prod_{i=1}^{j-1}(1 + \alpha_i y) = -\prod_{i=1}^{d}(1 + \alpha_i)\sum_{j=1}^{d}\tfrac{\alpha_j+\beta_j}{1+\alpha_j}\prod_{i=1}^{j-1}\tfrac{1+\alpha_i y}{1+\alpha_i}.$$

Hence,

$$\det(M_d) = (-1)^d\prod_{i=1}^{d}(1 + \alpha_i)\left(1 - \sum_{j=1}^{d}\frac{\alpha_j + \beta_j}{1 + \alpha_j}\prod_{i=1}^{j-1}\frac{1 + \alpha_i y}{1 + \alpha_i}\right).$$

Now, we consider the matrix N_d. If we add the first column to all other columns, then it is easy to see that $\det(N_d) = (-1)^d\prod_{i=1}^{d}(1 + \alpha_i)$. Therefore, with $\alpha_j = \frac{x^{2a_j}z^2}{1-x^{2a_j}z^2}$ and

$\beta_j = \frac{x^{a_j}z}{1+x^{a_j}z}$, we get (after algebraic simplification) that

$$\frac{\det(N_d)}{\det(M_d)} = \frac{1}{1 - \sum_{j=1}^{d}\left(x^{a_j}z\prod_{i=1}^{j-1}(1-(1-y)x^{2a_i}z^2)\right)}.$$

The case $|A| = \infty$ follows by taking the limit as $d \to \infty$. The proof for $C_{221}(x, y, z)$ follows with slight modifications. □

Setting $y = 0$ and $z = 1$ in Theorem 2.2, we get that the generating function for the number of compositions of n with parts in $\mathbb{N}$ that avoid the pattern 112 is given by

$$C_{112}(x, 0, 1) = \frac{1}{1 - \sum_{j\geq 1} x^j \prod_{i=1}^{j-1}(1-x^{2i})},$$

and the values of the corresponding sequence are 1, 1, 2, 4, 7, 13, 24, 43, 78, 142, 256, 463, 838, 1513, 2735, 4944, 8931, 16139, 29164, 52693, 95213 for $n = 0, 1, \ldots, 20$. The generating function for the number of compositions of n with parts in $\mathbb{N}$ that avoid the pattern 221 is given by

$$C_{221}(x, 0, 1) = \frac{1}{1 - \sum_{i\geq 1}\left(x^i \prod_{j\geq i+1}(1-x^{2j})\right)},$$

and the values of the corresponding sequence are 1, 1, 2, 4, 8, 15, 30, 58, 113, 220, 429, 835, 1627, 3169, 6172, 12023, 23419, 45616, 88853, 173073, 337118 for $n = 0, 1, \ldots, 20$. Note that there are a lot less compositions of n that avoid 112 than compositions that avoid 221. This notion can be made more precise using the formulas for the asymptotic behavior given in Theorem 3.1.

2.3 The pattern 123 (the statistic rise+rise) In the following theorem we will present the generating function for the number of compositions of n with j parts in A that contain the pattern 123 exactly r times.

Theorem 2.3. *Let A be any ordered subset of $\mathbb{N}$, with $|A| = d$. Then*

$$C_{123}(x, y, z) = \frac{1}{1 - t^1(A) - \sum\limits_{p=3}^{d}\sum\limits_{j=0}^{p-3}\binom{p-3}{j}t^{p+j}(A)(y-1)^{p-2}},$$

where $t^p(A) = \sum_{1\leq i_1<i_2<\cdots<i_p\leq d} z^p \prod_{j=1}^{p} x^{a_{i_j}}$.

Proof. Let σ be any composition of n with m parts in $A = \{a_1, \ldots, a_d\}$ that contains the pattern 123 exactly r times. To derive recursions for the generating function, we will break the composition into pieces, some of which have parts in the set A, and others that have parts from the set $A_k = \{a_{k+1}, a_{k+2}, \ldots, a_d\} = A\backslash\{a_1, \ldots, a_k\}$ (the index of A indicates the largest element excluded). To make this distinction for the generating functions, we will indicate the specific set from which the parts are selected as a superscript. Furthermore, since we want to split off the parts $a_1, a_2, \ldots$ successively to create recursive equations, we define $D^{A_k}(x, y, z)$ to be the generating function for the number of compositions σ of n with m parts in A_k such that for $a \notin A_k$, $a\sigma$ contains the pattern 123 exactly r times.

For any composition σ with parts in A, there are two possibilities: either σ does not contain a_1, in which case the generating function is given by $C_{123}^{A_1}(x,y,z)$, or the composition contains at least one occurrence of a_1, i.e., $\sigma = \bar{\sigma} a_1 \sigma_{k+1} \ldots \sigma_m$, where $\bar{\sigma}$ is a composition with parts from A_1, with generating function $C_{123}^{A_1}(x,y,z) C_{123}^{A}(a_1|x,y,z)$. Altogether, we have

$$C_{123}^{A}(x,y,z) = C_{123}^{A_1}(x,y,z) + C_{123}^{A_1}(x,y,z) C_{123}^{A}(a_1|x,y,z). \tag{5}$$

Now let us consider the compositions σ of n with m parts in A starting with a_1 which contain the pattern 123 exactly r times. Again, there are two cases: either σ contains exactly one occurrence of a_1, or the part a_1 occurs at least twice in σ. In the first case, the generating function is given by $x^{a_1} z\, D^{A_1}(x,y,z)$. If σ contains a_1 at least twice, then we split the composition into pieces according to the second occurrence of a_1, i.e., $\sigma = a_1 \bar{\sigma} a_1 \sigma_{k+1} \ldots \sigma_m$, where $\bar{\sigma}$ is a (possibly empty) composition with parts from A_1. Splitting off the initial part a_1 results in the generating function $x^{a_1} z\, D^{A_1}(x,y,z)\, C_{123}^{A}(a_1|x,y,z)$. Thus,

$$C_{123}^{A}(a_1|x,y,z) = x^{a_1} z D^{A_1}(x,y,z) + x^{a_1} z D^{A_1}(x,y,z) C_{123}^{A}(a_1|x,y,z).$$

