PARTITION REGULARITY OF MATRICES

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Abstract

This is a survey of results on partition regularity of matrices, beginning with the classic results of Richard Rado on kernel partition regularity, continuing with the groundbreaking results of Walter Deuber on image partition regularity, and leading up to the present day. Included are the largely settled world of finite matrices and the mostly unknown world of infinite matrices.

1. Introduction

I am deeply honored to be invited to speak on this occasion. I have known Ron Graham since 1972. Back then he was older than I. Well, I guess technically he still is, but then he *looked like* he was older than I. I suspect he has a picture in a closet somewhere.

It doesn't really have anything to do with the subject of this talk, but I would like to call attention to some of the greatest prose in the mathematical literature. As you may recall, if α, β, γ, and δ are cardinals and $[A]^\gamma = \{C \subseteq A : |C| = \gamma\}$, then the notation $\alpha \longrightarrow (\beta)^\gamma_\delta$ abbreviates the statement "whenevever A is a set with $|A| = \alpha$ and $[A]^\gamma$ is divided into δ classes, there is some $B \in [A]^\beta$ such that $[B]^\gamma$ is contained in one of these classes". In his lovely little book [11], Ron wrote "We will occasionally use this arrow notation unless there is danger of no confusion."

Ron is very much identified with the subject of *Ramsey Theory* and it is a portion of that subject with which I am concerned now, namely the partition regularity of matrices. In his famous 1933 paper [36] Richard Rado studied partition regularity of systems of linear equations.

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That is, given a system of equations

$$\begin{array}{ccccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \dots & + & a_{1,v}x_v & = & b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \dots & + & a_{2,v}x_v & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ a_{u,1}x_1 & + & a_{u,2}x_2 & + & \dots & + & a_{u,v}x_v & = & b_u \end{array}$$

and given a finite partition of the set $\mathbb{N}$ of positive integers, could one guarantee a solution set $\{x_1, x_2, \dots, x_v\}$ contained in one cell of the partition? In alternative coloring terminology, one is asking whether, whenever $\mathbb{N}$ is finitely colored, there must be a *monochromatic* solution set.

In matrix notation, the question being investigated was whether, given a finite coloring of $\mathbb{N}$, one could find $\vec{x}$ with monochromatic entries such that $A\vec{x} = \vec{b}$. (We will follow the usual custom of denoting the entries of a matrix by the lower case letter corresponding to the upper case name of the matrix.)

Most attention has been paid to the case where the system of equations is homogeneous, that is where $\vec{b} = \overline{0}$, and we shall address that first. In that case, the mapping $\vec{x} \mapsto A\vec{x}$ is a linear transformation. The terminology *kernel partition regular* was suggested by Walter Deuber and is based on consideration of this transformation.

Definition 1.1. Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Let S be a subsemigroup of $(\mathbb{R}, +)$. Then A is *kernel partition regular over* S (KPR/S) if and only if, whenever $S \setminus \{0\}$ is finitely colored, there must exist monochromatic $\vec{x} \in S^v$ such that $A\vec{x} = \overline{0}$.

We shall address the kernel partition regularity of a matrix A in Section 2.

In the case that $\vec{b} \neq \overline{0}$ one may, by moving $\vec{b}$ to the other side of the equation (and replacing $\vec{b}$ by $-\vec{b}$), talk about the kernel partition regularity of the affine transformation $\vec{x} \mapsto A\vec{x} + \vec{b}$.

Definition 1.2. Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$ and let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$. Let S be a subsemigroup of $(\mathbb{R}, +)$. Then the pair $(A, \vec{b}\,)$ is *kernel partition regular over* S (KPR/S) if and only if, whenever S is finitely colored, there must exist monochromatic $\vec{x} \in S^v$ such that $A\vec{x} + \vec{b} = \overline{0}$.

Notice that there is no point in requiring that 0 not be colored in the case that $\vec{b} \neq \overline{0}$. We shall address the kernel partition regularity of pairs $(A, \vec{b}\,)$ where $\vec{b} \neq \overline{0}$ in Section 3.

Call a subset B of $\mathbb{N}$ "large" if whenever A is KPR/$\mathbb{N}$ there must exist $\vec{x}$ with entries from B such that $A\vec{x} = \overline{0}$. Rado conjectured that large sets are partition regular. That is whenever a large set is partitioned into finitely many cells, one of these must be large. Deuber [3] proved this conjecture using what he called (m, p, c)-sets. These sets are the images of certain linear transformations and Deuber's results can be described in terms of the *image partition regularity* of certain matrices.

Definition 1.3. Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Let S be a subsemigroup of $(\mathbb{R}, +)$. Then A is *image partition regular over* S (IPR/S) if and only if, whenever $S \setminus \{0\}$ is finitely colored, there must exist $\vec{x} \in S^v$ such that the entries of $A\vec{x}$ are monochromatic.

We shall address the image partition regularity of a matrix A in Section 4.

Definition 1.4. Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$ and let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$. Let S be a subsemigroup of $(\mathbb{R}, +)$. Then the pair $(A, \vec{b}\,)$ is *image partition regular over* S (IPR/S) if and only if, whenever S is finitely colored, there must exist $\vec{x} \in S^v$ such that the entries of $A\vec{x} + \vec{b}$ are monochromatic.

We shall address the image partition regularity of pairs $(A, \vec{b}\,)$ where $\vec{b} \neq \overline{0}$ in Section 5.

We shall see in Sections 2 through 5 that the image and kernel partition regularity of finite matrices is largely settled. By way of contrast, the partition regularity of infinite matrices is a wide open field. There are many partial results, but no characterizations. We shall discuss what is known about infinite matrices in Section 6.

In a final section we will present some information about related topics.

2. Kernel Partition Regularity of Linear Transformations

In his original 1933 paper [36] Rado characterized kernel partition regularity of a finite matrix over $\mathbb{N}$ and he extended the result in his later paper [37] to cover other subsets of $\mathbb{R}$ (and even of $\mathbb{C}$). The characterization was in terms of something called the *columns condition*.

Definition 2.1. Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Denote the columns of A by $\vec{c_1}, \vec{c_2}, \ldots, \vec{c_v}$. Then A satisfies the *columns condition* if and only if there exist $m \in \{1, 2, \ldots, v\}$ and a partition $\{I_1, I_2, \ldots, I_m\}$ of $\{1, 2, \ldots, v\}$ into nonempty sets such that

(a) $\sum_{i \in I_1} \vec{c_i} = \overline{0}$ and

(b) for each $t \in \{2, 3, \ldots, m\}$ (if any), $\sum_{i \in I_t} \vec{c_i}$ is a linear combination with coefficients from $\mathbb{Q}$ of $\{\vec{c_i} : i \in \bigcup_{j=1}^{t-1} I_j\}$.

In his papers, Rado characterized kernel partition regularity over $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, as well as over the semigroups $\mathbb{Q}^+ = \{x \in \mathbb{Q} : x > 0\}$ and $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$.

Theorem 2.2 (Rado). *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. The following statements are equivalent.*

(a) A is KPR/$\mathbb{N}$.

(b) A is KPR/$\mathbb{Z}$.

(c) A is KPR/$\mathbb{Q}^+$.

(d) A is KPR/$\mathbb{Q}$.

(e) A is KPR/$\mathbb{R}^+$.

(f) A is KPR/$\mathbb{R}$.

(g) A satisfies the columns condition.

Proof. The implications in the following diagram are all trivial.

$$\begin{tikzcd}
 & \mathrm{KPR}/\mathbb{N} \arrow[dl] \arrow[dr] & & \\
\mathrm{KPR}/\mathbb{Z} \arrow[dr] & & \mathrm{KPR}/\mathbb{Q}^+ \arrow[dl] \arrow[dr] & \\
 & \mathrm{KPR}/\mathbb{Q} \arrow[dr] & & \mathrm{KPR}/\mathbb{R}^+ \arrow[dl] \\
 & & \mathrm{KPR}/\mathbb{R} &
\end{tikzcd}$$

To see that (f) implies (g), assume that A is KPR/$\mathbb{R}$. Then by [37, Theorem VII] A satisfies the version of Definition 2.1 which replaces (b) with

(b) for each $t \in \{2, 3, \ldots, m\}$ (if any), $\sum_{i \in I_t} \vec{c_i}$ is a linear combination with coefficients from $\mathbb{R}$ of $\{\vec{c_i} : i \in \bigcup_{j=1}^{t-1} I_j\}$.

But since a rational vector is in the linear span over $\mathbb{R}$ of a set of rational vectors if and only if it is in the linear span over $\mathbb{Q}$ of those same vectors, this tells us that A satisfies the columns condition.

That (g) implies (a) follows from the original version of Rado's Theorem ([36, Satz IV], or see [12, Theorem 3.5] or [22, Theorem 15.20]). $\square$

Notice that Rado's Theorem easily implies the classic theorem of Schur [39].

Theorem 2.3 (Schur). *Whenever $\mathbb{N}$ is finitely colored there exist x and y in $\mathbb{N}$ such that $\{x, y, x + y\}$ is monochromatic.*

Proof. This is just the assertion that the matrix $\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$ satisfies the columns condition. $\square$

Rado's Theorem also implies a far reaching generalization of Schur's Theorem (which was new at the time) involving arbitrarily large (but finite) sets of finite sums. We introduce some special notation for these. Given a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in $\mathbb{R}$, $FS(\langle x_n\rangle_{n=1}^{\infty}) = \{\sum_{n\in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}$, where for any set X, $\mathcal{P}_f(X)$ is the set of finite nonempty subsets of X. Also, given $m \in \mathbb{N}$ and a sequence $\langle x_n\rangle_{n=1}^{m}$ in $\mathbb{R}$, $FS(\langle x_n\rangle_{n=1}^{m}) = \big\{\sum_{n\in F} x_n : \emptyset \neq F \subseteq \{1, 2, \ldots, m\}\big\}$.

Before stating the finite Finite Sums Theorem in its generality, let us consider the case $m = 3$. This version says that whenever $\mathbb{N}$ is finitely colored, there must exist x_1, x_2, and x_3 such that $FS(\langle x_n\rangle_{n=1}^{3}) = \{x_1, x_2, x_1+x_2, x_3, x_1+x_3, x_2+x_3, x_1+x_2+x_3\}$ is monochromatic. If one labels the columnns of a matrix by the nonempty subsets of $\{1, 2, 3\}$ and the rows by the subsets of $\{1, 2, 3\}$ with at least two elements, then the coefficient matrix of the required equations is as follows.

$$\begin{array}{c c}
 & \begin{array}{ccccccc} \{1\} & \{2\} & \{3\} & \{1,2\} & \{1,3\} & \{2,3\} & \{1,2,3\} \end{array} \\
\begin{array}{c} \{1,2\} \\ \{1,3\} \\ \{2,3\} \\ \{1,2,3\} \end{array} & \begin{pmatrix} 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}
\end{array}$$

One then easily sees that this matrix satisfies the columns condition.

Corollary 2.4. *Let $m \in \mathbb{N}$. Whenever $\mathbb{N}$ is finitely colored, there exists a sequence $\langle x_n\rangle_{n=1}^{m}$ in $\mathbb{N}$ such that $FS(\langle x_n\rangle_{n=1}^{m})$ is monochromatic.*

Proof. We may assume that $m \geq 2$. Let A be a $(2^m - m - 1) \times (2^m - 1)$ matrix with columns indexed by the nonempty subsets of $\{1, 2, \ldots, m\}$ and rows indexed by the subsets of $\{1, 2, \ldots, m\}$ with at least two members. The entry in row F and column G is 1 if $G = \{n\}$ and $n \in F$, -1 if $F = G$, and 0 otherwise. Then A satisfies the columns condition with, for $t \in \{1, 2, \ldots, m\}$, $I_t = \{F \subseteq \{1, 2, \ldots, m\} : \min F = t\}$. □

A special case of Corollary 2.4 is the following theorem of Hilbert [13]. (In [13], Hilbert actually proved that there are infinitely many choices for a, a fact which does not follow from Rado's Theorem.)

