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ON MINIMAL COLORINGS WITHOUT MONOCHROMATIC SOLUTIONS TO A LINEAR EQUATION

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Abstract

For a ring R and system $\mathcal{L}$ of linear homogeneous equations, we call a coloring of the nonzero elements of R *minimal for* $\mathcal{L}$ if there are no monochromatic solutions to $\mathcal{L}$ and the coloring uses as few colors as possible. For a rational number q and positive integer n, let $E(q,n)$ denote the equation $\sum_{i=0}^{n-2} q^i x_i = q^{n-1}x_{n-1}$. We classify the minimal colorings of the nonzero rational numbers for each of the equations $E(q,3)$ with $q \in \{\frac{3}{2}, 2, 3, 4\}$, for $E(2,n)$ with $n \in \{3,4,5,6\}$, and for $x_1 + x_2 + x_3 = 4x_4$. These results lead to several open problems and conjectures on minimal colorings.

1. Introduction

The early developments in Ramsey theory focused mainly on partition regularity of systems of linear equations. A system $\mathcal{L}$ of linear homogeneous equations with coefficients in a ring R is called *r-regular* over R if, for every r-coloring of the nonzero elements of R, there is a monochromatic solution to $\mathcal{L}$. A system $\mathcal{L}$ of linear homogeneous equations is called *regular* over R if it is r-regular over R for all positive integers r.

In 1916, Schur [Sch16] proved that the equation $x + y = z$ is regular over $\mathbb{Z}$. In 1927, van der Waerden [vdW27] proved his celebrated theorem that every finite coloring of the positive integers contains arbitrarily long monochromatic arithmetic progressions. In his 1933 thesis, Rado [Rad33] generalized the theorems of Schur and van der Waerden by classifying those systems of linear homogeneous equations that are regular over $\mathbb{Z}$. In particular, a linear homogeneous equation with nonzero integer coefficients is regular over $\mathbb{Z}$ if and only if some nonempty subset of the coefficients sums to zero. In 1943, Rado [Rad43] generalized the theorem further by classifying those systems of linear equations that are regular over a subring of the complex numbers. More recently, analogues of Rado's theorem have been proven for abelian groups [Deu75], finite fields [BDH92], and commutative rings [BDHL94].

Some of the major remaining open problems on partition regularity concern the properties of colorings that are free of monochromatic solutions to a system of equations.

Definition. A coloring of the nonzero elements of a ring R (or more generally, a set of numbers S) is called *minimal for a system $\mathcal{L}$ of linear homogeneous equations* if it is free of monochromatic solutions to $\mathcal{L}$ and uses as few colors as possible.

Three basic questions arise for a given ring R and system $\mathcal{L}$ of linear homogeneous equations.

Question 1. What are the minimal colorings for $\mathcal{L}$?

Question 2. How many colors are used in a minimal coloring for $\mathcal{L}$?

Question 3. How many minimal colorings, up to isomorphism, are there for $\mathcal{L}$?

Rado made the following unresolved conjecture in his thesis on Question 2.

Conjecture 4 (Rado, [Rad33])**.** For all positive integers m and n, there exists a positive integer $k(m,n)$ such that if a system of m linear equations in n variables is $k(m,n)$-regular over $\mathbb{Z}$, then the system is regular over $\mathbb{Z}$.

Conjecture 4 is commonly known as Rado's Boundedness Conjecture [HLS03]. Rado proved that Conjecture 4 is true if it is true in the case when $m = 1$, that is, for single linear equations. Rado also settled his conjecture in the simple cases $n = 1$ and $n = 2$. Kleitman and the second author [FK05] recently proved Rado's Boundedness Conjecture for $n = 3$. They proved that if a linear equation in 3 variables is 36-regular over $\mathbb{Z}$, then it is regular over $\mathbb{Z}$.

Rado also made the following conjecture in his thesis.

Conjecture 5 (Rado, [Rad33])**.** For each positive integer n, there is a linear equation that is n-regular over $\mathbb{Z}$ but not $(n+1)$-regular over $\mathbb{Z}$.