Solving for $C_{123}^{A}(a_1|x,y,z)$ and substituting into (5) gives

$$C_{123}^{A}(x,y,z) = \frac{C_{123}^{A_1}(x,y,z)}{1 - x^{a_1} z D^{A_1}(x,y,z)}. \tag{6}$$

We now derive an expression for $D^{A_1}(x,y,z)$ by considering compositions σ with parts in A_1 such that $a_1\sigma$ contains the pattern 123 exactly r times. If σ does not contain the part a_2, the generating function for σ is given by $D^{A_2}(x,y,z)$. Otherwise, we write $\sigma = \bar{\sigma}^1 a_2 \bar{\sigma}^2 a_2 \bar{\sigma}^3 \ldots a_2 \bar{\sigma}^{\ell+2}$ with $\ell \geq 0$, where $\bar{\sigma}^j$ is a (possibly empty) composition with parts in A_2 for $j = 1, \ldots, \ell+2$. There are four subcases, depending on whether $\bar{\sigma}^1$ and $\bar{\sigma}^2$ are empty compositions or not. If $\bar{\sigma}^1$ and $\bar{\sigma}^2$ are both empty compositions, then $a_1\sigma = a_1 a_2$ or $a_1\sigma = a_1 a_2 a_2 \bar{\sigma}^3 \ldots a_2 \bar{\sigma}^{\ell+2}$, $\ell \geq 1$. In either case we can split off the initial part a_2 of σ which results in one less part, but no reduction in the occurrences of the pattern 123. Thus, the generating function for σ is given by

$$x^{a_2} z \sum_{\ell \geq 0} (x^{a_2} z\, D^{A_2}(x,y,z))^{\ell} = \frac{x^{a_2} z}{1 - x^{a_2} z\, D^{A_2}(x,y,z)}.$$

If $\bar{\sigma}^1$ is the empty composition and $\bar{\sigma}^2$ is not the empty composition, then $a_1\sigma = a_1 a_2 \bar{\sigma}^2$ or $a_1\sigma = a_1 a_2 \bar{\sigma}^2 a_2 \bar{\sigma}^3 \ldots a_2 \bar{\sigma}^{\ell+2}$, $\ell \geq 1$, and the generating function for σ is given by

$$x^{a_2} z\, y (D^{A_2}(x,y,z) - 1) \sum_{\ell \geq 0} (x^{a_2} z D^{A_2}(x,y,z))^{\ell} = \frac{x^{a_2} z\, y (D^{A_2}(x,y,z) - 1)}{1 - x^{a_2} z D^{A_2}(x,y,z)}.$$

If $\bar{\sigma}^1$ is not the empty composition and $\bar{\sigma}^2$ is the empty composition, then $\sigma = \bar{\sigma}^1 a_2$ or $\sigma = \bar{\sigma}^1 a_2 a_2 \bar{\sigma}^3 \ldots a_2 \bar{\sigma}^{\ell+2}$, $\ell \geq 1$, and the generating function for σ is given by

$$x^{a_2} z (D^{A_2}(x,y,z) - 1) \sum_{\ell \geq 0} (x^{a_2} z D^{A_2}(x,y,z))^{\ell} = \frac{x^{a_2} z (D^{A_2}(x,y,z) - 1)}{1 - x^{a_2} z D^{A_2}(x,y,z)}.$$

Finally, if both $\bar{\sigma}^1$ and $\bar{\sigma}^2$ are nonempty compositions, then the generating function for σ is given by

$$\frac{x^{a_2}z(D^{A_2}(x,y,z)-1)^2}{1-x^{a_2}zD^{A_2}(x,y,z)}.$$

Adding all four cases, using the shorthand D^A for $D^A(x,y,z)$, and solving for D^{A_1} we get

$$D^{A_1} = \frac{(1-x^{a_2}z(1-y))D^{A_2}+x^{a_2}z(1-y)}{1-x^{a_2}zD^{A_2}}. \tag{7}$$

We give an explicit expression for D^A in the following lemma.

Lemma 2.4. *Let $A=\{a_1,\dots,a_d\}$ and $t^p(A_k)=\sum_{k+1\le i_1<i_2<\cdots<i_p\le d} z^p\prod_{j=1}^p x^{a_{i_j}}$ for all p and $k=0,1,\dots,d$. Then*

$$D^A = \frac{1+\sum_{p=2}^d\sum_{j=0}^{p-2}\binom{p-2}{j}t^{p+j}(A)(y-1)^{p-1}}{1-t^1(A)-\sum_{p=3}^d\sum_{j=0}^{p-3}\binom{p-3}{j}t^{p+j}(A)(y-1)^{p-2}}. \tag{8}$$

Proof. Before we start proving (8), we will give an interpretation of $t^p(A_k)$ as the generating function for the number of partitions with p distinct parts from the set A_k, where A_k has $d-k$ elements, with $A_0=A$ and $0\le k\le d-1$. These partitions either contain the part a_{k+1} or not. In the first case, the generating function is given by $x^{a_{k+1}}z\,t^{p-1}(A_{k+1})$, and in the second case, by $t^p(A_{k+1})$. Thus,

$$t^p(A_k) = t^p(A_{k+1}) + x^{a_{k+1}}z\,t^{p-1}(A_{k+1}). \tag{9}$$

We now prove (8) by induction on d, the number of elements in A. For $d=0$ and $d=1$ we have $D^{\emptyset}=1$ and $D^{\{a_1\}}=\sum_{n\ge0}x^{n\,a_1}z^n=1/(1-x^{a_1}z)$, respectively, and thus, (8) holds. Now assume that (8) holds for $d-1$. Using (7) and the induction hypothesis for the set A_1 gives