Corollary 2.5 (Hilbert). *Let $m \in \mathbb{N}$. Whenever $\mathbb{N}$ is finitely colored, there exist $a \in \mathbb{N}$ and a sequence $\langle x_n\rangle_{n=1}^{m}$ in $\mathbb{N}$ such that $a + FS(\langle x_n\rangle_{n=1}^{m})$ is monochromatic.*

Proof. Take $\langle x_n\rangle_{n=1}^{m+1}$ as guaranteed by Corollary 2.4 and let $a = x_{m+1}$. □

In spite of the fact that van der Waerden's Theorem [42] was a major motivation for Rado's original paper—his introduction begins with a description of van der Waerden's Theorem—the situation is trickier with respect to this theorem. Consider the length five version of this theorem, which says that whenever $\mathbb{N}$ is finitely colored, there must exist a monochromatic length five arithmetic progression $\{a, a+d, a+2d, a+3d, a+4d\}$. If one lets

$$\begin{aligned} y_1 &= a \\ y_2 &= a+d \\ y_3 &= a+2d \\ y_4 &= a+3d \\ y_5 &= a+4d \end{aligned}$$

then a natural candidate for a set of equations describing this progression is

$$\begin{aligned} y_3 - y_2 &= y_2 - y_1 \\ y_4 - y_3 &= y_3 - y_2 \\ y_5 - y_4 &= y_4 - y_3 \end{aligned}$$

whose matrix of coefficients is

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{pmatrix}.$$

This matrix does satisfy the columns condition because the columns sum to $\overline{0}$. In this case the monochromatic solution that is guaranteed by Rado's Theorem might be constant, which is hardly what one has in mind when speaking of an "arithmetic progression". The standard fix to this problem is to add the requirement that the increment also be the same color (in which case it is necessarily positive). In the length five case one can let $y_6 = d$ and add the equation $y_6 = y_2 - y_1$. Then the coefficient matrix becomes

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

which satisfies the columns condition with $I_1 = \{1,2,3,4,5\}$ and $I_2 = \{6\}$.

One might hope to just derive van der Waerden's Theorem by a better choice of equations. We see in fact that it is impossible to prove van der Waerden's Theorem from Rado's Theorem without strengthening the conclusion.

Theorem 2.6. *Let $v \geq 3$. There do not exist u and a $u \times v$ matrix A with rational coefficients such that*

(a) A satisfies the columns condition and

(b) whenever $\vec{x} \in \mathbb{N}^v$ and $A\vec{x} = \overline{0}$, $\{x_1, x_2, \ldots, x_v\}$ is a (nontrivial) v-term arithmetic progression.

Proof. Suppose one has such u and A. Trivially no column of A can be $\overline{0}$. Since the columns of A do not sum to zero, pick $m \geq 2$ and $I_1, I_2, \ldots, I_m$ as guaranteed by the columns condition.

Pick $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} = \overline{0}$. Let $l = \max\left(\{2v(x_i - x_j) : i, j \in \{1, 2, \ldots, v\}\} \cup \{x_i : i \in \{1, 2, \ldots, v\}\}\right)$. Define $\vec{y} \in \mathbb{N}^v$ by

$$y_j = \begin{cases} x_j + l & \text{if } j \in I_1 \\ x_j & \text{if } j \in \{1, 2, \ldots, v\} \setminus I_1 \,. \end{cases}$$

Then $A\vec{y} = \overline{0}$ so pick $a, d \in \mathbb{N}$ such that $\{y_1, y_2, \ldots, y_v\} = \{a, a + d, \ldots, a + (v - 1)d\}$. Pick $i \neq j$ in I_1 and $k \in \{1, 2, \ldots, v\} \setminus I_1$ and pick $b, c, e \in \{0, 1, \ldots, v - 1\}$ such that $y_i = a + bd$, $y_j = a + cd$, and $y_k = a + ed$. Assume without loss of generality that $b < c$. Since $y_i > l \geq x_k$ we have that $e < b$.

Now $\frac{l}{2} > x_k - x_i$ so $\frac{l}{2} < x_i - x_k + l = y_i - y_k = (b - e)d < vd$ so $d > \frac{l}{2v}$. Thus $x_j - x_i = y_j - y_i \geq d > \frac{l}{2v}$ so $l < 2v(x_j - x_i)$, a contradiction. □

What one is really concerned about in deriving van der Waerden's Theorem is not that one is guaranteed a monochromatic solution to $A\vec{x} = \overline{0}$, but that one is guaranteed a monochromatic solution which is not constant. When this occurs has been characterized recently in joint research with Imre Leader.

Theorem 2.7. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. The following statements are equivalent.*

(a) *Whenever $\mathbb{N}$ is finitely colored there exists monochromatic nonconstant $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} = \overline{0}$.*

(b) *Whenever $\mathbb{Z}$ is finitely colored there exists monochromatic nonconstant $\vec{x} \in \mathbb{Z}^v$ such that $A\vec{x} = \overline{0}$.*

(c) *Whenever $\mathbb{Q}$ is finitely colored there exists monochromatic nonconstant $\vec{x} \in \mathbb{Q}^v$ such that $A\vec{x} = \overline{0}$.*

(d) *The matrix A satisfies the columns condition and there exists nonconstant $\vec{x} \in \mathbb{Q}^v$ such that $A\vec{x} = \overline{0}$.*

(e) *The matrix A satisfies the columns condition and if the sum of the columns of A is $\overline{0}$, then there exists nonempty $D \subsetneq \{1, 2, \ldots, v\}$ and for each $j \in D$ there exists $\alpha_j \in \mathbb{Q}\setminus\{0\}$ such that $\sum_{j \in D} \alpha_j \vec{c_j} = 0$, where $\vec{c_j}$ is column j of A.*

Proof. [16, Theorem 3.2]. □

3. Kernel Partition Regularity of Affine Transformations

In his original paper [36] Rado also solved the problem of the partition regularity of a non-homogeneous systems of linear equations over $\mathbb{Z}$ and $\mathbb{N}$. And in [37] he obtained a similar characterization for nonhomogeneous systems of linear equations over $\mathbb{Q}$. That is, he determined when the pair $(A,\vec{b}\,)$ is kernel partition regular over $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$.

We present proofs of (a) (which is [36, Satz VIII]) and (c) (which is [36, Satz V]) in the following theorem for the benefit of the reader who either has difficulty in getting access to [36] or has difficulty reading German.

Theorem 3.1 (Rado). *Let $u,v\in\mathbb{N}$, let A be a $u\times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b}\in\mathbb{Q}^u\setminus\{\overline{0}\}$.*

(a) The pair $(A,\vec{b}\,)$ is KPR/$\mathbb{Z}$ if and only if there exists $k\in\mathbb{Z}$ such that $A\overline{k}+\vec{b}=\overline{0}$.

(b) The pair $(A,\vec{b}\,)$ is KPR/$\mathbb{Q}$ if and only if there exists $k\in\mathbb{Q}$ such that $A\overline{k}+\vec{b}=\overline{0}$.

(c) The pair $(A,\vec{b}\,)$ is KPR/$\mathbb{N}$ if and only if either

(i) there exists $k\in\mathbb{N}$ such that $A\overline{k}+\vec{b}=\overline{0}$ or

(ii) there exists $k\in\mathbb{Z}$ such that $A\overline{k}+\vec{b}=\overline{0}$ and A satisfies the columns condition.

Proof. (a) The sufficiency is trivial.

For the necessity, note first that we can presume that the entries of A and $\vec{b}$ are integers. (Pick $k\in\mathbb{N}$ such that the entries of kA and $k\vec{b}$ are integers. If $kA\overline{d}+k\vec{b}=\overline{0}$, then $A\overline{d}+\vec{b}=\overline{0}$.) For $j\in\{1,2,\ldots,v\}$, let $\vec{c_j}$ be column j of A and let $\vec{\alpha}=\sum_{j=1}^{v}\vec{c_j}$ (so for $i\in\{1,2,\ldots,u\}$, $\alpha_i=\sum_{j=1}^{v}a_{i,j}$).

We note first that if $i\in\{1,2,\ldots,u\}$ and $\alpha_i=0$, then $b_i=0$. To see this, pick $n\in\mathbb{N}\setminus\{1\}$ such that $n>|b_i|$ and color $\mathbb{Z}$ by congruence mod n. Pick $d\in\{0,1,\ldots,n-1\}$ and $\vec{x}\in\mathbb{Z}^v$ such that $A\vec{x}+b=\overline{0}$ and $x_j\equiv d \pmod n$ for each $j\in\{1,2,\ldots,v\}$. For each $j\in\{1,2,\ldots,v\}$ pick $w_j\in\mathbb{Z}$ such that $x_j=nw_j+d$. Then $-b_i=n\sum_{j=1}^{v}a_{i,j}w_j+d\alpha_i$. Since $\alpha_i=0$, $n|b_i$ and thus $b_i=0$.

Let $G=\{i\in\{1,2,\ldots,u\}:\alpha_i\neq 0\}$. Since $\vec{b}\neq\overline{0}$, $G\neq\emptyset$. It suffices to find $d\in\mathbb{Z}\setminus\{0\}$ such that for each $i\in G$, $\alpha_i d=b_i$. We show first that for each $i\in G$, $\alpha_i|b_i$, so let $i\in G$ be given. If $|\alpha_i|=1$ we are done, so assume that $|\alpha_i|>1$ and let $n=|\alpha_i|$. Color $\mathbb{Z}$ by congruence mod n. Pick $d\in\{0,1,\ldots,n-1\}$ and $\vec{x}\in\mathbb{Z}^v$ such that $A\vec{x}+\vec{b}=\overline{0}$ and $x_j\equiv d \pmod n$ for each $j\in\{1,2,\ldots,v\}$. For each $j\in\{1,2,\ldots,v\}$ pick $w_j\in\mathbb{Z}$ such that $x_j=nw_j+d$. Then $-b_i=n\sum_{j=1}^{v}a_{i,j}w_j+d\alpha_i$, so $n|b_i$.

For each $i\in G$, pick $m_i\in\mathbb{Z}$ such that $b_i=m_i\alpha_i$. We show now that if n is a prime such that $n>|\alpha_i|$ and $n>|m_i|$ for each $i\in G$, then there exists $d_n\in\{0,1,\ldots,n-1\}$ such that $m_i=d_n$ if $m_i\geq 0$ and $m_i=d_n-n$ if $m_i<0$. To see this color $\mathbb{Z}$ by congruence mod n. Pick $d_n\in\{0,1,\ldots,n-1\}$ and $\vec{x}\in\mathbb{Z}^v$ such that $A\vec{x}+\vec{b}=\overline{0}$ and $x_j\equiv d_n \pmod n$ for each $j\in\{1,2,\ldots,v\}$. For each $j\in\{1,2,\ldots,v\}$ pick $w_j\in\mathbb{Z}$ such that $x_j=nw_j+d_n$. Then $m_i\alpha_i=b_i=-(n\sum_{j=1}^{v}a_{i,j}w_j+d_n\alpha_i)$ so n divides $(m_i-d_n)\alpha_i$ so n divides m_i-d_n. If $m_i\geq 0$, then $-n < m_i - d_n < n$ so $m_i = d_n$. If $m_i < 0$, then $-n < m_i - d_n + n < n$ so $m_i = d_n - n$.

If for each $i \in G$, $m_i \geq 0$, or for each $i \in G$, $m_i < 0$, we are done. So suppose that we have $s, t \in G$ such that $m_s \geq 0$ and $m_t < 0$. Pick distinct primes p and n such that $n > |\alpha_i|$, $p > |\alpha_i|$, $n > |m_i|$, and $p > |m_i|$ for each $i \in G$. Then $m_s = d_n = d_p$ and $m_t = d_n - n = d_p - p$, so $n = p$, a contradiction.

(b) The ideas needed for the proof are in [37]. See [16, Theorem 2.5] for the details.

(c) For the necessity, note that $(A, \vec{b}\,)$ is KPR/$\mathbb{Z}$ so pick by (a), $k \in \mathbb{Z}$ such that $A\overline{k} + \vec{b} = \overline{0}$. If $k \in \mathbb{N}$, we have that (i) holds, so assume that $k \leq 0$. To see that A satisfies the columns condition we show that A is KPR/$\mathbb{N}$ so that Theorem 2.2 applies. So let $r \in \mathbb{N}$ and let $\varphi : \mathbb{N} \to \{1, 2, \ldots, r\}$. Define $\psi : \mathbb{N} \to \{1, 2, \ldots, r\}$ by $\psi(x) = \varphi(x - k)$. Pick $\vec{y} \in \mathbb{N}^v$ such that $\vec{y}$ is monochromatic with respect to ψ and $A\vec{y} + \vec{b} = \overline{0}$. Let $\vec{x} = \vec{y} - \overline{k}$. Then $\vec{x}$ is monochromatic with respect to φ and $A\vec{x} = A\vec{y} - A\overline{k} = \overline{0}$.