While Conjecture 5 has remained open, Radoičić and the second author found a family of linear homogeneous equations that they conjecture verifies Conjecture 5. For a rational number q and positive integer n, let $E(q,n)$ denote the equation

$$x_0 + qx_1 + \cdots + q^{n-2}x_{n-2} = q^{n-1}x_{n-1}.$$

Definition. For a nonzero rational number q and prime number p, there is a unique representation of q as $q = p^v a/b$ with a, b, and v integers, b positive, $\gcd(a, b) = 1$, and $p \nmid a, b$. Define $v_p(q)$ to be the integer v and $w_p(q) \in \{1, \dots, p-1\}$ by $w_p(q) \equiv ab^{-1} \pmod{p}$.

For a prime p and positive integer n, let $c_{p,n} \colon \mathbb{Q} \setminus \{0\} \to \{0, 1, \dots, n-1\}$ be the n-coloring of the nonzero rational numbers defined by $c_{p,n}(q) \equiv v_p(q) \pmod{n}$.

To avoid any possible confusion, we now define what it means for two colorings of a set to be isomorphic.

Definition. Two colorings $c_1 \colon S \to C_1$ and $c_2 \colon S \to C_2$ of the same set S are *isomorphic* if there is a bijection $\phi \colon C_2 \to C_1$ such that $c_1 = \phi \circ c_2$.

Radoičić and the second author [FR05] proved that the n-coloring $c_{p,n}$ is free of monochromatic solutions to $E(p, n)$. Hence, the equation $E(p, n)$ is not n-regular over $\mathbb{Z}$. They also conjecture that $E(2, n)$ is $(n-1)$-regular over $\mathbb{Z}$, which would imply Conjecture 5. We make the following stronger conjecture.

Conjecture 6. For $n > 2$, $c_{2,n}$ is the only n-coloring, up to isomorphism, of the nonzero rational numbers without a monochromatic solution to $E(2, n)$.

In Section 2, we verify Conjecture 6 for $n = 3$ and $n = 4$. With a computer-generated proof, we have also verified Conjecture 6 for $n = 5$ and $n = 6$, and that $E(2, 7)$ is 6-regular. For brevity, we do not include these proofs. By the same technique, it can be shown that $c_{3,3}$ is the only minimal coloring of the nonzero rational numbers that is free of monochromatic solutions to $E(3, 3)$. We do, however, include a proof of the following result.

Proposition 7. The only 3-colorings, up to isomorphism, of the nonzero rational numbers without a monochromatic solution to $E(\frac{3}{2}, 3)$ are $c_{2,3}$ and $c_{3,3}$.

As it pertains to Question 3, the following definition is natural.

Definition. For a system $\mathcal{L}$ of linear homogeneous equations over a ring R, let $\Delta(\mathcal{L}; R)$ denote the number (as a cardinality) of minimal colorings, up to isomorphism, of the nonzero elements of R for $\mathcal{L}$.

For example, we have $\Delta(E(2, n); \mathbb{Q}) = 1$ for $n \in \{3, 4, 5, 6\}$ and $\Delta(E(\frac{3}{2}, 3); \mathbb{Q}) = 2$.

The following conjecture would reduce the problem of finding $\Delta(E(q, n); \mathbb{Q})$ to the case when q is a prime power.

Conjecture 8. If a, b, and n are integers, $n > 2$, $a > 1$, $|b| > 1$, and $\gcd(a, b) = 1$, then

$$\Delta(E(a, n); \mathbb{Q}) = \Delta(E(-a, n); \mathbb{Q})$$

and

$$\Delta(E(ab, n); \mathbb{Q}) = \Delta(E(a/b, n); \mathbb{Q}) = \Delta(E(a, n); \mathbb{Q}) + \Delta(E(b, n); \mathbb{Q}).$$

Let $\mathbb{Q} \setminus \{0\} = Q_0 \cup Q_1$ be the partition of the nonzero rational numbers given by $q \in Q_0$ if and only if $v_2(q)$ is even. Consider the 3-coloring c of Q_0 given by $c_0(q) = i$ if and only if $v_2(q) \equiv 2i \pmod 6$. For a permutation π of the set $\{0, 1, 2\}$, define the 3-coloring c_π of the nonzero rational numbers by $c_\pi(q) = c(q)$ for $q \in Q_0$ and $c_{\pi(q)} = \pi(c(2q))$ for $q \in Q_1$. It is easy to check that the six colorings of the form c_π are each minimal for the equation $E(4, 3)$. It can be shown that these are all the minimal colorings of the nonzero rational numbers for the equation $E(4, 3)$, but we leave it out for brevity. This construction can be easily generalized to give $(n!)^{r-1}$ different n-colorings that are free of monochromatic solutions to $E(2^r, n)$. It seems likely that these are the only minimal colorings for $E(2^r, n)$.