$$\begin{aligned} D^A &= \frac{(1-x^{a_1}z(1-y))D^{A_1}+x^{a_1}z(1-y)}{1-x^{a_1}zD^{A_1}}\\ &= \frac{(1-x^{a_1}z(1-y))\left(1+\sum\limits_{p=2}^{d-1}\sum\limits_{j=0}^{p-2}\binom{p-2}{j}t^{p+j}(A_1)(y-1)^{p-1}\right)}{1-t^1(A_1)-\sum\limits_{p=3}^{d-1}\sum\limits_{j=0}^{p-3}\binom{p-3}{j}t^{p+j}(A_1)(y-1)^{p-2}-x^{a_1}z\left(1+\sum\limits_{p=2}^{d-1}\sum\limits_{j=0}^{p-2}\binom{p-2}{j}t^{p+j}(A_1)(y-1)^{p-1}\right)}\\ &\quad+\frac{x^{a_1}z(1-y)\left(1-t^1(A_1)-\sum\limits_{p=3}^{d-1}\sum\limits_{j=0}^{p-3}\binom{p-3}{j}t^{p+j}(A_1)(y-1)^{p-2}\right)}{1-t^1(A_1)-\sum\limits_{p=3}^{d-1}\sum\limits_{j=0}^{p-3}\binom{p-3}{j}t^{p+j}(A_1)(y-1)^{p-2}-x^{a_1}z\left(1+\sum\limits_{p=2}^{d-1}\sum\limits_{j=0}^{p-2}\binom{p-2}{j}t^{p+j}(A_1)(y-1)^{p-1}\right)} = \frac{s_1}{s_2}. \end{aligned}$$

We first rewrite the denominator and obtain

$$\begin{aligned} s_2 = 1-t^1(A_1)-x^{a_1}z-\sum_{p=3}^{d-1}\sum_{j=0}^{p-3}\binom{p-3}{j}t^{p+j}(A_1)(y-1)^{p-2}\\ -\sum_{p=2}^{d-1}\sum_{j=0}^{p-2}\binom{p-2}{j}x^{a_1}z\,t^{p+j}(A_1)(y-1)^{p-1}. \end{aligned}$$

Combining the two double sums by reindexing the second one, adding the terms for $p = d$ to the first one (which by definition are all zero, as $t_{d-1}^{d+j}(A_1) = 0$ for $j \geq 0$) and applying (9) gives

$$s_2 = 1 - t^1(A) - \sum_{p=3}^{d} \sum_{j=0}^{p-3} \binom{p-3}{j} t^{p+j}(A)(y-1)^{p-2}.$$

Next we rewrite the numerator:

$$\begin{aligned} s_1 = 1 &+ x^{a_1} z(y-1)t^1(A_1) + \sum_{p=2}^{d-1} \sum_{j=0}^{p-2} \binom{p-2}{j} t^{p+j}(A_1)(y-1)^{p-1} \\ &+ x^{a_1} z \sum_{p=2}^{d-1} \sum_{j=0}^{p-2} \binom{p-2}{j} t^{p+j}(A_1)(y-1)^{p} + x^{a_1} z \sum_{p=3}^{d-1} \sum_{j=0}^{p-3} \binom{p-3}{j} t^{p+j}(A_1)(y-1)^{p-1}. \end{aligned}$$

We now look at the coefficient of $(y-1)^m$ and collect terms according to the respective power. For $m = 0$, the coefficient is 1. If $m = 1$, then the coefficient is given by $x^{a_1} z\, t^1(A_1) + t^2(A_1) = t^2(A)$ (using (9)). If $m = 2, 3, \ldots, d-1$, then the coefficient of $(y-1)^m$ is equal to

$$\begin{aligned} &\sum_{j=0}^{m-1} \binom{m-1}{j} t^{m+1+j}(A_1) + x^{a_1} z \sum_{j=0}^{m-2} \binom{m-2}{j} t^{m+j}(A_1) + x^{a_1} z \sum_{j=0}^{m-2} \binom{m-2}{j} t^{m+1+j}(A_1) \\ &= \sum_{j=0}^{m-1} \binom{m-1}{j} t^{m+1+j}(A_1) + x^{a_1} z \left(\sum_{j=0}^{m-2} \binom{m-2}{j} t^{m+j}(A_1) + \sum_{j=1}^{m-1} \binom{m-2}{j-1} t^{m+j}(A_1) \right) \end{aligned}$$

which, using the identity $\binom{a}{b-1} + \binom{a}{b} = \binom{a+1}{b}$ and the fact that $\binom{m-2}{-1} = \binom{m-2}{m-1} = 0$,

$$\begin{aligned} &= \sum_{j=0}^{m-1} \binom{m-1}{j} t^{m+1+j}(A_1) + x^{a_1} z \sum_{j=0}^{m-1} \binom{m-1}{j} t^{m+j}(A_1) \\ &= \sum_{j=0}^{m-1} \binom{m-1}{j} \left(t^{m+1+j}(A_1) + x^{a_1} z\, t^{m+j}(A_1) \right) = \sum_{j=0}^{m-1} \binom{m-1}{j} t^{m+j+1}(A), \end{aligned}$$

where the last equality follows once more from using (9). Thus,

$$\begin{aligned} s_1 &= 1 + t^2(A)(y-1) + \sum_{m=2}^{d-1} \sum_{j=0}^{m-1} \binom{m-1}{j} t^{m+j+1}(A)(y-1)^m \\ &= 1 + \sum_{m=1}^{d-1} \sum_{j=0}^{m-1} \binom{m-1}{j} t^{m+j+1}(A)(y-1)^m. \end{aligned}$$

Reindexing the sum and combining this result with the result for s_2 completes the proof of the lemma. □

We now can obtain an exact formula for $C_{123}^{A}(x, y, z)$ as follows. Using (6) and Lemma 2.4 results in

$$C_{123}^{A}(x,y,z) = \frac{C_{123}^{A_1}(x,y,z)}{1 - x^{a_1} z \frac{1 + \sum_{p=2}^{d-1} \sum_{j=0}^{p-2} \binom{p-2}{j} t^{p+j}(A_1)(y-1)^{p-1}}{1 - t^1(A_1) - \sum_{p=3}^{d-1} \sum_{j=0}^{p-3} \binom{p-3}{j} t^{p+j}(A_1)(y-1)^{p-2}}}.$$

Using the same arguments as in the proof of the above lemma we get that

$$C_{123}^{A}(x,y,z) = C_{123}^{A_1}(x,y,z)\frac{1-t^1(A_1)-\sum_{p=3}^{d-1}\sum_{j=0}^{p-3}\binom{p-3}{j}t^{p+j}(A_1)(y-1)^{p-2}}{1-t^1(A)-\sum_{p=3}^{d}\sum_{j=0}^{p-3}\binom{p-3}{j}t^{p+j}(A)(y-1)^{p-2}}.$$