For the sufficiency, if (i) holds the conclusion is trivial, so assume that (i) fails (and thus (ii) holds). Let $r \in \mathbb{N}$ and let $\varphi : \mathbb{N} \to \{1, 2, \ldots, r\}$. Pick $k \in \mathbb{Z}$ such that $k \leq 0$ and $A\overline{k} + \vec{b} = \overline{0}$. Define $\psi : \mathbb{N} \to \{1, 2, \ldots, r - k\}$ by

$$\psi(x) = \begin{cases} \varphi(x + k) & \text{if } x > -k \\ r + x & \text{if } x \leq -k\,. \end{cases}$$

Since A satisfies the columns condition, pick by Theorem 2.2 some $\vec{y} \in \mathbb{N}^v$ such that $\vec{y}$ is monochromatic with respect to ψ and $A\vec{y} = \overline{0}$. We cannot have that the constant color of the entries of $\vec{y}$ is greater than r, because then we would have that the columns of A would sum to $\overline{0}$ and thus $A\overline{k} + \vec{b} = \overline{0} + \vec{b} \neq \overline{0}$. Let $\vec{x} = \vec{y} + \overline{k}$. Then $\vec{x}$ is monochromatic with respect to φ and $A\vec{x} + \vec{b} = A\vec{y} + A\overline{k} + \vec{b} = \overline{0}$. □

Notice two things about Theorem 3.1. First, while trivially for a pair $(A, \vec{b}\,)$

$$\text{KPR}/\mathbb{N} \Rightarrow \text{KPR}/\mathbb{Z} \Rightarrow \text{KPR}/\mathbb{Q}$$

none of the implications are reversible. The pair $\big((2), 1\big)$ is kernel partition regular over $\mathbb{Q}$ but not over $\mathbb{Z}$ (since the only solution to $2x + 1 = 0$ is not an integer) and the pair $\big((1), 1\big)$ is kernel partition regular over $\mathbb{Z}$ but not over $\mathbb{N}$.

Second, with the exception of Theorem 3.1(c), the conclusion is that the only way a pair can be kernel partition regular is for it to be trivially so. This certainly explains why Theorem 2.2 is very famous, while hardly anyone has heard of Theorem 3.1.

However, the picture changes when one considers nonconstant kernel partition regularity.

Theorem 3.2. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$. If $S = \mathbb{Z}$ or $S = \mathbb{Q}$, then the following statements are equivalent.*

(a) *Whenever S is finitely colored there exists monochromatic nonconstant $\vec{x} \in S^v$ such that $A\vec{x} + \vec{b} = \overline{0}$.*

(b) *There exists $d \in S$ such that $A\overline{d} + \vec{b} = \overline{0}$, A satisfies the columns condition, and there exists nonconstant $\vec{x} \in \mathbb{Q}^v$ such that $A\vec{x} + \vec{b} = \overline{0}$.*

Proof. [16, Theorem 3.3]. □

Notice that as a consequence of Theorem 3.2, we have that the pair $\big(\begin{pmatrix} 2 & -2 & 2 \end{pmatrix}, (1)\big)$ is nonconstantly kernel partition regular over $\mathbb{Q}$ but not over $\mathbb{Z}$.

The following theorem establishes that nonconstant kernel partition regularity of $(A, \vec{b})$ over $\mathbb{Z}$ and $\mathbb{N}$ are equivalent. It is interesting that there does not seem to be a trivial proof of this equivalence. (By way of contrast, the equivalence of Theorem 2.2 (a) and (b) is trivial—given $\varphi : \mathbb{N} \to \{1, 2, \ldots, r\}$ define $\psi : \mathbb{Z} \setminus \{0\} \to \{1, 2, \ldots, 2r\}$ by $\psi(x) = \varphi(x)$ if $x > 0$ and $\psi(x) = r + \varphi(-x)$ if $x < 0$.)

Theorem 3.3. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$. The following statements are equivalent.*

(a) *Whenever $\mathbb{N}$ is finitely colored there exists nonconstant monochromatic $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} + \vec{b} = \overline{0}$.*

(b) *Whenever $\mathbb{Z}$ is finitely colored there exists nonconstant monochromatic $\vec{x} \in \mathbb{Z}^v$ such that $A\vec{x} + \vec{b} = \overline{0}$.*

(c) *There exists $d \in \mathbb{Z}$ such that $A\overline{d} + \vec{b} = \overline{0}$, A satisfies the columns condition, and there exists nonconstant $\vec{x} \in \mathbb{Q}^v$ such that $A\vec{x} + \vec{b} = \overline{0}$.*

Proof. [16, Theorem 3.4]. □

4. Image Partition Regularity of Linear Transformations

Recall from the introduction that Rado called a subset B of $\mathbb{N}$ *large* provided that whenever A is KPR/$\mathbb{N}$ there must exist $\vec{x}$ with entries from B such that $A\vec{x} = \overline{0}$ and he conjectured that whenever a large set is finitely colored there must be a monochromatic large set. Deuber [3] proved this conjecture, using what he called "(m, p, c)-sets".

Definition 4.1. Let $m, p, c \in \mathbb{N}$ with $p \geq c$. Then B is an (m, p, c)*-set* if and only if There exists $\vec{x} \in \mathbb{N}^m$ such that $B = \{\sum_{i=1}^{m} \lambda_i x_i$: each $\lambda_i \in \{-p, -p+1, \ldots, p-1, p\}$ and if $t = \min\{i : \lambda_i \neq 0\}$, then $\lambda_t = c\}$.

Notice that each (m, p, c)-set is the image of a *first entries matrix*.

Definition 4.2. Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix. Then A is a *first entries matrix* if and only if

(1) the entries of A are from $\mathbb{Q}$,

(2) no row of A is $\overline{0}$,

(3) the first (leftmost) nonzero entry of each row is positive, and

(4) the first nonzero entries of any two rows are equal if they occur in the same column.

A number c is a *first entry* of A if c is the first nonzero entry of some row of A.

While Deuber's proof of Rado's conjecture used the terminology of (m, p, c)-sets, the proofs are valid for images of arbitrary first entries matrices, and we shall phrase them in that fashion. We write $\omega = \mathbb{N} \cup \{0\}$.

Theorem 4.3 (Deuber). *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$ which satisfies the columns condition. There exists $m \in \{1, 2, \ldots, v\}$ and a $v \times m$ first entries matrix B with entries from ω such that $AB = \mathbf{O}$.*

Proof. The ideas needed for the proof are in [3]. For the details see [22, Lemma 15.15]. □

Theorem 4.4 (Deuber). *Let $v, m, r \in \mathbb{N}$ and let B be a $v \times m$ first entries matrix. There exist $n, q \in \mathbb{N}$ and an $n \times q$ first entries matrix C such that, whenever $\vec{y} \in \mathbb{Z}^q$ and the entries of $C\vec{y}$ are r-colored, there exists $\vec{w} \in \mathbb{Z}^m$ such that the entries of $B\vec{w}$ are contained in the entries of $C\vec{y}$ and are monochromatic.*

Proof. [3, Satz 3.1]. Or see [23, Theorem 3.4]. □

Theorem 4.5 (Deuber). *Let $n, q \in \mathbb{N}$ and let C be an $n \times q$ first entries matrix. There exist $s, t \in \mathbb{N}$ and an $s \times t$ matrix D which is KPR/$\mathbb{N}$ such that whenever $\vec{x} \in \mathbb{N}^t$ and $D\vec{x} = \overline{0}$, there exists some $\vec{y} \in \mathbb{Z}^q$ such that all entries of $C\vec{y}$ are included in the entries of $\vec{x}$.*

Proof. [3, Satz 2.2]. □

With these tools, we are able to spell out the simple argument establishing Rado's Conjecture.

Corollary 4.6 (Deuber). *Let X be a large subset of $\mathbb{N}$, let $r \in \mathbb{N}$, and let $\varphi : X \to \{1, 2, \ldots, r\}$. There exists a large monochromatic subset of X.*

Proof. Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$ which is KPR/$\mathbb{N}$. Then by Theorem 2.2 A satisfies the columns condition. By Theorem 4.3 pick $m \in \{1, 2, \ldots, v\}$ and a $v \times m$ first entries matrix B with entries from ω such that $AB = \mathbf{O}$. Pick $n, q \in \mathbb{N}$ and a first entries matrix C as guaranteed by Theorem 4.4. Finally, pick $s, t \in \mathbb{N}$ and an $s \times t$ matrix D as guaranteed by Theorem 4.5.

Pick $\vec{x} \in \mathbb{N}^t$ such that the entries of $\vec{x}$ are contained in X and $D\vec{x} = \overline{0}$. Pick $\vec{y} \in \mathbb{Z}^q$ such that all entries of $C\vec{y}$ are included in the entries of $\vec{x}$. Pick $\vec{w} \in \mathbb{Z}^m$ such that the entries of $B\vec{w}$ are contained in the entries of $C\vec{y}$ and are monochromatic. Let $\vec{z} = B\vec{w}$. Then $A\vec{z} = \overline{0}$. □

Notice that, as a consequence of Theorem 4.5, first entries matrices are image partition regular. Several Ramsey Theoretic results are trivially equivalent to the image partition regularity of first entries matrices. For example, the length 5 version of van der Waerden's Theorem and the case $m = 3$ of the Finite Sums Theorem (Corollary 2.4) are naturally represented by the following matrices.

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Notice that in representing these theorems by first entries matrices one does not need to think or come up with equations that need to be solved.

Since the fact that first entries matrices are image partition regular had been known since the publication of [3] in 1973, and since image partition regularity is such a natural concept, I was surprised to find out in the early 1990's that it was not known which finite matrices are image partition regular. Imre Leader and I solved that problem in [15]. Since then, several other characterizations have been obtained in collaboration with Leader, Dona Strauss, and Irene Moshesh. Some of these involve the notion of *central* sets, which we pause now to define.

Central sets were introduced by Furstenberg [9] and defined in terms of notions of topological dynamics. These sets enjoy very strong combinatorial properties. (See [9, Proposition 8.21] or [22, Chapter 14].) See Chapter 19 and notes on that chapter in [22] for a description of how the equivalence with the algebraic definition given below was arrived at. (The idea that they might be equivalent is due to my long time collaborator Vitaly Bergelson. I have no idea how he came up with that idea.)

Let $(S, +)$ be an infinite discrete semigroup. We take the points of βS to be the ultrafilters on S, the principal ultrafilters being identified with the points of S. Given a set $A \subseteq S$, $\overline{A} = \{p \in \beta S : A \in p\}$. The set $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets (as well as a basis for the closed sets) of βS.

There is a natural extension of the operation $+$ of S to βS, making βS a compact right topological semigroup with S contained in its topological center. This says that for each $p \in \beta S$ the function $\rho_p : \beta S \to \beta S$ is continuous and for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$ is continuous, where $\rho_p(q) = q + p$ and $\lambda_x(q) = x + q$. See [22] for an elementary introduction to the semigroup βS.

Any compact Hausdorff right topological semigroup $(T, +)$ has a smallest two sided ideal $K(T)$ which is the union of all of the minimal left ideals of T, each of which is closed [22, Theorem 2.8] and any compact right topological semigroup contains idempotents. Since the minimal left ideals are themselves compact right topological semigroups, this says in particular that there are idempotents in the smallest ideal. There is a partial ordering of the idempotents of T determined by $p \leq q$ if and only if $p = p + q = q + p$. An idempotent p is minimal with respect to this order if and only if $p \in K(T)$ [22, Theorem 1.59]. Such an idempotent is called simply "minimal"

Definition 4.7. Let $(S, +)$ be an infinite discrete semigroup. A set $A \subseteq S$ is *central* if and only if there is some minimal idempotent p in βS such that $A \in p$.

Notice in particular that if S is partitioned into finitely many pieces, one of them must be central.

Theorem 4.8. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. The following statements are equivalent.*

(a) A is IPR/$\mathbb{N}$.

(b) For every central set C in $\mathbb{N}$, there exists $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} \in C^u$.

(c) For every central set C in $\mathbb{N}$, $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ is central in $\mathbb{N}^v$.

(d) There exist $m \in \mathbb{N}$, a $v \times m$ matrix G with non-negative rational entries and no row equal to $\overline{0}$, and a $u \times m$ first entries matrix B, with non-negative entries and all its first entries equal to 1, such that $AG = B$.

(e) There exist $m \in \mathbb{N}$, a $v \times m$ matrix G with non-negative rational entries and no row equal to $\overline{0}$, and a $u \times m$ first entries matrix B, with all its first entries equal to 1, such that $AG = B$.

(f) There exist $m \in \mathbb{N}$, a $v \times m$ matrix G with entries from ω and no row equal to $\overline{0}$, a $u \times m$ first entries matrix B with entries from ω, and $c \in \mathbb{N}$ such that c is the only first entry of B and $AG = B$.

(g) There exist $m \in \mathbb{N}$, a $u \times m$ first entries matrix B with all entries from ω, and $c \in \mathbb{N}$ such that c is the only first entry of B and for each $\vec{y} \in \mathbb{N}^m$ there exists $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} = B\vec{y}$.

(h) There exist $m \in \mathbb{N}$ and a $u \times m$ first entries matrix B such that for each $\vec{y} \in \mathbb{N}^m$ there exists $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} = B\vec{y}$.