For each odd prime p and integer $n > 2$, we can find $(n!)^{\frac{p-3}{2}}$ different n-colorings of the nonzero rational numbers that are free of monochromatic solutions to $E(p, n)$. These n-colorings c are given by the following two properties:

1. For every nonzero rational number q, we have $c(q) = c(p^j q)$ if and only if j is a multiple of n.

2. If $v_p(q_1) = v_p(q_2)$ and $w_p(q_1) \equiv \pm w_p(q_2) \pmod p$, then $c(q_1) = c(q_2)$.

For an odd prime p, positive integers n and r with $n > 2$, the observations above can be generalized to construct a family of $(n!)^{r+\frac{p-5}{2}}$ different n-colorings of the nonzero rational numbers that are free of monochromatic solutions to $E(p^r, n)$. It seems plausible, though we have shown little evidence to support it, that these are all the minimal colorings for $E(p^r, n)$. This would imply that $\Delta(E(p^r; n); \mathbb{Q}) = (n!)^{r+\frac{p-5}{2}}$ for $n > 2$, p an odd prime, and r a positive integer.

The total number of colorings of the nonzero rational numbers is $2^{\aleph_0}$. Hence, for every system $\mathcal{L}$ of equations, we have $\Delta(\mathcal{L}; \mathbb{Q}) \leq 2^{\aleph_0}$. This upper bound can be achieved, as the following proposition demonstrates.

Proposition 9. We have

$$\Delta(x_1 + x_2 + x_3 = 4x_4; \mathbb{Q}) = 2^{\aleph_0}.$$

In fact, we classify all of the $2^{\aleph_0}$ minimal colorings for $x_1 + x_2 + x_3 = 4x_4$ in Section 3.

For a prime number p, let C_p be the $(p - 1)$-coloring of the nonzero rational numbers defined by $C_p(q) = w_p(q)$. For any set $A = \{a_1, \ldots, a_n\}$ such that no non-empty subset of A sums to zero, Rado proved that the equation $a_1x_1 + \cdots + a_nx_n = 0$ is not regular by showing that if p is a sufficiently large prime number, then the coloring C_p is free of monochromatic solutions to $a_1x_1 + \cdots + a_nx_n = 0$. It turns out that the 4-coloring C_5 is minimal for the equation $x_1 + x_2 + x_3 = 4x_4$.

Let $\Pi_p = (\pi_n)_{n\in\mathbb{Z}}$ be a sequence of permutations of the set $\{1, \ldots, p - 1\}$ of nonzero elements of $\mathbb{Z}_p$ such that π_0 is the identity permutation. For each such sequence Π_p, we define the coloring c_{Π_p} of the nonzero rational numbers by $c_{\Pi_p}(q) = \pi_{v_p(q)}(w_p(q))$. In particular, the coloring c_{Π_p} is the same as C_p if π_n is the identity permutation for all integers n. It is a straightforward exercise to show that each of the $(p-1)$-colorings c_{Π_p} is free of monochromatic solutions to the equation $x_1 + \cdots + x_{p-2} = (p-1)x_{p-1}$. For $p = 5$, we can say even more.

Proposition 10. Each of the 4-colorings c_{Π_5} is minimal for $x_1 + x_2 + x_3 = 4x_4$, and there are no other minimal colorings for $x_1 + x_2 + x_3 = 4x_4$.

There are exactly $2^{\aleph_0}$ colorings of the form c_{Π_5}, which establishes Proposition 9.

In all the examples above, either an equation has finitely many or $2^{\aleph_0}$ minimal colorings of the nonzero rational numbers. It seems likely that this is always the case.

Conjecture 11. For every nonregular finite system $\mathcal{L}$ of linear homogeneous equations, either $\Delta(\mathcal{L}, \mathbb{Q})$ is finite or $2^{\aleph_0}$. In particular, there is no $\mathcal{L}$ satisfying $\Delta(\mathcal{L}, \mathbb{Q}) = \aleph_0$.

1.1. Minimal Colorings Over the Real Numbers

When studying colorings of the real numbers, we must be careful about which axioms we choose for set theory. In this subsection, we assume that q is any rational number other than -1, 0, or 1.