Iterating this equation and using that $C_{123}^{A_d}(x,y,z) = C_{123}^{\emptyset}(x,y,z) = 1$ results in:

$$C_{123}^{A}(x,y,z) = \prod_{k=0}^{d-1}\frac{1-t^1(A_{k+1})-\sum_{p=3}^{d-1}\sum_{j=0}^{p-3}\binom{p-3}{j}t^{p+j}(A_{k+1})(y-1)^{p-2}}{1-t^1(A_k)-\sum_{p=3}^{d}\sum_{j=0}^{p-3}\binom{p-3}{j}t^{p+j}(A_k)(y-1)^{p-2}}.$$

Simplifying and using that $t^p(A_d) = t^p(\emptyset) = 0$ gives the desired result. □

To apply Theorem 2.3 to $A = \mathbb{N}$, we first show that $t^p(\mathbb{N}) = x^{\binom{p+1}{2}}z^p/(x;x)_p$, where we use the customary notation $(a;q)_n = \prod_{j=0}^{n-1}(1-a\,q^j)$. Clearly, the formula for $t^p(\mathbb{N})$ holds for $p = 0$. For $p \geq 1$, we get from the definition of $t^p(\mathbb{N})$ with $a_i = i$ for $i \geq 1$ that

$$\begin{aligned}
t^p(\mathbb{N}) &= z^p \sum_{1\leq i_1<i_2<\cdots<i_{p-1}<i_p} x^{i_1+i_2+\cdots+i_p} = z^p \sum_{1\leq i_1<i_2<\cdots<i_{p-1}} x^{i_1+i_2+\cdots+i_{p-1}} \sum_{i>i_{p-1}} x^i \\
&= z^p \sum_{1\leq i_1<i_2<\cdots<i_{p-1}} x^{i_1+i_2+\cdots+i_{p-1}} \frac{x^{i_{p-1}+1}}{(1-x)} \\
&= z^p \frac{x}{(1-x)} \sum_{1\leq i_1<i_2<\cdots<i_{p-2}} x^{i_1+i_2+\cdots+i_{p-2}} \sum_{i>i_{p-2}} x^{2i} \\
&= \cdots = x^{\binom{p+1}{2}} z^p \prod_{j=1}^{p}(1-x^j)^{-1} = x^{\binom{p+1}{2}} z^p/(x;x)_p.
\end{aligned}$$

Setting $y = 0$ and $z = 1$ in Theorem 2.3 we obtain the generating function for the number of compositions with parts in $\mathbb{N}$ that avoid 123 as

$$C_{123}^{\mathbb{N}}(x,0,1) = \frac{1}{1-\frac{x}{1-x}-\sum\limits_{p\geq 3}\sum\limits_{j=0}^{p-3}\binom{p-3}{j}\frac{x^{\binom{p+1+j}{2}}}{(x;x)_{p+j}}(-1)^{p-2}},$$

and the sequence for the number of 123 pattern-avoiding compositions with parts in $\mathbb{N}$ for $n = 0$ to $n = 20$ is given by 1, 1, 2, 4, 8, 16, 31, 61, 119, 232, 453, 883, 1721, 3354, 6536, 12735, 24813, 48344, 94189, 183506, 357518. Note that the first time the pattern 123 can occur is for $n = 6$, as the composition 123.

2.4 The patterns $\{121, 132, 231\}$ and $\{212, 213, 312\}$ (the statistics *peak* = rise + drop and *valley* = drop+rise) We will now look at the set of patterns $\{121, 132, 231\}$ together, as they constitute the statistic *peak*, and likewise for the set $\{212, 213, 312\}$. For ease of use, we will refer to these sets of patterns collectively as the patterns *peak* and *valley*, respectively, and define $C_{peak}^{A}(x,y,z)$ and $C_{valley}^{A}(x,y,z)$ accordingly. Before we can state the result for the respective generating functions, we need a few definitions. For any set $B \subseteq A$

and for $s \geq 1$, we define

$$P^s(B) = \{(i_1, \ldots, i_s) |\, a_{i_j} \in B, j = 1, \ldots, s, \text{ and } i_{2\ell-1} < i_{2\ell} \leq i_{2\ell+1} \text{ for } 1 \leq \ell \leq \lfloor s/2 \rfloor\},$$

$$Q^s(B) = \{(i_1, \ldots, i_s) |\, a_{i_j} \in B, j = 1, \ldots, s, \text{ and } i_{2\ell-1} \leq i_{2\ell} < i_{2\ell+1} \text{ for } 1 \leq \ell \leq \lfloor s/2 \rfloor\}$$

and

$$M^s(B) = \sum_{(i_1,\ldots,i_s)\in P^s(B)} z^p \prod_{j=1}^{s} b_{i_j} \quad \text{and} \quad N^s(B) = \sum_{(i_1,\ldots,i_s)\in Q^s(B)} z^p \prod_{j=1}^{s} b_{i_j}.$$

On route to the explicit expressions given in Theorem 2.5 we will express the generating functions for the patterns *peak* and *valley* as continued fractions, for which we will use the notation $[c_0, c_1, c_2, \ldots, c_{n-1}, c_n] = c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{\ddots + \cfrac{1}{c_{n-1} + 1/c_n}}}}.$

Theorem 2.5. *Let $A = \{a_1, \ldots, a_d\}$, $P^s(A)$, $Q^s(A)$, $M^s(A)$, and $N^s(A)$ be defined as above. Then*

$$C^A_{peak}(x, y, z) = \frac{1 + \sum_{j\geq1} M^{2j}(A)(1-y)^j}{1 + \sum_{j\geq1} M^{2j}(A)(1-y)^j - \sum_{j\geq0} M^{2j+1}(A)(1-y)^j}, \quad \textit{and}$$

$$C^A_{valley}(x, y, z) = \frac{1 + \sum_{j\geq1} M^{2j}(A)(1-y)^j}{1 + \sum_{j\geq1} M^{2j}(A)(1-y)^j - \sum_{j\geq0} N^{2j+1}(A)(1-y)^j}.$$