(i) There exist $t_1, t_2, \ldots, t_v \in \mathbb{Q}^+$ such that the matrix

$$\begin{pmatrix} t_1 a_{1,1} & t_2 a_{1,2} & t_3 a_{1,3} & \cdots & t_v a_{1,v} & -1 & 0 & 0 & \cdots & 0 \\ t_1 a_{2,1} & t_2 a_{2,2} & t_3 a_{2,3} & \cdots & t_v a_{2,v} & 0 & -1 & 0 & \cdots & 0 \\ t_1 a_{3,1} & t_2 a_{3,2} & t_3 a_{3,3} & \cdots & t_v a_{3,v} & 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t_1 a_{u,1} & t_2 a_{u,2} & t_3 a_{u,3} & \cdots & t_v a_{u,v} & 0 & 0 & 0 & \cdots & -1 \end{pmatrix}$$

is KPR/$\mathbb{N}$.

(j) There exist $t_1, t_2, \ldots, t_v \in \mathbb{Q}^+$ such that the matrix

$$\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ t_1a_{1,1} & t_2a_{1,2} & t_3a_{1,3} & \ldots & t_va_{1,v} \\ t_1a_{2,1} & t_2a_{2,2} & t_3a_{2,3} & \ldots & t_va_{2,v} \\ t_1a_{3,1} & t_2a_{3,2} & t_3a_{3,3} & \ldots & t_va_{3,v} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_1a_{u,1} & t_2a_{u,2} & t_3a_{u,3} & \ldots & t_va_{u,v} \end{pmatrix}$$

is IPR/$\mathbb{N}$.

(k) There exist $b_1, b_2, \ldots, b_v \in \mathbb{Q}^+$ such that the matrix

$$\begin{pmatrix} b_1 & 0 & 0 & \cdots & 0 \\ 0 & b_2 & 0 & \cdots & 0 \\ 0 & 0 & b_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_v \\ & & A & & \end{pmatrix}$$

is IPR/$\mathbb{N}$.

(l) There exist $b_1, b_2, \ldots, b_v \in \mathbb{Q}^+$ such that the matrix

$$\begin{pmatrix} b_1 & 0 & 0 & \cdots & 0 \\ 0 & b_2 & 0 & \cdots & 0 \\ 0 & 0 & b_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_v \\ & & A & & \end{pmatrix}$$

is IPR/$\mathbb{Z}$.

(m) For each $\vec{r} \in \mathbb{Q}^v \setminus \{\vec{0}\}$ there exists $b \in \mathbb{Q} \setminus \{0\}$ such that

$$\begin{pmatrix} b\vec{r} \\ A \end{pmatrix}$$

is IPR/$\mathbb{N}$.

(n) Whenever $m \in \mathbb{N}$, $\phi_1, \phi_2, \ldots, \phi_m$ are nonzero linear mappings from $\mathbb{Q}^v$ to $\mathbb{Q}$, there exists $\vec{b} \in \mathbb{Q}^m$ such that, whenever C is central in $\mathbb{N}$, there exists $\vec{x} \in \mathbb{N}^v$ for which $A\vec{x} \in C^u$ and, for each $i \in \{1, 2, \ldots, m\}$, $b_i\phi_i(\vec{x}) \in C$, and in particular $\phi_i(\vec{x}) \neq 0$.

(o) For every central set C in $\mathbb{N}$, there exists $\vec{x} \in \mathbb{N}^v$ such that $\vec{y} = A\vec{x} \in C^u$, all entries of $\vec{x}$ are distinct, and for all $i, j \in \{1, 2, \ldots, u\}$, if rows i and j of A are unequal, then $y_i \neq y_j$.

(p) Given any column $\vec{c} \in \mathbb{Q}^u$, the matrix $\begin{pmatrix} A & \vec{c} \end{pmatrix}$ is IPR/$\mathbb{N}$.

(q) *Whenever* $m \in \mathbb{N}$, $\phi_1, \phi_2, \ldots, \phi_m$ *are nonzero linear mappings from* $\mathbb{Q}^v$ *to* $\mathbb{Q}$, *and* C *is a central subset of* $\mathbb{N}$, *there exist positive* $b_1, b_2, \ldots, b_m$ *in* $\mathbb{Q}$ *such that* $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u$ and for each $i \in \{1, 2, \ldots, m\}$, $b_i\phi_i(\vec{x}) \in C\}$ *is central in* $\mathbb{N}^v$.

(r) *Whenever* $m \in \mathbb{N}$ *and* C *is a central subset of* $\mathbb{N}$ $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u$, all entries of $\vec{x}$ are distinct, and entries of $A\vec{x}$ corresponding to distinct rows of A are distinct$\}$ *is central in* $\mathbb{N}^v$.

Proof. [17, Theorem 2.10], [15, Theorem 3.1], and [21, Theorem 4.1]. □

Notice that condition (i) of Theorem 4.8 is a computable condition. (Also, condition (l) leads to another computable condition via Theorem 4.13 below.)

Condition (p) of Theorem 4.8 is from [21], and is somewhat surprising. While it follows easily from some of the other characterizations, it seems counterintuitive in terms of the definition of image partition regularity that one can add *any* column to an image partition regular matrix, and it will remain image partition regular.

Note that by Theorem 4.8(n), any matrix which is IPR/$\mathbb{N}$ is automatically nonconstantly IPR/$\mathbb{N}$.

For Theorem 4.13 it will be convenient to assume that, if $l = \text{rank}(A)$, then the first l rows of A are linearly independent. Since rearranging the rows of A clearly does not affect its image partition regularity, there is no loss of generality in this assumption.

Definition 4.9. Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$ with $\text{rank}(A) = l < u$. Assume that the first l rows of A are linearly independent and denote the rows of A by $\vec{r_1}, \vec{r_2}, \ldots, \vec{r_u}$. For $i \in \{l+1, l+2, \ldots, u\}$ and $j \in \{1, 2, \ldots, l\}$, let $\gamma_{i,j} \in \mathbb{Q}$ be determined by $\vec{r_i} = \sum_{j=1}^{l} \gamma_{i,j}\vec{r_j}$. Then $D(A)$ is the $(u-l) \times u$ matrix defined by, for $i \in \{1, 2, \ldots, u-l\}$ and $j \in \{1, 2, \ldots, u\}$,

$$d_{i,j} = \begin{cases} \gamma_{l+i,j} & \text{if } j \leq l \\ -1 & \text{if } j = l+i \\ 0 & \text{otherwise.} \end{cases}$$

Several of the equivalences in the following theorem have not been recorded before. To prove these we shall need three lemmas. The first one is a minor modification of [17, Lemma 2.2].

Lemma 4.10. *Let* A *be a* $u \times v$ *matrix with entries from* $\mathbb{Z}$, *define* $\varphi : \mathbb{Z}^v \to \mathbb{Z}^u$ *by* $\varphi(\vec{x}) = A\vec{x}$, *and let* $\widetilde{\varphi} : \beta(\mathbb{Z}^v) \to (\beta\mathbb{Z})^u$ *be its continuous extension. Let* p *be a minimal idempotent in* $\beta\mathbb{N}$ *with the property that for every* $C \in p$ *there exists* $\vec{x} \in \mathbb{Z}^v$ *such that* $A\vec{x} \in C^u$ *and let* $\overline{p} = (p \quad p \quad \ldots \quad p)^T$. *Then there is a minimal idempotent* $q \in \beta(\mathbb{Z}^v)$ *such that* $\widetilde{\varphi}(q) = \overline{p}$.

Proof. By [22, Exercise 4.3.5 and Theorem 1.65] $p \in K(\beta\mathbb{Z})$ and so by [22, Theorem 2.23], $\overline{p} \in K\big((\beta\mathbb{Z})^u\big)$. By [22, Corollary 4.22], $\widetilde{\varphi} : \beta(\mathbb{Z}^v) \to (\beta\mathbb{Z})^u$ is a homomorphism.

We claim that $\overline{p} \in \widetilde{\varphi}[\beta(\mathbb{Z}^v)]$ so suppose instead that $\overline{p} \notin \widetilde{\varphi}[\beta(\mathbb{Z}^v)]$, which is closed, and pick a neighborhood U of $\overline{p}$ such that $U \cap \widetilde{\varphi}[\beta(\mathbb{Z}^v)] = \emptyset$. Pick $D \in p$ such that $\overline{D}^u \subseteq U$ and pick $\vec{x} \in \mathbb{Z}^v$ such that $A\vec{x} \in D^u$. Then $\varphi(\vec{x}) \in U \cap \widetilde{\varphi}[\beta(\mathbb{Z}^v)]$, a contradiction.

Let $M = \{q \in \beta(\mathbb{Z}^v) : \widetilde{\varphi}(q) = \overline{p}\}$. Then M is a compact subsemigroup of $\beta(\mathbb{Z}^v)$, so pick an idempotent $w \in M$ by [22, Theorem 2.5]. By [22, Theorem 1.60], pick a minimal idempotent $q \in \beta(\mathbb{Z}^v)$ with $q \leq w$. Since $\widetilde{\varphi}$ is a homomorphism, $\widetilde{\varphi}(q) \leq \widetilde{\varphi}(w) = \overline{p}$ so, since $\overline{p}$ is minimal in $(\beta\mathbb{Z})^u$, we have that $\widetilde{\varphi}(q) = \overline{p}$. □

Lemma 4.11. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ first entries matrix with entries from $\mathbb{Q}$, and let C be a central subset of $\mathbb{N}$. Then there exists $\vec{x} \in \mathbb{N}^v$ for which $A\vec{x} \in C^u$.*

Proof. [17, Lemma 2.8]. □

Lemma 4.12. *Let A be a central subset of $\mathbb{N}$ and let $\alpha \in \mathbb{Q}^+$. Then $\mathbb{N} \cap \alpha A$ is central in $\mathbb{N}$.*

Proof. This fact is presented as [1, Lemma 3.8]. That proof uses the dynamical characterization of central sets. An algebraic proof is in [17, Lemma 2.1]. □

Theorem 4.13. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Let $l = \text{rank}(A)$ and assume that the first l rows of A are linearly independent. The following statements are equivalent.*

(a) A is IPR/$\mathbb{Z}$.

(b) A is IPR/$\mathbb{Q}$.

(c) A is IPR/$\mathbb{R}$.

(d) Either $l = u$ or $D(A)$ is KPR/$\mathbb{N}$.

(e) There exist $t_1, t_2, \ldots, t_v \in \mathbb{Q} \setminus \{0\}$ such that the matrix

$$\begin{pmatrix} t_1a_{1,1} & t_2a_{1,2} & t_3a_{1,3} & \cdots & t_va_{1,v} & -1 & 0 & 0 & \cdots & 0 \\ t_1a_{2,1} & t_2a_{2,2} & t_3a_{2,3} & \cdots & t_va_{2,v} & 0 & -1 & 0 & \cdots & 0 \\ t_1a_{3,1} & t_2a_{3,2} & t_3a_{3,3} & \cdots & t_va_{3,v} & 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t_1a_{u,1} & t_2a_{u,2} & t_3a_{u,3} & \cdots & t_va_{u,v} & 0 & 0 & 0 & \cdots & -1 \end{pmatrix}$$

is KPR/$\mathbb{Z}$.

(f) For each $\vec{r} \in \mathbb{Q}^v \setminus \{\vec{0}\}$ there exists $b \in \mathbb{Q} \setminus \{0\}$ such that

$$\begin{pmatrix} b\vec{r} \\ A \end{pmatrix}$$

is IPR/$\mathbb{Z}$.

(g) *There exist* $b_1, b_2, \ldots, b_v \in \mathbb{Q} \setminus \{0\}$ *such that the matrix*

$$\begin{pmatrix} b_1 & 0 & 0 & \cdots & 0 \\ 0 & b_2 & 0 & \cdots & 0 \\ 0 & 0 & b_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_v \\ & & A & & \end{pmatrix}$$

is IPR/$\mathbb{Z}$.

(h) *There exist* $m \in \mathbb{N}$ *and a* $u \times m$ *first entries matrix* B *such that for each* $\vec{y} \in \mathbb{Z}^m$ *there exists* $\vec{x} \in \mathbb{Z}^v$ *such that* $A\vec{x} = B\vec{y}$.

(i) *For every central set* C *in* $\mathbb{N}$, *there exists* $\vec{x} \in \mathbb{Z}^v$ *such that* $A\vec{x} \in C^u$.

(j) *For every central set* C *in* $\mathbb{N}$, $\{\vec{x} \in \mathbb{Z}^v : A\vec{x} \in C^u\}$ *is central in* $\mathbb{Z}^v$.

(k) *Given any column* $\vec{c} \in \mathbb{Q}^u$, *the matrix* $(\,A \quad \vec{c}\,)$ *is IPR/*$\mathbb{Z}$.

(l) *There exist* $t_1, t_2, \ldots, t_v \in \mathbb{Q} \setminus \{0\}$ *such that the matrix*

$$\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ t_1a_{1,1} & t_2a_{1,2} & t_3a_{1,3} & \ldots & t_va_{1,v} \\ t_1a_{2,1} & t_2a_{2,2} & t_3a_{2,3} & \ldots & t_va_{2,v} \\ t_1a_{3,1} & t_2a_{3,2} & t_3a_{3,3} & \ldots & t_va_{3,v} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_1a_{u,1} & t_2a_{u,2} & t_3a_{u,3} & \ldots & t_va_{u,v} \end{pmatrix}$$

is IPR/$\mathbb{Z}$.