Radoičić and the second author [FR05] showed that it is independent of Zermelo-Fraenkel (ZF) set theory that the equation $E(q, 3)$ is 3-regular over $\mathbb{R}$. They also showed that it is independent of ZF that $E(2, 4)$ is 4-regular over $\mathbb{R}$. We extend these results by showing that for $n = 5$ and $n = 6$, it is independent of ZF that $E(2, n)$ is n-regular over $\mathbb{R}$. Also, we show that it is independent of ZF that $x_1 + x_2 + x_3 = 4x_4$ is 4-regular over $\mathbb{R}$.

They also show that in the Zermelo-Fraenkel-Choice (ZFC) system of axioms, if r is a positive integer and $\mathcal{L}$ is a finite system of linear homogeneous equations with rational coefficients, then $\mathcal{L}$ is r-regular over $\mathbb{R}$ if and only if $\mathcal{L}$ is r-regular over $\mathbb{Z}$. It follows, in ZFC, that the equation $E(q, 3)$ is not 3-regular over $\mathbb{R}$. In Section 4, we prove in ZFC that for every integer $n \geq 3$, there are $2^{2^{\aleph_0}}$ different n-colorings of the nonzero real numbers without a monochromatic solution to $E(q, n)$. Hence, in ZFC, we have $\Delta(E, \mathbb{R}) = 2^{2^{\aleph_0}}$ for $E = E(q, 3)$ or $E = E(2, n)$ with $n \in \{3, 4, 5, 6\}$.

Solovay [Sol70] proved that the following axiom is relatively consistent with ZF.

Axiom 12. Every subset of the real numbers is Lebesgue measurable.

We will use LM to denote Axiom 12. Notice that the axiom LM is not consistent with ZFC because, with the axiom of choice, there are subsets of $\mathbb{R}$ that are not Lebesgue measurable.

The following lemma is useful in proving in ZF+LM that a given linear equation is r-regular over $\mathbb{R}$ for appropriate r.

Lemma 13. Suppose c is a coloring of the nonzero real numbers that uses at most countably many colors and there are positive real numbers a, b, d such that $\log_a b$ is irrational and $c(x) = c(ax) = c(bx) \neq c(dx)$ for all nonzero real numbers x. Then there is a color class of c that is not Lebesgue measurable.

In ZF+LM, if $n \in \{3, 4, 5, 6\}$, then for every n-coloring c of the nonzero real numbers without a monochromatic solution to $E(2, n)$, we have $c(x) = c(3x) = c(5x) \neq c(2x)$ for all non-zero rational numbers x, and $\log_3 5$ is irrational. Hence, by Lemma 13, the equation $E(2, n)$ is n-regular in ZF+LM for $n \in \{3, 4, 5, 6\}$. For every 4-coloring c of the nonzero real numbers without a monochromatic solution to the equation $x_1 + x_2 + x_3 = 4x_3$, we have $c(x) = c(6x) = c(11x) \neq c(2x)$ for all nonzero rational numbers x and $\log_6 11$ is irrational. Hence, by Lemma 13, the equation $x_1 + x_2 + x_3 = 4x_4$ is 4-regular in ZF+LM.

2. Minimal Colorings Over the Rationals

The following straightforward lemma [FK05] is useful in proving that certain colorings are free of monochromatic solutions to particular linear equations.

Lemma 14. Suppose $t_1, \ldots, t_n$ are nonzero rational numbers and p is a prime number such that $v_p(t_1) \leq v_p(t_2) \leq \cdots \leq v_p(t_n)$ and $\sum_{i=1}^{n} t_i = 0$. Then $v_p(t_1) = v_p(t_2)$.

The following proposition is due to the second author and Radoičić [FR05].

Proposition 15. For each integer $n > 1$, the n-coloring $c_{2,n}$ is free of monochromatic solutions to $E(2, n)$.

Proof. For $n > 1$, if $x_0 + 2x_1 + \cdots + 2^{n-2}x_{n-2} - 2^{n-1}x_{n-1} = 0$ is a solution to $E(2, n)$ in the nonzero rationals, then we have by Lemma 14 that for some i and j with $0 \leq i < j < n$,

$$v_2(2^i x_i) = v_2(2^j x_j).$$

It follows that $v_2(x_i) = (j - i) + v_2(x_j)$ and $v_2(x_j) - v_2(x_i) \in \{1, \ldots, n-1\}$. Therefore, $v_2(x_j) - v_2(x_i)$ is not a multiple of n. Hence, x_i and x_j are not the same color, and the coloring $c_{2,n}$ is free of monochromatic solutions to $E(2, n)$. □

They [FR05] also prove the following structural result for 3-colorings of the nonzero rational numbers without a monochromatic solution to $E(q, 3)$.