Proof. We prove the result for the pattern *peak*. To derive first a recursion, and then an explicit formula for $C^A_{peak}(x, y, z)$, we concentrate on occurrences of a_d, the largest part in the set $A = \{a_1, a_2, \ldots, a_d\}$. If a_d is surrounded by smaller parts on both sides, then a peak occurs. Let σ be any composition with parts in A, and define $\bar{A}_k = \{a_1, \ldots, a_k\}$ (the index for the set indicates the largest element included). Note that $A = \bar{A}_d$.
We now look at the different possibilities for occurrences of a_d. If σ does not contain a_d, then the generating function is given by $C^{\bar{A}_{d-1}}_{peak}(x, y, z)$. If there is at least one occurrence, then we need to look at three cases for the first occurrence of a_d. If the first occurrence is at the beginning of the composition, i.e., $\sigma = a_d\sigma'$, where σ' is a (possibly empty) composition with parts in A, then no peak occurs, and the generating function is given by $x^{a_d} z\, C^A_{peak}(x, y, z)$. If the first (and only) occurrence of a_d is at the end of the composition, i.e., $\sigma = \bar{\sigma}a_d$, where $\bar{\sigma}$ is a non-empty composition with parts in $\bar{A}_{d-1}$, then the generating function is given by $x^{a_d} z(C^{\bar{A}_{d-1}}_{peak}(x, y, z) - 1)$. Finally, if the first occurrence of a_d is in the interior of the composition, then $\sigma = \bar{\sigma}a_d\sigma'$, where σ' is a non-empty composition with parts in A. If σ' starts with a_d, then $\sigma = \bar{\sigma}a_d\, a_d\, \sigma'$, and both a_d's can be split off without decreasing the number of occurrences of peaks; the generating function is given by $(C^{\bar{A}_{d-1}}_{peak}(x, y, z) - 1)x^{2a_d} z^2 C^A_{peak}(x, y, z)$. If σ' does not start with a_d, then a peak occurs and the generating

function is given by

$$(\underbrace{C_{peak}^{\bar{A}_{d-1}}(x,y,z)-1}_{\bar{\sigma}\text{ non-empty}})x^{a_d}z\,y(\underbrace{C_{peak}^{A}(x,y,z)-1}_{\sigma'\text{ non-empty}}\underbrace{-x^{a_d}z\,C_{peak}^{A}(x,y,z)}_{\text{does not start with }a_d}).$$

Now, let $C^A = C_{peak}^A(x,y,z)$. Combining the three cases above, we get

$$\begin{aligned} C^A = {} & C^{\bar{A}_{d-1}} + x^{a_d}z\,C^A + x^{a_d}z\,(C^{\bar{A}_{d-1}}-1) + x^{2a_d}z^2C^A(C^{\bar{A}_{d-1}}-1) \\ & + x^{a_d}zy(C^A - 1 - x^{a_d}zC^A)(C^{\bar{A}_{d-1}}-1), \end{aligned}$$

or, equivalently,

$$\begin{aligned} C^A &= \frac{(1+x^{a_d}z(1-y))C^{\bar{A}_{d-1}} - x^{a_d}z(1-y)}{1-x^{a_d}z(1-x^{a_d}z)(1-y) - x^{a_d}z(x^{a_d}z(1-y)+y)C^{\bar{A}_{d-1}}} \\ &= \cfrac{1}{1-x^{a_d}z-\cfrac{C^{\bar{A}_{d-1}}-1}{(1+x^{a_d}z(1-y))C^{\bar{A}_{d-1}}-x^{a_d}z(1-y)}} \qquad (10) \\ &= \cfrac{1}{1-x^{a_d}z-\cfrac{1}{[x^{a_d}z(1-y),1-1/C^{\bar{A}_{d-1}}]}}. \end{aligned}$$

Hence, by induction on d and using the fact that $C^{\bar{A}_1} = \frac{1}{1-x^{a_1}z}$, we can express the generating function $C_{peak}^A(x,y,z)$ as a continued fraction.

Lemma 2.6. *For $A=\{a_1,\dots,a_d\}$, $b_i = x^{a_i}z$, and $C^A = C_{peak}^A(x,y,z)$,*

$$C^A = \cfrac{1}{1-b_d-\cfrac{1}{[b_d(1-y),b_{d-1},b_{d-1}(1-y),\dots,b_2,b_2(1-y),b_1]}}.$$

Now we derive an explicit formula for $C_{peak}^A(x,y,z)$ based on recursions for $M^s(\bar{A}_d) = M^s(A)$ for odd and even s; if s is odd, both the last and second-to-last elements can equal d, whereas in the case s even, the second-to-last element can be at most $d-1$. By separating the elements of $P^s(A)$ according to whether the last element equals d or is less than d, we get the following two recursions:

$$\begin{aligned} M^{2s+1}(A) &= b_d\,M^{2s}(A) + M^{2s+1}(\bar{A}_{d-1}) \quad \text{and} \\ M^{2s}(A) &= b_d\,M^{2s-1}(\bar{A}_{d-1}) + M^{2s}(\bar{A}_{d-1}). \end{aligned} \qquad (11)$$

Define $G_d = \frac{1}{[b_d(1-y),b_{d-1},b_{d-1}(1-y),\dots,b_2,b_2(1-y),b_1]}$, i.e., G_d consists of the portion of C^A in the continued fraction expansion which has a repeating pattern. We now derive an expression for G_d in terms of the $M^s(A)$.

Lemma 2.7. *For all $d \geq 2$, $G_d = \dfrac{\sum_{j\geq 0} M^{2j+1}(A)(1-y)^j}{1+\sum_{j\geq 1} M^{2j}(A)(1-y)^j} = \dfrac{G_d^1}{G_d^2}$.*

Proof. We prove the statement by induction on d. For $d = 2$, $G_2^1 = b_1$ (only the term $j = 0$ contributes), $G_2^2 = 1 + b_1\, b_2\, (1 - y)$ (only the term $j = 1$ contributes) and therefore, the lemma holds. Now let $d \geq 3$ and assume that the lemma holds for $d - 1$, i.e., $G_{d-1} = \frac{G_{d-1}^1}{G_{d-1}^2}$. By the definition of G_d it is easy to see that $G_d = \frac{1}{b_d(1-y)+1/(b_{d-1}+G_{d-1})}$. Substituting the induction hypothesis for $d - 1$ into the expression for G_d and simplifying yields

$$G_d = \frac{b_{d-1}G_{d-1}^2 + G_{d-1}^1}{G_{d-1}^2 + b_d(1-y)(b_{d-1}G_{d-1}^2 + G_{d-1}^1)}.$$