(m) *Whenever* $m \in \mathbb{N}$, $\phi_1, \phi_2, \ldots, \phi_m$ *are nonzero linear mappings from* $\mathbb{Q}^v$ *to* $\mathbb{Q}$, *there exists* $\vec{b} \in \mathbb{Q}^m$ *such that, whenever* C *is central in* $\mathbb{N}$, *there exists* $\vec{x} \in \mathbb{Z}^v$ *for which* $A\vec{x} \in C^u$ *and, for each* $i \in \{1, 2, \ldots, m\}$, $b_i\phi_i(\vec{x}) \in C$, *and in particular* $\phi_i(\vec{x}) \neq 0$.

(n) *For every central set* C *in* $\mathbb{N}$, *there exists* $\vec{x} \in \mathbb{Z}^v$ *such that* $\vec{y} = A\vec{x} \in C^u$, *all entries of* $\vec{x}$ *are distinct, and for all* $i, j \in \{1, 2, \ldots, u\}$, *if rows* i *and* j *of* A *are unequal, then* $y_i \neq y_j$.

(o) *Whenever* $m \in \mathbb{N}$, $\phi_1, \phi_2, \ldots, \phi_m$ *are nonzero linear mappings from* $\mathbb{Q}^v$ *to* $\mathbb{Q}$, *and* C *is a central subset of* $\mathbb{N}$, *there exist* $b_1, b_2, \ldots, b_m$ *in* $\mathbb{Q} \setminus \{0\}$ *such that* $\{\vec{x} \in \mathbb{Z}^v : A\vec{x} \in C^u$ and for each $i \in \{1, 2, \ldots, m\}$, $b_i\phi_i(\vec{x}) \in C\}$ *is central in* $\mathbb{Z}^v$.

(p) *Whenever* $m \in \mathbb{N}$ *and* C *is a central subset of* $\mathbb{N}$, $\{\vec{x} \in \mathbb{Z}^v : A\vec{x} \in C^u$, all entries of $\vec{x}$ are distinct, and entries of $A\vec{x}$ corresponding to distinct rows of A are distinct$\}$ *is central in* $\mathbb{Z}^v$.

Proof. That statements (a), (b), and (c) are equivalent follows from [25, Theorem 2.4]. (What we are calling IPR was called WIPR there.) That statements (a), (d), (e), (f), (g), and (h) are equivalent is [15, Theorem 2.2], so we have that statements (a) through (h) are equivalent.

To see that (h) implies (i), pick B as guaranteed by (h) and let C be a central subset of $\mathbb{N}$. Pick by Lemma 4.11 some $\vec{y} \in \mathbb{N}^m$ such that $B\vec{y} \in C^u$. Pick $\vec{x} \in \mathbb{Z}^v$ such that $A\vec{x} = B\vec{y}$.

To see that (i) implies (j), pick $d \in \mathbb{N}$ such that all entries of $B = dA$ are in $\mathbb{Z}$. Let C be a central subset of $\mathbb{N}$. By Lemma 4.12, dC is central in $\mathbb{N}$ so pick a minimal idempotent p in $\beta\mathbb{N}$ such that $dC \in p$. Define $\varphi : \mathbb{Z}^v \to \mathbb{Z}^u$ by $\varphi(\vec{x}) = B\vec{x}$ and let $\widetilde{\varphi} : \beta(\mathbb{Z}^v) \to (\beta\mathbb{Z})^u$ be its continuous extension. We claim that for each $D \in p$ there exists $\vec{x} \in \mathbb{Z}^v$ such that $B\vec{x} \in D^u$. Indeed, given $D \in p$, $d^{-1}D = \{z \in \mathbb{N} : dz \in D\}$ is central in $\mathbb{N}$ by Lemma 4.12, so by assumption, there is some $\vec{x} \in \mathbb{Z}^v$ such that $A\vec{x} \in (d^{-1}D)^u$ so $B\vec{x} \in D^u$. By Lemma 4.10, pick a minimal idempotent q of $\beta(\mathbb{Z}^v)$ such that $\widetilde{\varphi}(q) = \overline{p}$. Now $(\,\overline{dC}\,)^u$ is a neighborhood of $\overline{p}$ so pick $E \in q$ such that $\widetilde{\varphi}[\,\overline{E}\,] \subseteq (\,\overline{dC}\,)^u$. Then $E \subseteq \{\vec{x} \in \mathbb{Z}^v : B\vec{x} \in (dC)^u\} = \{\vec{x} \in \mathbb{Z}^v : A\vec{x} \in C^u\}$, so $\{\vec{x} \in \mathbb{Z}^v : A\vec{x} \in C^u\}$ is central in $\mathbb{Z}^v$.

Trivially (j) implies (a) so we have now established that statements (a) through (j) are equivalent.

Trivially (k) implies (a). To see that (h) implies (k), let $\vec{c} \in \mathbb{Q}^u$ be given. Pick a first entries matrix B as guaranteed by (h) and let $D = (\,B \quad \vec{c}\,)$. Then D is a first entries matrix. Given $\vec{w} \in \mathbb{Z}^{m+1}$, let $\vec{y}$ consist of the first m entries of $\vec{w}$ and pick $\vec{z} \in \mathbb{Z}^v$ such that $A\vec{z} = B\vec{y}$. Let

$$\vec{x} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_v \\ w_{m+1} \end{pmatrix} .$$

Then $(\,A \quad \vec{c}\,)\,\vec{x} = D\vec{y}$ so $(\,A \quad \vec{c}\,)$ satisfies statement (h) and is therefore IPR/$\mathbb{Z}$.

Now note that trivially (o) implies (m) and (m) implies (i). Also trivially (p) implies (n) and (n) implies (i). To complete the proof we shall show that (a) implies (o) and (o) implies (p).

So assume that A is IPR/$\mathbb{Z}$, let $m \in \mathbb{N}$, let $\phi_1, \phi_2, \ldots, \phi_m$ be nonzero linear mappings from $\mathbb{Q}^v$ to $\mathbb{Q}$, and let C be a central subset of $\mathbb{N}$. For each $i \in \{1, 2, \ldots, m\}$ pick a row $\vec{r_i} \in \mathbb{Q}^v \setminus \{\overline{0}\}$ such that $\phi_i(\vec{x}) = \vec{r_i} \cdot \vec{x}$ for all $\vec{x} \in \mathbb{Q}^v$. By applying statement (f) m times in succession (using the fact that at each stage the new matrix satisfies (f) because (a) implies (f)) choose $b_1, b_2, \ldots, b_m$ in $\mathbb{Q} \setminus \{0\}$ such that the matrix

$$\begin{pmatrix} b_1\vec{r_1} \\ b_2\vec{r_2} \\ \vdots \\ b_m\vec{r_m} \\ A \end{pmatrix}$$

is IPR/$\mathbb{Z}$. The conclusion then follows because this matrix satisfies statement (j).

Finally, assume that statment (o) holds. For $i \neq j$ in $\{1, 2, \ldots, v\}$, let $\overrightarrow{\phi_{i,j}}$ be the linear mapping from $\mathbb{Q}^v$ to $\mathbb{Q}$ taking $\vec{x}$ to $x_i - x_j$. For $i \neq j$ in $\{1, 2, \ldots, u\}$, if row i and row j of A are distinct, let $\overrightarrow{\psi_{i,j}}$ be the linear mapping from $\mathbb{Q}^v$ to $\mathbb{Q}$ taking $\vec{x}$ to $\sum_{t=1}^{v}(a_{i,t} - a_{j,t}) \cdot x_t$. Applying statement (o) to the set $\big\{\phi_{i,j} : i \neq j \text{ in } \{1, 2, \ldots, v\}\big\} \cup \big\{\psi_{i,j} : \text{row } i \text{ and row } j \text{ of } A$ are distinct $\big\}$, we reach the desired conclusion. □

Notice that IPR/$\mathbb{N}$ is a strictly stronger condition than IPR/$\mathbb{Z}$. To see this consider

$$A = \begin{pmatrix} 1 & -1 \\ 3 & 2 \\ 4 & 6 \end{pmatrix}.$$

Then $D(A) = \begin{pmatrix} -2 & 2 & -1 \end{pmatrix}$ so A is IPR/$\mathbb{Z}$ by Theorem 4.13(d). To see that A is not IPR/$\mathbb{N}$, notice that the only choice of t_1 and t_2 for which the matrix

$$\begin{pmatrix} t_1 & -t_1 & -1 & 0 & 0 \\ 3t_1 & 2t_1 & 0 & -1 & 0 \\ 4t_1 & 6t_1 & 0 & 0 & -1 \end{pmatrix}$$

satisfies the columns condition is $t_1 = \frac{3}{5}$ and $t_2 = -\frac{2}{5}$, so by Theorem 4.8(i) A is not IPR/$\mathbb{N}$. (And, by Theorem 4.13(e) one has a second verification that A is IPR/$\mathbb{Z}$.)

When defining kernel partition regularity of A over S for a subsemigroup S of $(\mathbb{R}, +)$, there is only one reasonable definition, namely the one given in Definition 1.1. Since the entries of $\vec{x}$ are to be monochrome, they must come from the set being colored. And if 0 were not excluded from the set being colored, one would allow the trivial solution $\vec{x} = \overline{0}$ and so all matrices would be KPR/S. (One might argue for the requirement that S be colored and the entries of $\vec{x}$ should be monochromatic and not constantly 0. But then, by assigning 0 to its own color, one sees that this is equivalent to the definition given.)

By contrast, when defining image partition regularity, there are several reasonable choices that can be made. Let T be the subgroup of $(\mathbb{R}, +)$ generated by S. If $0 \in S$, then one may color S or $S \setminus \{0\}$ and one may demand that one gets the entries of $A\vec{x}$ monochromatic with $\vec{x} \in (S \setminus \{0\})^v$, $\vec{x} \in S^v \setminus \{\vec{0}\}$, $\vec{x} \in (T \setminus \{0\})^v$, or $\vec{x} \in T^v \setminus \{\vec{0}\}$. If $0 \notin S$ one may demand that one gets the entries of $A\vec{x}$ monochromatic with $\vec{x} \in S^v$, $\vec{x} \in (T \setminus \{0\})^v$, or $\vec{x} \in T^v \setminus \{\vec{0}\}$. (We note that there is never a point in allowing $\vec{x} = \vec{0}$. If $S \setminus \{0\}$ is colored, then $\vec{x} = \vec{0}$ is impossible, and if $0 \in S$ and S is colored, then $\vec{x} = \vec{0}$ yields a trivial solution for any matrix.)

In [25] all of these possible versions are considered for the semigroups $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Q}^+$, and $\mathbb{R}^+$. For finite matrices these lead to a total of four distinct notions while for infinite matrices there are 15 distinct notions.

5. Image Partition Regularity of Affine Transformations

In this section we present characterizations of the image partition regularity of pairs $(A, \vec{b})$ where $\vec{b} \neq \overline{0}$. Most of the results of this section are from the dissertation of Irene Moshesh [30] and will be appearing in [21].

The situation with respect to image partition reguarity of the pair $(A, \vec{b})$ over $\mathbb{Z}$ and $\mathbb{Q}$ is essentially identical to that for kernel partition regularity. That is, the pair is image partition regular if and only if it is trivially so.

Theorem 5.1. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$. The pair $(A, \vec{b})$ is IPR/$\mathbb{Z}$ if and only if there exist $k \in \mathbb{Z}$ and $\vec{x} \in \mathbb{Z}^v$ such that $A\vec{x} + \vec{b} = \overline{k}$.*

Proof. [21, Theorem 3.3]. □

Theorem 5.2. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$. The pair $(A, \vec{b})$ is IPR/$\mathbb{Q}$ if and only if there exist $k \in \mathbb{Q}$ and $\vec{x} \in \mathbb{Q}^v$ such that $A\vec{x} + \vec{b} = \overline{k}$.*

Proof. [21, Theorem 3.2]. □

And, as in the case of kernel partition regularity, things get a little more interesting when image partition regularity over $\mathbb{N}$ is considered.

Theorem 5.3. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$. The following statements are equivalent.*

- *(a) The pair $(A, \vec{b})$ is IPR/$\mathbb{N}$.*
- *(b) Either*
  - *(i) there exist $k \in \mathbb{N}$ and $\vec{y} \in \mathbb{N}^v$ such that $A\vec{y} + \vec{b} = \overline{k}$ or*
  - *(ii) there exist $k \in \mathbb{Z}$ and $\vec{y} \in \mathbb{Z}^v$ such that $A\vec{y} + \vec{b} = \overline{k}$, A is IPR/$\mathbb{N}$, and A has at least two distinct rows.*

Proof. [21, Theorem 3.5]. □

We did not introduce separately the notion *weakly image partition regular over* $\mathbb{N}$ for a linear transformation (meaning that $\mathbb{N}$ is colored and one wants $\vec{x} \in \mathbb{Z}^v$) because it is equivalent to IPR/$\mathbb{Z}$. However, as we shall see, that equivalence no longer holds for affine transformations.