Lemma 16. Let x and q be nonzero rational numbers with $q \neq \pm 1$, m and n be integers, and let $a = \frac{q+1}{q^2}$ and $b = q(q-1)$. For every 3-coloring c of the nonzero rational numbers without a monochromatic solution to $E(q, 3)$, we have $c(x) = c(a^m b^n x)$ if and only if $m + n$ is a multiple of 3.

Proof. Since $x + q(x) = q^2(ax)$, ax must be a different color than x. Since $bx + q(x) = q^2(x)$, then bx must be a different color than x. Since $x + q(abx) = q^2(x)$, then abx must be a different color than x. Hence, $c(x) \neq c(rx)$ for $r \in \{a, b, ab\}$.

Recall that for a group G and subset $S \subset G$, the Cayley graph $\Gamma(G,S)$ has vertex set G and two elements x and y are adjacent if there is an element $s \in S$ such that $x = sy$ or $y = sx$.

We associate each rational number $a^m b^n x$ (with m and n integers) with the lattice point (m,n). Let $S = \{(1,0),(0,1),(1,1)\}$ and consider the Cayley graph $\Gamma(\mathbb{Z}^2, S)$. Define the 3-coloring χ of $\mathbb{Z}^2$ by $\chi(m,n) = c(a^m b^n x)$. Since $c(rx) \neq c(x)$ for $r \in \{a, b, ab\}$, then χ is a proper 3-coloring of the vertices of $\Gamma(\mathbb{Z}^2, S)$. By induction, it is a straightforward check that there is only one proper coloring of $\Gamma(\mathbb{Z}^2, S)$ up to isomorphism and this coloring is given by $\chi(m,n) \equiv m+n \pmod 3$. Hence, $c(x) = c(a^m b^n x)$ if and only if $m+n$ is a multiple of 3. □

2.1. The Minimal Coloring for $E(2,3)$

We prove that $c_{2,3}$ is the only minimal coloring of the nonzero rational numbers, up to isomorphism, for the equation $E(2,3)$. By Proposition 15, we know that the 3-coloring $c_{2,3}$ is free of monochromatic solutions to $E(2,3)$. By Lemma 16, the numbers 2, 3, and 4 must be all different colors, so $E(2,3)$ is 2-regular. Hence $c_{2,3}$ is a minimal coloring of the nonzero rational numbers for $E(2,3)$.

Proposition 17. The only minimal coloring of the nonzero rational numbers for $E(2,3)$ is $c_{2,3}$.

Proof. Suppose c is a 3-coloring of the nonzero rational numbers without a monochromatic solution to $E(2,3)$. By Lemma 16, for every nonzero rational number x and integers m and n, we have $c((\frac{3}{4})^m 2^n x) = c(x)$ if and only if $m+n$ is a multiple of 3. Equivalently, for every nonzero rational number x and integers m and n, $c(2^m 3^n x) = c(x)$ if and only if m is a multiple of 3.

For nonzero rational numbers x and r with r positive, we now show that $c(x) = c(rx)$ if and only if $v_2(r) \equiv 0 \pmod 3$. By induction, it suffices to prove that $c(x) = c(px)$ for every odd positive integer p and nonzero rational number x. For $p = 1$ or 3, we have already established that $c(x) = c(px)$ for every nonzero rational number x. So let p be an odd integer greater than 3 and suppose that for every positive odd integer $p' < p$ and rational number x, $c(p'x) = c(x)$. The numbers px and $4x$ are different colors because otherwise $(x_0, x_1, x_2) = (4(p-2)x, 4x, px)$ is a monochromatic solution to $E(2,3)$. We have px and $2x$ are different colors because otherwise $(16x, 2(p-4)x, px)$ is a monochromatic solution to $E(2,3)$. Since $c(x)$, $c(2x)$, and $c(4x)$ are all different colors, then $c(px) = c(x)$. By induction, for nonzero rational numbers x and r with r positive, we have $c(x) = c(rx)$ if and only if $v_2(r) \equiv 0 \pmod 3$.