Using the definitions of G_{d-1}^1 and G_{d-1}^2 and (11) yields $b_{d-1}G_{d-1}^2 + G_{d-1}^1 = G_d^1$, and this result together with the definitions of G_{d-1}^1 and G_{d-1}^2 and (11) yields $G_{d-1}^2 + b_d(1-y)(b_{d-1}G_{d-1}^2 + G_{d-1}^1) = G_d^2$, which completes the proof of Lemma 2.7. □

Now we use Lemma 2.7 and $C^A = \frac{1}{1-b_d-G_d}$ to get (after simplification) that C^A equals

$$\frac{1 + \sum_{j\geq 1} M^{2j}(A)(1-y)^j}{1 - b_d - M^1(A) + \sum_{j\geq 1} M^{2j}(A)(1-y)^j - \sum_{j\geq 1}(b_d M^{2j}(A) + M^{2j+1}(A))(1-y)^j}.$$

Using (11) we get that

$$\begin{aligned} C^A &= \frac{1 + \sum_{j\geq 1} M^{2j}(A)(1-y)^j}{1 - M^1(A) + \sum_{j\geq 1} M^{2j}(A)(1-y)^j - \sum_{j\geq 1} M^{2j+1}(A)(1-y)^j} \\ &= \frac{1 + \sum_{j\geq 1} M^{2j}(A)(1-y)^j}{1 + \sum_{j\geq 1} M^{2j}(A)(1-y)^j - \sum_{j\geq 0} M^{2j+1}(A)(1-y)^j}. \end{aligned}$$

This completes the proof for the pattern *peak* since the formula also holds for the case $d = \infty$. The result for the pattern *valley* follows with minor modifications, focusing on the smallest rather than the largest part, replacing $\bar{A}_k$ with $A_k = \{a_{k+1}, a_{k+2}, \ldots, a_d\}$, and using the recursions

$$M^s(A_k) = b_{k+1}\, N^{s-1}(A_{k+1}) + M^s(A_{k+1}) \text{ and } N^s(A_k) = b_{k+1} M^{s-1}(A_k) + N^s(A_{k+1}),$$

which are obtained by separating the elements of $P^s(A_k)$ according to whether the first element equals $k + 1$ or is greater than $k + 1$. □

We now apply Theorem 2.5 to $A = \mathbb{N}$. Similar to the derivation of $t^p(\mathbb{N})$, but with extra care since there are both $<$ and $\leq$ constraints (resulting in only odd powers of x in the numerator), we get for all $s \geq 1$,

$$\begin{aligned} M^{2s}(\mathbb{N}) &= \sum_{1\leq i_1<i_2\leq i_3\cdots i_{2s-2}\leq i_{2s-1}<i_{2s}} x^{i_1+\cdots+i_{2s}} z^{2s} = x^{s(s+2)} z^{2s}/(x;x)_{2s}, \\ M^{2s+1}(\mathbb{N}) &= \sum_{1\leq i_1<i_2\leq i_3\cdots i_{2s-1}<i_{2s}\leq i_{2s+1}} x^{i_1+\cdots+i_{2s}} z^{2s} = x^{s^2+3s+1} z^{2s+1}/(x;x)_{2s+1}, \text{ and} \\ N^{2s+1}(\mathbb{N}) &= \sum_{1\leq i_1\leq i_2<i_3\cdots i_{2s-1}\leq i_{2s}<i_{2s+1}} x^{i_1+\cdots+i_{2s+1}} z^{2s+1} = x^{(s+1)^2} z^{2s+1}/(x;x)_{2s+1}. \end{aligned}$$

Substituting these expressions into Theorem 2.5 gives the generating functions for the number of compositions of n with parts in $\mathbb{N}$ without *peaks* and *valleys*, respectively, as

$$C^{\mathbb{N}}_{peak}(x,0,1) = \frac{1+\sum_{j\geq 1}\frac{x^{j(j+2)}}{(x;x)_{2j}}}{1+\sum_{j\geq 1}\frac{x^{j(j+2)}}{(x;x)_{2j}} - \sum_{j\geq 0}\frac{x^{j^2+3j+1}}{(x;x)_{2j+1}}}$$

and

$$C^{\mathbb{N}}_{valley}(x,0,1) = \frac{1+\sum_{j\geq 1}\frac{x^{j(j+2)}}{(x;x)_{2j}}}{1+\sum_{j\geq 1}\frac{x^{j(j+2)}}{(x;x)_{2j}} - \sum_{j\geq 0}\frac{x^{(j+1)^2}}{(x;x)_{2j+1}}}.$$

Rewriting the generating function as a geometric series allows us to compute the sequence for the number of *peak*-avoiding compositions with parts in $\mathbb{N}$. The terms for $n = 0$ to $n = 20$ are given by 1, 1, 2, 4, 7, 13, 22, 38, 64, 107, 177, 293, 481, 789, 1291, 2110, 3445, 5621, 9167, 14947, 24366. The corresponding sequence for *valley*-avoiding compositions is given by 1, 1, 2, 4, 8, 15, 28, 52, 96, 177, 326, 600, 1104, 2032, 3740, 6884, 12672, 23327, 42942, 79052, 145528. Note that the first time a peak can occur is for $n = 4$ (121), and the first time a valley can occur is for $n = 5$ (212).

Remark: Even though the statistics *peak* and *valley* are in some sense symmetric, one cannot obtain the number of *valley*-avoiding compositions from the number of *peak*-avoiding compositions. However, there is a connection, namely the number of *valleys* in the compositions of n with m parts are equal to the number of *peaks* in the compositions of $m(n+1) - n$ with m parts. This can easily be seen as follows: In each composition of n with m parts, replace each part σ_i by $(n+1) - \sigma_i$, which results in a composition of $m(n+1) - \sum_{i=1}^{m}\sigma_i = m(n+1) - n$. This connection will be important in Section **??**, when we apply the results derived for the various patterns to words on k letters.