Definition 5.4. Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$. The pair (A, b) is *weakly image partition regular over* $\mathbb{N}$ (WIPR/$\mathbb{N}$) if and only if whenever $\mathbb{N}$ is finitely colored, there exists $\vec{x} \in \mathbb{Z}^v$ such that the entries of $A\vec{x} + \vec{b}$ are monochromatic.

Theorem 5.5. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$. The following statements are equivalent.*

(a) *The pair $(A, \vec{b})$ is WIPR/$\mathbb{N}$.*

(b) *Either*

(i) *there exist $k \in \mathbb{N}$ and $\vec{y} \in \mathbb{Z}^v$ such that $A\vec{y} + \vec{b} = \overline{k}$ or*

(ii) *there exist $k \in \mathbb{Z}$ and $\vec{y} \in \mathbb{Z}^v$ such that $A\vec{y} + \vec{b} = \overline{k}$ and A is IPR/$\mathbb{Z}$.*

Proof. [21, Theorem 3.4]. □

With our experience dealing with kernel partition regularity behind us, we realize that we should probably be asking also about nonconstant image partition regularity.

Definition 5.6. Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$.

(a) The pair (A, b) is *nonconstantly weakly image partition regular over* $\mathbb{N}$ (NCWIPR/$\mathbb{N}$) if and only if whenever $\mathbb{N}$ is finitely colored, there exists $\vec{x} \in \mathbb{Z}^v$ such that the entries of $A\vec{x} + \vec{b}$ are monochromatic and nonconstant.

(b) Let S be any of $\mathbb{N}$, $\mathbb{Z}$, or $\mathbb{Q}$. The pair (A, b) is *nonconstantly image partition regular over* S (IPR/S) if and only if whenever S is finitely colored, there exists $\vec{x} \in S^v$ such that the entries of $A\vec{x} + \vec{b}$ are monochromatic and nonconstant.

Theorem 5.7. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$. The following statements are equivalent.*

(a) *The pair $(A, \vec{b})$ is NCIPR/$\mathbb{N}$.*

(b) *There exist $k \in \mathbb{Z}$ and $\vec{y} \in \mathbb{Z}^v$ such that $A\vec{y} + \vec{b} = \overline{k}$, A is IPR/$\mathbb{N}$, and A has at least two distinct rows.*

Proof. [21, Theorem 4.4]. □

Theorem 5.7. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$. The following statements are equivalent.*

(a) *The pair $(A, \vec{b})$ is NCWIPR/$\mathbb{N}$.*

(b) *The pair $(A, \vec{b})$ is NCIPR/$\mathbb{Z}$.*

(c) *There exist $k \in \mathbb{Z}$ and $\vec{y} \in \mathbb{Z}^v$ such that $A\vec{y} + \vec{b} = \overline{k}$, A is IPR/$\mathbb{Z}$, and A has at least two distinct rows.*

Proof. [21, Theorem 3.6]. □

Theorem 5.8. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$. The following statements are equivalent.*

(a) *The pair* $(A, \vec{b})$ *is NCIPR/*$\mathbb{Q}$.

(b) *Either*

(i) $\text{rank}(A) = u > 1$ *or*

(i) $\text{rank}(A) < u$ *and the pair* $(D, -D\vec{b})$ *is nonconstantly KPR/*$\mathbb{Q}$*, where* $D = D(A)$.

Proof. [21, Theorem 3.5]. □

Theorem 5.9. *Let* $u, v \in \mathbb{N}$*, let* A *be a* $u \times v$ *matrix with entries from* $\mathbb{Q}$*, and let* $\vec{b} \in \mathbb{Q}^u \setminus \{\vec{0}\}$*. All of the implications in the following diagram hold and the only implications that hold among these notions are those shown or ones that follow by transitivity.*

$$\begin{array}{ccccccc}
\text{NCIPR}/\mathbb{N} & \Rightarrow & \text{NCWIPR}/\mathbb{N} & \Leftrightarrow & \text{NCIPR}/\mathbb{Z} & \Rightarrow & \text{NCIPR}/\mathbb{Q} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\text{IPR}/\mathbb{N} & \Rightarrow & \text{WIPR}/\mathbb{N} & \Rightarrow & \text{IPR}/\mathbb{Z} & \Rightarrow & \text{IPR}/\mathbb{Q}
\end{array}$$

Proof. All of the implications shown follow from the results of this section. We shall list examples establishing that none of the missing implications hold, leaving the verification to the reader.

The pair $\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix}\right)$ is IPR/$\mathbb{N}$ but not NCIPR/$\mathbb{Q}$.

The pair $\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}\right)$ is WIPR/$\mathbb{N}$ but not IPR/$\mathbb{N}$.

The pair $\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}\right)$ is IPR/$\mathbb{Z}$ but not WIPR/$\mathbb{N}$.

The pair $\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \\ -5/2 \end{pmatrix}\right)$ is IPR/$\mathbb{Q}$ but not IPR/$\mathbb{Z}$.

The pair $\left(\begin{pmatrix} 1 & -1 \\ 3 & 2 \\ 4 & 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}\right)$ is NCIPR/$\mathbb{Z}$ but not IPR/$\mathbb{N}$.

The pair $\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}\right)$ is NCIPR/$\mathbb{Q}$ but not IPR/$\mathbb{Z}$. □

6. Partition Regularity of Infinite Matrices

In contrast with the situation with finite matrices, there are only piecemeal results for infinite matrices. I only know of one result involving kernel partition regularity of infinite matrices. By contrast, there are many sufficient conditions know for image partition regularity (over $\mathbb{N}$) of infinite matrices, but nothing close to a characterization. In this section I will try to discuss some of the major themes involving image partition regularity of infinite matrices as well as some of the contrasts with the finite situation.

The infinite matrices with which we will be concerned have countably many rows, all entries are rational, and each row has finitely many nonzero entries. We define IPR/$\mathbb{N}$ and IPR/$\mathbb{Z}$ as in Definition 1.3 allowing u and v to be ω, which is the cardinal of countable infinity. Probably the simplest example of an infinite matrix which is IPR/$\mathbb{N}$ is given by the infinite Finite Sums Theorem ([14] or see [22, Corollary 5.10]). This is a matrix consisting of all rows with entries of 0's and 1's with finitely many 1's in each row. A more general version is given by the Milliken-Taylor Theorem, which we will present as Theorem 6.2.

Definition 6.1. Let $\vec{a} = \langle a_i \rangle_{i=1}^n$ be a finite sequence in $\mathbb{Z} \setminus \{0\}$ with no adjacent repeated terms and let A be an $\omega \times \omega$ matrix. Then A is a *$MT(\vec{a})$-matrix* if and only if whenever $\langle F_i \rangle_{i=1}^n$ is a sequence in $\mathcal{P}_f(\mathbb{N})$ with $\max F_i < \min F_{i+1}$ for all $i \in \{1, 2, \ldots, n-1\}$ (if any), there is a row $\vec{r}$ of A with $r_j = a_i$ if $j \in F_i$ and $r_j = 0$ if $j \notin \bigcup_{i=1}^n F_i$. Further all rows of A are of this form.

Theorem 6.2 (Milliken and Taylor). *Let $\vec{a}$ be a sequence in $\mathbb{Z}\setminus\{0\}$ with no adjacent repeated terms and final term positive and let A be a $MT(\vec{a})$-matrix. Then A is IPR/$\mathbb{N}$.*

Proof. This is an immediate consequence of [29, Theorem 2.2] or [41, Lemma 2.2]. Or see [22, Theorem 18.8]. For the details of the derivation see [5, Theorem 2.5]. □

The restriction that the final term of $\vec{a}$ be positive is needed to guarantee that for sufficiently fast growing $\vec{x} \in \mathbb{N}^\omega$ the entries of $A\vec{x}$ are in $\mathbb{N}$.

We know from Theorem 4.8 that any finite matrix which is IPR/$\mathbb{N}$ has the property that images of that matrix can be found inside any central subset of $\mathbb{N}$. As a consequence, we know that if A and B are finite matrices which are IPR/$\mathbb{N}$, so is $\begin{pmatrix} A & \mathbf{O} \\ \mathbf{O} & B \end{pmatrix}$.

We see however that this is far from true for infinite matrices. (If A and B are $\omega\times\omega$ matrices, then $\begin{pmatrix} A & \mathbf{O} \\ \mathbf{O} & B \end{pmatrix}$ is an $(\omega + \omega) \times (\omega + \omega)$ matrix. We shall blissfully ignore this distinction.)

The following result is a result of collaboration with Deuber, Leader, Lefmann, and Strauss.

Theorem 6.3. *Let $\vec{a}$ and $\vec{b}$ be sequences in $\mathbb{Z}\setminus\{0\}$ with no adjacent repeated terms and final term positive, let A be a $MT(\vec{a})$-matrix, and let B be a $MT(\vec{b})$-matrix. The matrix $\begin{pmatrix} A & \mathbf{O} \\ \mathbf{O} & B \end{pmatrix}$ is IPR/$\mathbb{N}$ if and only if there is a positive rational α such that $\vec{b} = \alpha\cdot\vec{a}$.*

Proof. This was established for entries of $\vec{a}$ and $\vec{b}$ positive in [5, Theorems 3.2 and 3.3]. The general case follows from [19, Theorem 3.1]. □

One does however have the following result, whose proof was supplied by V. Rödl.

Theorem 6.4 (Rödl). *Let A be a finite matrix which is IPR/$\mathbb{N}$ and let B be an infinite martix which is IPR/$\mathbb{N}$. Then the matrix $\begin{pmatrix} A & \mathbf{O} \\ \mathbf{O} & B \end{pmatrix}$ is IPR/$\mathbb{N}$.*

Proof. [18, Lemma 2.3].

In [18] we attempted to restore some of the order of Theorem 4.8 to infinite image partition regularity by defining a stronger notion.

Definition 6.5. Let A be an infinite matrix with entries from $\mathbb{Q}$ and finitely many nonzero entries in each row. Then A is *centrally image partition regular over* $\mathbb{N}$ if and only if whenever C is a central subset of $\mathbb{N}$, there exists $\vec{x}\in\mathbb{N}^\omega$ such that all entries of $A\vec{x}$ are in C.

One immediately obtains the fact that if A and B are centrally image partition regular, then $\begin{pmatrix} A & \mathbf{O} \\ \mathbf{O} & B \end{pmatrix}$ is centrally image partition regular.

Unfortunately, the condition of Theorem 4.8(o) need not hold. So we introduced an even stronger requirement.

Definition 6.6. Let A be an infinite matrix with entries from $\mathbb{Q}$ and finitely many nonzero entries in each row. Then A is *strongly centrally image partition regular* if and only if whenever C is a central set in $\mathbb{N}$, there exists $\vec{x}\in\mathbb{N}^\omega$ such that all entries of $\vec{y} = A\vec{x}$ are in C and for all $i, j\in\omega$, if rows i and j of A are unequal, then $y_i \neq y_j$.

Given a row of a matrix with finitely many nonzero entries, we shall refer to the sequence obtained by deleting all 0's and then deleting adjacent repetitions as the *compressed form* or the row. For example, the compressed form of $(0\ \ -2\ \ 0\ \ -2\ \ 3\ \ 0\ \ 0\ \ 3\ \ 1\ \ 0\ \ 0\ \ \ldots)$ is $\langle -2, 3, 1\rangle$.

Theorem 6.7. *Let $k\in\mathbb{N}$, let $\langle a_1, a_2, \ldots, a_k\rangle$ be a compressed sequence in $\mathbb{Z}\setminus\{0\}$ with $a_k > 0$, and let $m\in\mathbb{Z}\setminus\{0\}$. Let M be a matrix, with finitely many nonzero entries in each row, such that* *(i) the compressed form of each row is* $\langle a_1, a_2, \ldots, a_k\rangle$ *and*

(ii) the sum of each row is m.

Then M *is strongly centrally image partition regular.*

Proof. [18, Theorem 3.7].

As an example, if

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 1 & 0 & 0 & \ldots \\ 0 & 1 & 1 & 0 & 0 & \ldots \\ 1 & 0 & 0 & 1 & 0 & \ldots \\ 0 & 1 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 1 & 1 & 0 & \ldots \\ 1 & 0 & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 2 & 0 & 0 & \ldots \\ 0 & 1 & 2 & 0 & 0 & \ldots \\ 1 & 0 & 0 & 2 & 0 & \ldots \\ 0 & 1 & 0 & 2 & 0 & \ldots \\ 0 & 0 & 1 & 2 & 0 & \ldots \\ 1 & 0 & 0 & 0 & 2 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

then A and B are strongly centrally image partition regular and thus $\begin{pmatrix} A & \mathbf{O} \\ \mathbf{O} & B \end{pmatrix}$ is IPR/$\mathbb{N}$. (In fact it is strongly centrally image partition regular.)