To finish the proof, we need to show that $c(x) = c(-x)$ for every nonzero rational number x. Since $(10x, -x, 2x)$ is a solution to $E(2,3)$ and $10x$ and $2x$ are the same color, then $-x$ and $2x$ are different colors. Since $(12x, -8x, -x)$ is a solution to $E(2,3)$, $-8x$ and $-x$ are the same color, and $12x$ is the same color as $4x$, then $-x$ and $4x$ are differently colored. Since

x, $2x$, and $4x$ are all different colors, then $c(-x) = c(x)$. Therefore, $c_{2,3}$ is the only minimal coloring, up to isomorphism, of the nonzero rational numbers without a monochromatic solution to $E(2, 3)$. □

Using a very similar argument to the proof of Proposition 17, it is not difficult to show that there are only two minimal colorings of the positive integers for the equation $E(2, 3)$. These two minimal colorings consist of $c_{2,3}$ with its domain restricted to the positive integers and a coloring $c'_{2,3}$ that is identical to $c_{2,3}$ with its domain restricted to the positive integers except that the color of 1 is different. Formally, $c'_{2,3}\colon \mathbb{N} \to \{0, 1, 2\}$ is the coloring of the positive integers such that $c'_{2,3}(1) = 2$ and $c'_{2,3}(n) = c_{2,3}(n)$ for $n > 1$.

Proposition 18. The only two minimal colorings of the positive integers for the equation $E(2, 3)$ are the colorings $c_{2,3}$ with its domain restricted to the positive integers and $c'_{2,3}$.

2.2. The Minimal Coloring for $E(2, 4)$

In this subsection we prove that the only minimal coloring of the nonzero rational numbers, up to isomorphism, for $E(2, 4)$ is $c_{2,4}$. The proof uses the following lemma from [FR05], proven below.

Lemma 19. For every 4-coloring c of the nonzero rational numbers without a monochromatic solution to $E(2, 4)$ and for nonzero rational number x and integers m and n, we have $c(x) = c(2^m 3^n x)$ if and only if m is a multiple of 4.

In particular, Lemma 19 implies that $E(2, 4)$ is 3-regular since in every 4-coloring of the nonzero rational numbers without a monochromatic solution to $E(2, 4)$, the numbers 1, 2, 4, and 8 are different colors. It follows that $c_{2,4}$ is minimal for $E(2, 4)$. Moreover, we have the following proposition.

Proposition 20. The coloring $c_{2,4}$ is the only minimal coloring of the nonzero rational numbers without a monochromatic solution to $E(2, 4)$.

Proof of Lemma 19. By induction on m and n, it suffices to prove that for all nonzero rational numbers q, we have $c(q) = c(3q) = c(16q)$ and $c(q) \notin \{c(2q), c(4q), c(8q)\}$. By considering all solutions to $E(2, 4)$ with exactly two distinct variables, for $r \in \{\frac{n+1}{n} : n \in \mathbb{Z} \text{ and } 1 \leq n \leq 7\}$, we have that q and rq are different colors (call these ratios r *forbidden*). Note immediately that $c(x) \neq c(2x)$ by the forbidden ratio 2.

We begin by showing that $c(x) \neq c(4x)$. Proceeding by contradiction, suppose $c(x) = c(4x)$ instead for some x. By forbidden ratios amongst themselves, we have that $c(x) = c(4x)$, $c(2x)$, $c(3x)$, and $c(\frac{3}{2}x)$ are distinct colors. Then $c(\frac{9}{4}x) = c(2x)$ by forbidden ratios from $3x$ and $\frac{3}{2}x$ and because of the solution $(x, 4x, \frac{9}{4}x, \frac{9}{4}x)$. Similarly, $c(6x) = c(\frac{3}{2}x)$ because of forbidden ratios from $4x$ and $3x$ and because of the solution $(6x, 2x, 2x, \frac{9}{4}x)$. Finally, $\frac{18}{7}x$ cannot be colored with any colors because of forbidden ratios from $\frac{9}{4}x$ and $3x$ as well as the solutions $(\frac{18}{7}x, x, 4x, \frac{18}{7}x)$ and $(\frac{18}{7}x, 6x, \frac{3}{2}x, \frac{18}{7}x)$.