3. Asymptotics for the number of compositions avoiding τ

We will now use methods from Complex Analysis to compute the asymptotics for the number of compositions with parts in $\mathbb{N}$ which avoid a given pattern τ. We think of the generating function as a complex function, and indicate this fact by using the variable z instead of the variable x. Thus, we look at the function $C_\tau(z) = C_\tau(z,0,1) = \sum_{n\geq 0} C_\tau(n,0)z^n$. Since $C_\tau(z)$ is meromorphic, the asymptotic behavior of $C_\tau(n,0)$ is determined by the dominant pole of the function $C_\tau(z) = 1/f(z)$, i.e., the smallest positive root of $f(z)$ (see for example [26]). Using Theorem 5.2.1 [26] and the discussions preceding it, we obtain the following result.

Theorem 3.1. *The asymptotic behavior for τ-avoiding compositions with parts in $\mathbb{N}$ is given by*

$$\begin{aligned}
C_{111}(n,0) &= 0.499301 \cdot 1.91076^n + O((10/7)^n)\\
C_{112}(n,0) &= 0.692005 \cdot 1.80688^n + O((10/7)^n)\\
C_{221}(n,0) &= 0.545362 \cdot 1.94785^n + O((10/7)^n)\\
C_{123}(n,0) &= 0.576096 \cdot 1.94823^n + O((10/7)^n)\\
C_{peak}(n,0) &= 1.394560 \cdot 1.62975^n + O((10/7)^n)\\
C_{valley}(n,0) &= 0.728207 \cdot 1.84092^n + O((10/7)^n).
\end{aligned}$$

Proof. Let ρ be the smallest positive root of $f(z)$. If ρ is a simple pole, then the residue is given by $1/f'(\rho)$. Since $C_\tau(n,0) \le 2^{n-1}$, the number of unrestricted compositions with parts in $\mathbb{N}$, we know that the radius of convergence of $C_\tau(z) > 0.5$, and therefore, $\rho > 0.5$ for all patterns τ. Using Mathematica and Maple, we compute both ρ and $1/f'(\rho)$ for all patterns. To verify that we are dealing with simple poles in each case, we use the "Principle of the Argument" (see Theorem 4.10a, [20]), which states that the number of zeros of a function $f(z)$ is equal to the winding number of the transformed curve $f(\Gamma)$ around the origin, where Γ is a simple closed curve. We use as Γ the circle $r = |z| = 0.7$. Figure 1 shows the six graphs.

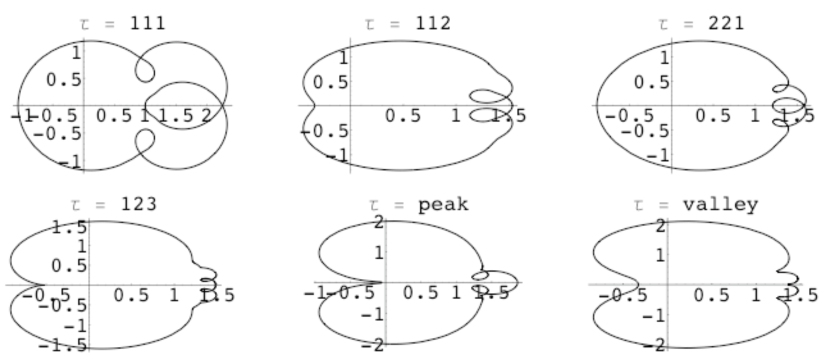

Figure 1. The image of the circle $|z| = 0.7$ under the respective generating functions.

Clearly, the winding number is 1 in each case, confirming that ρ is a simple pole. Thus, we obtain from Theorem 5.2.1 [26] that $C_\tau(n,0) = K \cdot v^n + O((1/r)^n)$, where $K = -1/(\rho f'(\rho))$ and $v = 1/\rho$, which completes the proof. $\square$

Note that Theorem 3.1 for $\tau = 111$ gives the asymptotics for the number of 2-Carlitz compositions. Asymptotics for the Carlitz compositions were given in [23].

4. Counting occurrences of subword patterns

Several authors (for example, see [3, 4] and references therein) have studied the occurrence of subword patterns in words on k letters. We will apply the results derived in the previous sections to words on k letters, and thus obtain previous results as special cases.

Let $[k] = \{1, 2, \ldots, k\}$ be a (totally ordered) alphabet on k letters. We call the elements of $[k]^n$ *words*. A word σ *contains* a pattern τ if σ contains a subsequence isomorphic to τ. Otherwise, we say that σ *avoids* τ. The *reversal* of τ, denoted by $r(\tau)$, is the pattern τ read from right to left, and the *complement* of τ, denoted by $c(\tau)$, is the pattern obtained by replacing τ_i by $k+1-\tau_i$. The set $\{\tau, r(\tau), c(\tau), c(r(\tau))\}$ is called the *symmetry* class of τ. It is easy to see that patterns from the same symmetry class occur an equal number of times in all the words of length m.

The connection between compositions and words is as follows: If $C_\tau^A(x, y, z)$ is the generating function for the number of compositions of n with m parts in the set A and r occurrences of the pattern or statistic τ, then $C_\tau^A(1, y, z)$ is the generating function for the number of words of length m on the alphabet A with r occurrences of the subword or statistic τ. We are now ready to apply our results from the previous sections to words on k letters.

Theorem 2.1 gives

$$C_{111}^{[k]}(1, y, z) = \frac{1}{1 - \frac{kz(1+(1-y)z)}{1+z(1+z)(1-y)}} = \frac{1 + z(1+z)(1-y)}{1 - (k-1+y)z - (k-1)(1-y)z^2},$$

i.e., we obtain the results of Example 2.2 [4] and Theorem 3.1 [3].

Theorem 2.2 gives (after simplification)

$$C_{112}^{[k]}(1, y, z) = C_{221}^{[k]}(1, y, z) = \frac{(1-y)z}{(1-y)z - 1 + (1-(1-y)z^2)^k},$$

i.e., we obtain Theorem 3.2 [3]. In addition, for $y = 0$ we obtain the generating function for the words of length m that avoid the pattern 112, given in Theorem 3.10 [4]:

$$C_{112}^{[k]}(1, 0, z) = C_{221}^{[k]}(1, 0, z) = \frac{z}{z - 1 + (1-z^2)^k}.$$

Note that necessarily $C_{221}^{[k]}(1, y, z) = C_{112}^{[k]}(1, y, z)$, as 112 and $221 = c(112)$ are in the same symmetry class, which explains why the respective generating functions for compositions are similar in structure.