Of course, dealing as it does with central sets, the proof of Theorem 6.7 relies heavily on the algebraic structure of $\beta\mathbb{N}$. Also, Theorem 6.7 does not apply if the row sums are 0.

Leader and Russell [26] provide an elementary proof of the following theorem, which includes the above example as a special case. Notice that one is not assuming that the specified sequences are compressed.

Theorem 6.8 (Leader and Russell). *Let* $\langle a_1, a_2, \ldots, a_k\rangle$ *and* $\langle b_1, b_2, \ldots, b_n\rangle$ *be sequences in* $\mathbb{Z} \setminus \{0\}$ *with* $a_k > 0$, $b_n > 0$, *and* $\sum_{i=1}^k b_i \neq 0$. *Let* A *be a matrix with the property that the nonzero entries of each of its rows are* $a_1, a_2, \ldots, a_k$ *in order and let* B *be a matrix with the property that the nonzero entries of each of its rows are* $b_1, b_2, \ldots, b_n$ *in order. Then* $\begin{pmatrix} A & \mathbf{O} \\ \mathbf{O} & B \end{pmatrix}$ *is IPR/*$\mathbb{N}$.

Proof. [26, Theorem 2]. □

As a consequence of Theorem 6.8, if

$$A = \begin{pmatrix} 1 & -1 & -1 & 1 & 0 & 0 & \ldots \\ 1 & -1 & -1 & 0 & 1 & 0 & \ldots \\ 1 & -1 & 0 & -1 & 1 & 0 & \ldots \\ 1 & 0 & -1 & -1 & 1 & 0 & \ldots \\ 0 & 1 & -1 & -1 & 1 & 0 & \ldots \\ 1 & -1 & -1 & 0 & 0 & 1 & \ldots \\ 1 & -1 & 0 & -1 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, B = \begin{pmatrix} -1 & 1 & -1 & 1 & 0 & 0 & \ldots \\ -1 & 1 & -1 & 0 & 1 & 0 & \ldots \\ -1 & 1 & 0 & -1 & 1 & 0 & \ldots \\ -1 & 0 & 1 & -1 & 1 & 0 & \ldots \\ 0 & -1 & 1 & -1 & 1 & 0 & \ldots \\ -1 & 1 & -1 & 0 & 0 & 1 & \ldots \\ -1 & 1 & 0 & -1 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and

$$C = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & \dots \\ 1 & 0 & 2 & 0 & 0 & \dots \\ 0 & 1 & 2 & 0 & 0 & \dots \\ 1 & 0 & 0 & 2 & 0 & \dots \\ 0 & 1 & 0 & 2 & 0 & \dots \\ 0 & 0 & 1 & 2 & 0 & \dots \\ 1 & 0 & 0 & 0 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

then $\begin{pmatrix} A & \mathbf{O} \\ \mathbf{O} & C \end{pmatrix}$ is IPR/$\mathbb{N}$ and $\begin{pmatrix} B & \mathbf{O} \\ \mathbf{O} & C \end{pmatrix}$ is IPR/$\mathbb{N}$. But it is a consequence of [18, Theorems 3.14 and 3.17] that $\begin{pmatrix} A & \mathbf{O} \\ \mathbf{O} & B \end{pmatrix}$ is not IPR/$\mathbb{N}$.

An early result in image partition regularity came in collaboration with Walter Deuber. He wanted to know whether, in his terminology, given a finite coloring of $\mathbb{N}$, one could choose for each triple (m, p, c), an (m, p, c)-set which was monochromatic and had the property that all finite sums taken by choosing at most one element from each (m, p, c) set were also monochromatic. The answer was yes, and can be easily represented in terms of the image partition regularity of a particular infinite matrix.

Notice that there are countably many first entries matrices and enumerate them as $\langle A_n \rangle_{n=1}^{\infty}$. Assume that for each n, A_n is a $u_n \times v_n$ matrix, let $\alpha_0 = 0$ and for $n \in \mathbb{N}$ let $\alpha_n = \sum_{t=1}^{n} v_n$. Let M be a matrix with the property that each row of M has only finitely many nonzero entries and for each $n \in \mathbb{N}$, the entries of that row from $\alpha_{n-1} + 1$ through α_n are either all 0 or are the entries of a row of A_n. Assume further that every row of this kind appears in M.

Theorem 6.9. *Let M be a matrix described above. Then M is IPR/$\mathbb{N}$.*

Proof. [4]. □

Then Hanno Lefmann and I established the following stronger result.

Theorem 6.10. *Let M be a matrix as described before Theorem 6.9. Let $\vec{x} \in \mathbb{N}^{\omega}$ have the property that all entries of $M\vec{x}$ are in $\mathbb{N}$ and let C be the set of entries of $M\vec{x}$. If C is finitely colored, then there exists $\vec{y} \in \mathbb{N}^{\omega}$ such that the entries of $M\vec{y}$ are contained in C and are monochromatic.*

Proof. [20, Theorem 2.7]. □

Several results dealing with image partition regularity along the lines of Theorem 6.10 are in [24]. We do not have space here to describe them in detail.

7. Restricted and Sparse Results

In this section we address an issue related to partition regularity, which deals with sets which are partition regular with respect to certain kernel or image partition regular matrices, but not with respect to others. While this may be viewed as an entirely peripheral subject, one view of its origins is the conjecture of Rado regarding "large" sets, which led through Deuber's Theorem to the subject of image partition regular matrices. The subject of restricted and sparce Ramsey theoretic results is one on which I have little expertise, and I am grateful to Imre Leader for providing me assistance with this section.

To introduce the notions of *restricted* and *sparse* we will use a subject which has nothing at all to do with partition regular matrices, namely the simplest nontrivial version of Ramsey's Theorem. This says that whenever the edges of a complete graph on 6 vertices (a K_6) are colored with two colors, there must be a monochromatic triangle. Erdős and Hajnal [6] asked for a graph which contains no K_6 but has the property that whenever its edges are 2-colored there must be a monochromatic triangle. It is easy to see that if a 9-cycle is deleted from a K_{10} the resulting graph has this property. (I do not know who first observed this fact.) A minimal example was provided by Ron Graham.

Theorem 7.1 (Graham). *If a graph on 7 vertices contains no K_6, then there is a 2-coloring of the edges with no monochromatic triangle. If a 5-cycle is deleted from a K_8, the resulting graph contains no K_6 and has the property that whenever its edges are 2-colored there is a monochromatic triangle.*

Proof. [10]. □

In the same problem, Erdős and Hajnal conjectured that for each $r \in \mathbb{N}$ there exists some graph which contains no K_4 and has the property that whenever its edges are r-colored there must exist some monochromatic triangle. The case $r = 2$ was proved by Folkman and the general case by Nešetřil and Rödl.

Theorem 7.2 (Folkman). *Let $n_1, n_2 \in \mathbb{N}$ and let $\Gamma(n_1, n_2)$ be the set of finite graphs with the property that whenever its edges are partitioned into classes C_1 and C_2, either C_1 contains the edges of a K_{n_1} or C_2 contains the edges of a K_{n_2}. Let $r = \max\{n_1, n_2\}$. There is a graph in $\Gamma(n_1, n_2)$ which contains no K_{r+1}.*

Proof. [7]. □

Theorem 7.3 (Nešetřil and Rödl). *Let $m, r \in \mathbb{N}$ and let G be a finite graph which contains no K_m. Then there is a finite graph which contains no K_m and has the property that whenever its edges are r-colored, there must exist a monochromatic copy of G.*

Proof. [31]. □

Theorems 7.1, 7.2, and 7.3 are examples of *restricted* Ramsey-type theorems. To talk about *sparse* Ramsey-type theorems, we need to say what we mean by a cycle of triangles.

Definition 7.4. Let G be a graph and let $n \in \mathbb{N}$. A *cycle of triangles of length* n *in* G is a sequence $T_1, e_1, T_2, e_2, \ldots, T_n, e_n, T_{n+1}$ where each T_i is a triangle, each e_i is an edge of T_i and of T_{i+1}, $e_i \neq e_j$ and $T_i \neq T_j$ for $i \neq j$ in $\{1, 2, \ldots, n\}$, and $T_{n+1} = T_1$.

A graph is then *sparse* (with respect to triangles) provided there are no short cycles of triangles.

Lemma 7.5. *Let* $G = (V, E)$ *be a graph with the property that whenever its edges are two colored there must exist a monochromatic triangle. Then there must be some cycle of triangles in* G *of odd length.*

Proof. Let $\langle S_\alpha \rangle_{\alpha \in I}$ enumerate the triangles of G, viewed as edge sets. (So each S_α is a set of three elements of E.) For each $\alpha \in I$ pick a $T_\alpha \subseteq S_\alpha$ with $|T_\alpha| = 2$. Let $E' = \{T_\alpha : \alpha \in I\}$, let $V' = \bigcup E'$, and let $G' = (V', E')$. Notice that $V' \subseteq E$. We claim that G' has chromatic number at least 3. For, given a two coloring of V', extend it arbitrarily to a two coloring of E and pick $\alpha \in I$ such that S_α is monochromatic. Then T_α is monochromatic with respect to the original coloring of V'. But then, as is well known, G' must have an odd cycle and this induces a cycle of triangles in G of odd length. □

Theorem 7.6 (Nešetřil and Rödl). *Let* $n, r \in \mathbb{N}$. *There is a graph which contains no cycle of triangles of length less than or equal to* n *and has the property that whenever its edges are* r*-colored, there must be a monochromatic triangle.*

Proof. This is a corollary of a much more general result [32, Theorem 1.4]. □

Now we turn our attention to restricted and sparse theorems related to image partition regularity of matrices, dealing with the restricted theorems first.

Consider first van der Waerden's theorem. As we have seen, this is precisely the assertion that for each $n \in \mathbb{N}$, the matrix

$$V_n = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & n \end{pmatrix}$$

is image partition regular.

Theorem 7.7 (Spencer). *For each $k, n \in \mathbb{N}$, there is a finite subset $S_{k,n}$ of $\mathbb{N}$ with the property that*

(a) whenever $S_{k,n}$ is k-colored, there is a monochromatic image of V_n and

(b) $S_{k,n}$ contains no image of V_{n+1}.

Proof. [40]. □

Notice that Theorem 7.7 easily implies a uniform version of itself.

Corollary 7.8. *For each $n \in \mathbb{N} \setminus \{1\}$ there is a subset T_n of $\mathbb{N}$ such that*

(a) whenever T_n is finitely colored, there is a monochromatic image of V_n and

(b) T_n contains no image of V_{n+1}.

Proof. Given $S_{k,n}$ as guaranteed by Theorem 7.7, $a + S_{k,n}$ has the same properties for all $a \in \mathbb{N}$ so simply choose a sequence $\langle a_k \rangle_{k=1}^{\infty}$ such that any length three progression in $\bigcup_{k=1}^{\infty}(a_k + S_{k,n})$ is contained in some $a_k + S_{k,n}$ and let $T_n = \bigcup_{k=1}^{\infty}(a_k + S_{k,n})$. □

Now consider the (finite) Finite Sums Theorem (Corollary 2.4). For each $n \in \mathbb{N}$, let FS_n be the $(2^n - 1) \times n$ matrix with rows indexed by $\mathcal{P}_f(\{1, 2, \ldots, n\})$ and columns indexed by $\{1, 2, \ldots, n\}$ with an entry in row F and column k equal to 1 if $k \in F$ and 0 otherwise. Then the Finite Sums Theorem is the assertion that each FS_n is image partition regular.

Theorem 7.9 (Nešetřil and Rödl). *Let $k, n \in \mathbb{N}$. There is a finite subset $S_{k,n}$ of $\mathbb{N}$ such that*

(a) whenever $S_{k,n}$ is k-colored, there is a monochromatic image of FS_n and

(b) $S_{k,n}$ contains no image of FS_{n+1}.

Proof. This is a consequence of [33] where the corresponding version of the Finite Unions Theorem is established. □

Again, this result implies a uniform version of itself using the fact that if $S_{k,n}$ is as guaranteed by Theorem 7.9, then $aS_{k,n}$ has the same properties for each $a \in \mathbb{N}$.

We now present some restricted Ramsey-type theorems about quite general image partition regular matrices and kernel partition regular matrices. These were obtained in collaboration with Vitaly Bergelson and Imre Leader.

Definition 7.10. Let $n \in \mathbb{N}$. A finite matrix A is an *n-sparse monic first entries matrix* if and only if it is a first entries matrix all of whose first entries are equal to 1 such that for each column j, the number of rows with first entry in column j and more than one nonzero entry is at most n.