We now show that $c(x) = c(3x)$. Assume otherwise, so that $c(x) \neq c(3x)$ for some x. Once again, by forbidden ratios amongst themselves, we have that $c(x)$, $c(3x)$, $c(2x)$, and $c(4x)$ are distinct; furthermore, $c(\frac{3}{2}x) = c(4x)$ also by forbidden ratios. We have $c(\frac{4}{3}x) = c(3x)$ by forbidden ratios from x and $2x$ as well as the solution $(4x, \frac{4}{3}x, \frac{4}{3}x, \frac{3}{2}x)$. Next, $c(\frac{5}{3}x) = c(x)$ by forbidden ratios from $\frac{4}{3}x$ and $2x$ as well as the solution $(4x, \frac{5}{3}x, \frac{3}{2}x, \frac{5}{3}x)$. Similarly, $c(6x) = c(2x)$ by forbidden ratios from $3x$ and $\frac{3}{2}x$ and the solution $(6x, \frac{5}{3}x, x, \frac{5}{3}x)$. Finally, $\frac{9}{4}x$ cannot be colored with any colors, because of forbidden ratios from $3x$ and $\frac{3}{2}x$ as well as the solutions $(x, \frac{5}{3}x, \frac{9}{4}x, \frac{5}{3}x)$ and $(6x, 2x, 2x, \frac{9}{4}x)$.

Clearly, $c(x) \neq c(8x)$ since otherwise $(8x, \frac{8}{3}x, \frac{8}{3}x, 3x)$ would be a monochromatic solution. Completing the proof, both $c(x)$ and $c(16x)$ must be different from all of $c(2x)$, $c(4x)$, and $c(8x)$ (which are distinct), so $c(x) = c(16x)$. □

Proof of Proposition 20. The proof is similar to the proof of Proposition 17. Suppose c is a 4-coloring of the nonzero rational numbers without a monochromatic solution to $E(2, 4)$. By Lemma 19, for every nonzero rational number x and integers m and n, we have $c(x) = c(2^m 3^n x)$ if and only if m is a multiple of 4. We now show that for every positive rational number r with $v_2(r) \equiv 0 \pmod 4$, we have $c(x) = c(rx)$. By induction, it suffices to prove that $c(x) = c(px)$ for every positive odd integer p and nonzero rational number x. For $p = 1$ or 3, Lemma 19 implies that $c(x) = c(px)$ for every nonzero rational number x.

For $p = 5$, we have $(18x, 5x, 5x, 6x)$ is a solution to $E(2, 4)$ and $c(18x) = c(2x) = c(6x)$, so $5x$ and $2x$ are different colors. Also, $(12x, 4x, 5x, 5x)$ is a solution to $E(2, 4)$ and $c(12x) = c(4x)$, so $5x$ and $4x$ are different colors. Finally, $(5x, \frac{3}{2}x, 8x, 5x)$ is a solution to $E(2, 4)$ and $c(\frac{3}{2}x) = c(8x)$, so $c(5x)$ and $c(8x)$ are different colors. Since x, $2x$, $4x$, and $8x$ are all different colors, then $c(5x)$ must be the color of $c(x)$.

The rest of the proof is by induction on p. Suppose p is an odd integer greater than 5 and that for all positive odd $p' < p$ and rational x, $c(p'x) = c(x)$. Then for all x, we know that px and $2x$ are different colors since $(\frac{32}{3}x, \frac{32}{3}x, 2(p-4)x, px)$ would otherwise be a monochromatic solution. Similarly, px and $4x$ are different colors because of $(\frac{64}{5}x, 4(p-2)x, \frac{4}{5}x, px)$. Finally, px and $8x$ are different colors because of $(8(p-2)x, \frac{8}{3}x, \frac{8}{3}x, px)$. Since $c(x)$, $c(2x)$, $c(4x)$, and $c(8x)$ are all distinct, it follows that $c(px) = c(x)$ as desired.

To finish the proof, we need to show that $c(x) = c(-x)$ for every nonzero rational number x. Since $(10x, -x, 2x, 2x)$ is a solution to $E(2, 4)$ and $c(10x) = c(2x)$, then $c(-x)$ and $c(2x)$ are different colors. Since $(28x, -16x, -x, -x)$ is a solution to $E(2, 4)$, $c(-16x) = c(-x)$, and $c(28x) = c(4x)$, then $-x$ and $4x$ are different colors. Since $(88x, -16x, -16x, -x)$ is a solution to $E(2, 4)$, $c(-16x) = c(-x)$, and $c(88x) = c(8x)$, then $-x$ and $8x$ are different colors. Since x, $2x$, $4x$, and $8x$ are all different colors, then $c(-x) = c(x)$. Therefore, $c_{2,4}$ is the only minimal coloring, up to isomorphism, of the nonzero rational numbers without a monochromatic solution to $E(2, 4)$. □