Theorem 2.3 for $x = 1$ yields $t_k^p([k]) = \binom{k}{p} z^p$ and thus,

$$C_{123}^{[k]}(1, y, z) = \frac{1}{1 - k\,z - \sum_{p=3}^{k} \sum_{j=0}^{p-3} \binom{p-3}{j} \binom{k}{p+j} z^{p+j} (y-1)^{p-2}}.$$

This generating function for the number of words of length m that contain the pattern 123 exactly r times was given in a different form in Theorem 3.3 of [3]:

Theorem 3.3 (Burnstein and Mansour, [3]) *For* $k \geq 2$,

$$F_{123}(z,y;k) = \frac{1}{1 - k\,z - \sum_{j=3}^{k}(-z)^j\binom{k}{j}(1-y)^{\lfloor j/2\rfloor}U_{j-3}(y)},$$

where $U_0(y) = U_1(y) = 1, U_{2n}(y) = (1-y)U_{2n-1}(y) - U_{2n-2}(y)$, and $U_{2n+1}(y) = U_{2n}(y) - U_{2n-1}(y)$. Furthermore, the generating function for $U_n(y)$ is given by

$$\sum_{n\geq 0} U_n(y)z^n = \frac{1+z+z^2}{1+(1+y)z^2+z^4}.$$

We can prove the equivalence of the two generating functions by substituting $x = 1$ into equations (6) and (7) which yields the expressions given in [3] for $F_\tau(z,y;k)$ and $D_\tau(z,y;k)$. Comparison of the initial conditions then shows that $C_{123}^{[k]}(1,y,z) = F_{123}(z,y;k)$.

Substituting $y = 0$ and $x = 1$ in Theorem 2.3 we get the generating function for the number of words of length m avoiding the subword 123:

$$C_{123}^{[k]}(1,0,z) = \frac{1}{1 - k\,z - (-1)^p\sum_{p=3}^{k}\sum_{j=0}^{p-3}\binom{p-3}{j}\binom{k}{p+j}z^{p+j}}.$$

This can be shown to give the result of Theorem 3.13 [4], namely

$$F_{123}(z,0;k) = \frac{1}{\sum_{j=0}^{k} a_j\binom{k}{j}z^j}, \tag{12}$$

where $a_{3\ell} = 1$, $a_{3\ell+1} = -1$ and $a_{3\ell+2} = 0$. We use the form of Theorem 3.3 [3] for $y = 0$, which gives

$$F_{123}(z,0;k) = \frac{1}{1 - k\,z + \sum_{j=3}^{k}(-1)^{j+1}z^j\binom{k}{j}U_{j-3}(0)}.$$

Note that for $y = 0$ and $m \geq 0$, $U_m(0) = U_{m-1}(0) - U_{m-2}(0)$, and thus, $U_m(0) = 1$ for $m \equiv 0$ or $1 \pmod 6$, $U_m(0) = -1$ for $m \equiv 3$ or $4 \pmod 6$, and $U_m(0) = 0$ otherwise. For $j \geq 3$ and $j = 3m$, $U_{j-3}(0) = 1$ and $(-1)^{j+1} = 1$. If $j \geq 3$ and $j = 3m+1$, then $U_{j-3}(0)$ and $(-1)^{j+1}$ will have opposite signs, resulting in a coefficient of -1. Finally, if $j \geq 3$ and $j = 3m+2$, then $U_{j-3}(0) = 0$. Note that formula (12) also holds for $j = 0, 1, 2$, and therefore we have shown the equivalence of the two results.

Now we apply our results to the statistics *peak* and *valley*, which will give new results. Using Theorem 2.5 for $A = [k]$ and $x = 1$, we need to determine the products $M^s([k]) = \sum_{(i_1,i_2,\ldots,i_s)\in P^s([k])}\prod_{j=1}^{s} b_{i_j} = z^s|P^s([k])|$. To determine $|P^s([k])|$ for $s = 2\ell+1$ and $s = 2\ell$, we rewrite the sequence of alternating $\leq$ and $<$ signs in the definition of $P^s([k])$ as strict inequalities and obtain (with $j_n = i_n + \lfloor (n-1)/2\rfloor$)

$$|P^{2\ell+1}([k])| = |\{(j_1,j_2,\ldots,j_{2\ell+1})|1 \leq j_1 < j_2 < \cdots < j_{2\ell+1} \leq k+\ell\}| = \binom{k+\ell}{2\ell+1}.$$

Using the same argument, we obtain $|P^{2\ell}([k])| = \binom{k-1+\ell}{2\ell}$. Substituting $M^{2\ell}([k]) = z^{2\ell}\binom{k-1+\ell}{2\ell}$ and $M^{2\ell+1}([k]) = z^{2\ell+1}\binom{k+\ell}{2\ell+1}$ into Theorem 2.5 gives

$$C_{peak}^{[k]}(1,y,z) = \frac{\sum_{j\geq 0} z^{2j}(1-y)^j\binom{k-1+j}{2j}}{\sum_{j\geq 0} z^{2j}(1-y)^j\binom{k-1+j}{2j} - \sum_{j\geq 0} z^{2j+1}(1-y)^j\binom{k+j}{2j+1}}.$$

Since $c(peak) = valley$, we have $C_{valley}^{[k]}(1,y,z) = C_{peak}^{[k]}(1,y,z)$, and, setting $y=0$, the generating function for the number of words of length m on the alphabet $[k]$ without peaks (valleys) is given by

$$C_{peak}^{[k]}(1,0,z) = C_{valley}^{[k]}(1,0,z) = \frac{\sum_{j\geq 0} z^{2j}\binom{k-1+j}{2j}}{\sum_{j\geq 0} z^{2j}\binom{k-1+j}{2j} - \sum_{j\geq 0} z^{2j+1}\binom{k+j}{2j+1}}.$$

Note that once again the symmetry structure of words explains the fact that the generating functions for the number of compositions that avoid *valleys* and *peaks*, respectively, have similar structure.

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