For $n \in \mathbb{N}$ let

$$V_n^+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & n \end{pmatrix}.$$

The fact that V_n^+ is image partition regular for each n yields the strengthened version of van der Waerden's Theorem which has the increment the same color as the terms of the arithmetic progression. Notice that V_n^+ is an n-sparse monic first entries matrix.

Theorem 7.11. *Let $n \in \mathbb{N}$. There is a set $E \subseteq \mathbb{N}$ such that*

(a) whenever E is finitely colored and A is an n-sparse monic first entries matrix, E contains a monochromatic image of A and

(b) E contains no image of V_{n+1}^+.

Proof. [2, Theorem 3.6]. □

Corollary 7.12. *Let $(m, p, c) \in \mathbb{N}^3$ with $c \leq p$ and let $n = (2p + 1)^{m-1} - 1$. There is a set $E \subseteq \mathbb{N}$ such that*

(a) whenever E is finitely colored E contains a monochromatic (m, p, c)-set and

(b) E contains no image of V_{n+1}^+.

Proof. Pick E as guaranteed by Theorem 7.11 for n. Then cE has the same properties. Let A be a matrix such that any image of A is an (m, p, c)-set and let $B = \frac{1}{c}A$. Then B is an n-sparse monic first entries matrix. Let cE be finitely colored and pick $\vec{x} \in \mathbb{N}^m$ such that the entries of $B\vec{x}$ are monochromatic. Notice that each entry of $\vec{x}$ is an entry of $B\vec{x}$ and is thus a multiple of c. Let $\vec{y} = \frac{1}{c}\vec{x}$. Then $A\vec{y}$ is a monochromatic (m, p, c)-set. □

The following theorem, also obtained in collaboration with Bergelson and Leader, establishes not only do the specified matrices have monochromatic solutions, but all can be found in one given color. Given $n \in \mathbb{N}$, the matrix

$$W_n = \begin{pmatrix} 1 & 1 & -1 & 0 & \ldots & 0 \\ 2 & 1 & 0 & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n & 1 & 0 & 0 & \ldots & -1 \end{pmatrix}$$

is a kernel partition regular matrix corresponding to the image partition regular matrix V_n^+. That is any solution to $W_n\vec{x} = \overline{0}$ is a length $n+1$ arithmetic progression together with its increment.

Theorem 7.13. *Let $n \in \mathbb{N}$. There is a set $E \subseteq \mathbb{N}$ such that*

(a) whenever E is finitely colored there is one color which contains a solution to $A\vec{x} = \overline{0}$ for every kernel partition regular matrix A with at most n rows and

(b) E contains no solution to $W_{n+1}\vec{x} = \overline{0}$.

Proof. [2, Theorem 4.2]. □

The proof of [2, Theorem 4.2] in fact establishes that a similar strengthening of Theorem 7.11 holds.

We now turn our attention to sparse Ramsey-type theorems involving partition regular matrices, first dealing with a sparse version of van der Waerden's theorem. By a *cycle of length k of arithmetic progressions of length n* we mean a sequence $T_1, x_1, T_2, \ldots, T_k, x_k, T_{k+1}$ where each T_i is a length n arithmetic progression, each $x_i \in T_i \cap T_{i+1}$, $x_i \neq x_j$ and $T_i \neq T_j$ for $i \neq j$ in $\{1, 2, \ldots, n\}$, and $T_{n+1} = T_1$.

Theorem 7.14 (Rödl). *Let $n, k \in \mathbb{N}$. There is a subset S of $\mathbb{N}$ such that*

(a) whenever S is finitely colored there is a monochromatic arithmetic progression of length n and

(b) S does not contain any cycle of length less than or equal to k of length n arithmetic progressions.

Proof. [38]. □

A different proof of Theorem 7.14 was obtained by Prömel and Voigt [35] using ideas of Frankl, Graham, and Rödl [8]. (The date on [38] is misleading. It circulated as a manuscript for several years before it was published.)

In [34] Nešetřil and Rödl generalized Theorems 7.6 and 7.14 (as well as several other theorems that I have not mentioned) "beyond belief". (The quoted words are from Imre Leader.)

Very recently, Leader and Russell established a sparse version of Deuber's Theorem. A cycle of (m,p,c)-sets is defined like a cycle of arithmetic progressions.

Theorem 7.15 (Leader and Russell). *Let $m,p,c,k,g \in \mathbb{N}$ with $c \le p$. There is a set $S \subseteq \mathbb{N}$ such that*

(a) whenever S is k-colored there is a monochromatic (m,p,c)-set and

(b) there is no cycle of (m,p,c)-sets of length less than or equal to g,

Proof. [27, Theorem 16]. □

We conclude with a somewhat related result. Given a matrix A and a subset C of $\mathbb{N}$, we say that C is kernel partition regular for A provided that whenever C is finitely colored there must be a monochromatic solution to $A\vec{x} = \overline{0}$. Let A and B be kernel partition regular matrices. Say that A *Rado-dominates* B if every set which is kernel partition regular for A is kernel partition regular for B. Say that A *solution-dominates* B if every solution to $A\vec{x} = \overline{0}$ contains a solution to $B\vec{y} = \overline{0}$. In [2] we conjectured that the only way A could Rado-dominate B is for A to solution-dominate B. In a very recent result, Leader and Russell proved the simplest non-trivial instance of this conjecture, namely for 1×3 matrices, and then showed that it is indeed false for a 1×3 matrix and a 1×4 matrix. I will include the proof of Theorem 7.16(b) because it is so short and pretty.

Theorem 7.16 (Leader and Russell).

(a) If A and B are 1×3 matrices and A Rado-dominates B, then A solution dominates B.

(b) Any set which is kernel partition regular for $\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$ is kernel partition regular for $\begin{pmatrix} 1 & 1 & 1 & -1 \end{pmatrix}$.

Proof. (a) [28, Theorem 2].

(b) This is [28, Theorem 12]. Let C be kernel partition regular for $\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$ and let $r \in \mathbb{N}$ and let $\varphi : C \to \{1, 2, \ldots, r\}$. Define $\psi : C \to \{1, 2, \ldots, 2r\}$ by, for $z \in C$,

$$\psi(z) = \begin{cases} \varphi(z) & \text{if there exist } x, y \in C \text{ such that } \varphi(x) = \varphi(y) = \varphi(z) \text{ and } x + y = z \\ r + \varphi(z) & \text{otherwise.} \end{cases}$$

Pick $x, y, z \in C$ such that $\varphi(x) = \varphi(y) = \varphi(z)$ and $x + y = z$. Then $\varphi(x) = \varphi(z) \le r$ so pick $u, v \in C$ such that $\varphi(u) = \varphi(v) = \varphi(x)$ and $u + v = x$. Then $u + v + y = z$. □

References

[1] V. Bergelson, N. Hindman, and B. Kra, *Iterated spectra of numbers — elementary, dynamical, and algebraic approaches*, Trans. Amer. Math. Soc. **348** (1996), 893-912.

[2] V. Bergelson, N. Hindman, and I. Leader, *Sets partition regular for n equations need not solve $n+1$*, Proc. London Math. Soc. **73** (1996), 481-500.

[3] W. Deuber, *Partitionen und lineare Gleichungssysteme*, Math. Zeit. **133** (1973), 109-123.

[4] W. Deuber and N. Hindman, *Partitions and sums of (m,p,c)-sets*, J. Comb. Theory (Series A) **45** (1987), 300-302.

[5] W. Deuber, N. Hindman, I. Leader, and H. Lefmann, *Infinite partition regular matrices*, Combinatorica **15** (1995), 333-355.

[6] P. Erdős and A. Hajnal, *Research Problem* 2-5, J. Comb. Theory **2** (1967), 105.

[7] J. Folkman, *Graphs with monochromatic complete subgraphs in every edge coloring*, SIAM J. Appl. Math. **18** (1970), 19-24.

[8] P. Frankl, R. Graham, and V. Rödl, *Induced restricted Ramsey theorems for spaces*, J. Comb. Theory (Series A) **44** (1987),

[9] H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, Princeton University Press, Princeton, 1981.

[10] R. Graham, *On edgewise 2-colorable graphs with monochromatic triangles and containing no complete hexagon*, J. Comb. Theory **4** (1968), 300.

[11] R. Graham, *Rudiments of Ramsey theory*, American Mathematical Society, Providence, 1981.

[12] R. Graham, B. Rothschild, and J. Spencer, *Ramsey Theory*, Wiley, New York, 1990.

[13] D. Hilbert, *Über die Irreduzibilität ganzer rationaler Funktionen mit ganzzahlingen Koeffizienten*, J. Reine Angew. Math. **110** (1892), 104-129.

[14] N. Hindman, *Finite sums from sequences within cells of a partition of* $\mathbb{N}$, J. Comb. Theory (Series A) **17** (1974), 1-11.

[15] N. Hindman and I. Leader, *Image partition regularity of matrices*, Comb. Prob. and Comp. **2** (1993), 437-463.

[16] N. Hindman and I. Leader, *Nonconstant monochromatic solutions to systems of linear equations*, in Topics in Discrete Mathematics, Springer, Berlin, to appear*.

[17] N. Hindman, I. Leader, and D. Strauss, *Image partition regular matrices – bounded solutions and preservation of largeness*, Discrete Math. **242** (2002), 115-144.

[18] N. Hindman, I. Leader, and D. Strauss, *Infinite partition regular matrices – solutions in central sets*, Trans. Amer. Math. Soc. **355** (2003), 1213-1235.

[19] N. Hindman, I. Leader, and D. Strauss, *Separating Milliken-Taylor systems with negative entries*, Proc. Edinburgh Math. Soc. **46** (2003), 45-61.

[20] N. Hindman and H. Lefmann, *Partition regularity of $(\mathcal{M},\mathcal{P},\mathcal{C})$-systems*, J. Comb. Theory (Series A) **64** (1993), 1-9.

[21] N. Hindman and I. Moshesh, *Image partition regularity of affine transformations*, manuscript*.

* (Currently (2006) available at `http://members.aol.com/nhindman/`.)

[22] N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification: theory and applications*, de Gruyter, Berlin, 1998.

[23] N. Hindman and D. Strauss, *An algebraic proof of Deuber's Theorem*, Comb. Prob. and Comp. **7** (1998), 167-180.

[24] N. Hindman and D. Strauss, *Infinite partition regular matrices, II — extending the finite results*, Topology Proceedings **25** (2000), 217-255.

[25] N. Hindman and D. Strauss, *Image partition regularity over the integers, rationals and reals*, New York J. Math. **11** (2005), 519-538.

[26] I. Leader and P. Russell, *Consistency for partition regular equations*, Discrete Math., to appear**.

[27] I. Leader and P. Russell, *Sparse partition regularity*, Proc. London Math. Soc., to appear**.

[28] I. Leader and P. Russell, *Independence for partition regular equations*, manuscript**.

[29] K. Milliken, *Ramsey's Theorem with sums or unions*, J. Comb. Theory (Series A) **18** (1975), 276-290.

[30] I. Moshesh, *Image partition regularity of affine transformations*, Dissertation, Howard University, 2006.

[31] J. Nešetřil and V. Rödl, *The Ramsey property for graphs with forbidden complete subgraphs*, J. Comb. Theory (Series B) **20** (1976), 243-249.

[32] J. Nešetřil and V. Rödl, *Sparse Ramsey graphs*, Combinatorica **4** (1984), 71-78.

[33] J. Nešetřil and V. Rödl, *Finite union theorems with restrictions*, Graphs and Combinatorics **2** (1980), 357-361.

[34] J. Nešetřil and V. Rödl, *Partite constructions and Ramsey space systems*, in Mathematics of Ramsey Theory, Springer-Verlag, Berlin, 1990.

[35] H.-J. Prömel and B. Voigt, *A sparse Graham-Rothschild theorem*, Trans. Amer. Math. Soc. **309** (1988), 113-137.

[36] R. Rado, *Studien zur Kombinatorik*, Math. Zeit. **36** (1933), 242-280.

[37] R. Rado, *Note on combinatorial analysis*, Proc. London Math. Soc. **48** (1943), 122-160.

[38] V. Rödl, *On Ramsey families of sets*, Graphs and Combinatorics **6** (1990), 187-195.

[39] I. Schur, *Über die Kongruenz* $x^m + y^m = z^m (\mathrm{mod}\ p)$, Jahresbericht der Deutschen Math.-Verein. **25** (1916), 114-117.

[40] J. Spencer, *Restricted Ramsey configurations*, J. Comb. Theory (Series A) **19** (1975), 278-286.

[41] A. Taylor, *A canonical partition relation for finite subsets of* ω, J. Comb. Theory (Series A) **21** (1976), 137-146.

[42] B. van der Waerden, *Beweis einer Baudetschen Vermutung*, Nieuw Arch. Wiskunde **19** (1927), 212-216.

** (Currently (2006) available at `http://www.dpmms.cam.ac.uk/~par31/preprints/`.)