2.3. The Minimal Colorings for $E(\frac{3}{2}, 3)$

In this subsection we prove that the two minimal colorings of the nonzero rational numbers for $E(\frac{3}{2}, 3)$ are $c_{2,3}$ and $c_{3,3}$. By a proof similar to Lemma 15, it is clear that $c_{2,3}$ and $c_{3,3}$ are free of monochromatic solutions to $E(\frac{3}{2}, 3)$. The equation $E(\frac{3}{2}, 3)$ is 2-regular since in any coloring of the nonzero rational numbers without a monochromatic solution to $E(\frac{3}{2}, 3)$, the numbers 9, 10, and 12 are all different colors (by Lemma 16). Hence $c_{2,3}$ and $c_{3,3}$ are minimal colorings for $E(\frac{3}{2}, 3)$. Again from Lemma 16, it follows that for every nonzero rational number x and integers m and n, we have $c(x) = c((\frac{6}{5})^m(\frac{10}{9})^n x)$ if and only if $m+n$ is a multiple of 3. In particular, $c(x) = c(rx)$ for $r \in \{\frac{8}{5}, \frac{64}{27}, \frac{40}{27}\}$ and $c(x) \neq c(rx)$ for $r \in \{\frac{6}{5}, \frac{10}{9}, \frac{4}{3}\}$.

For the rest of this subsection, we suppose that c is a 3-coloring that is free of monochromatic solutions to $E(\frac{3}{2}, 3)$ and we will deduce that c is either $c_{2,3}$ or $c_{3,3}$. We first build up structural properties about c if there is a nonzero rational number x such that $c(x) = c(2x)$ and deduce that c is the coloring $c_{3,3}$. We then prove properties about c if $c(x) \neq c(2x)$ for every nonzero rational number x. We deduce in this case that c is the coloring $c_{2,3}$.

Lemma 21. If x is a nonzero rational number such that $c(2x) = c(x)$, then $c(2^n x) = c(x)$ for every nonnegative integer n.

Lemma 22. If x is a nonzero rational number such that $c(2x) = c(x)$, then for all nonnegative integers k, m, and n we have $c(2^k 3^m 5^n x) = c(x)$ if and only if m is a multiple of 3.

Lemma 23. If x is a nonzero rational number such that $c(2x) = c(x)$, then $c(3x) = c(6x)$.

From Lemma 22 and Lemma 23, we have the following result.

Lemma 24. If c is a 3-coloring of the nonzero rational numbers without a monochromatic solution to $E(\frac{3}{2}, 3)$ and x is a rational number such that $c(2x) = c(x)$, then for all integers m_1, n_1, p_1, m_2, n_2, p_2, we have $c(2^{n_1}3^{m_1}5^{p_1}x) = c(2^{n_2}3^{m_2}5^{p_2}x)$ if and only if $m_1 \equiv m_2 \pmod 3$.

The following lemma gets us much closer to proving that $c_{3,3}$ is the only 3-coloring of the nonzero rational numbers for which there is a nonzero rational number x such that $c(x) = c(2x)$.

Lemma 25. If x is a nonzero rational number such that $c(x) = c(2x)$, then for every positive integer n, we have $c(nx) = c(x)$ if and only if $v_3(n)$ is a multiple of 3.

Finally, to finish the proof that $c_{3,3}$ is the only 3-coloring c of the nonzero rational numbers without a monochromatic solution to $E(\frac{3}{2}, 3)$ and for which there is a rational number x such that $c(x) = c(2x)$, it suffices to prove that $c(y) = c(-y)$ for all y.

Lemma 26. The only 3-coloring c of the nonzero rational numbers for which there is a rational number x such that $c(x) = c(2x)$ is $c_{3,3}$.

Having completed the case when there is a nonzero rational number x such that $c(x) = c(2x)$, we now look at those 3-colorings for which $c(x) \neq c(2x)$ for all nonzero rational numbers x.

Lemma 27. If $c(x) \neq c(2x)$ for every nonzero rational number x, then $c(2^n y) = c(y)$ holds for integer n and nonzero rational number y if and only if n is a multiple of 3.

Lemma 28. If $c(x) \neq c(2x)$ for every nonzero rational x, then $c(2^m 3^n 5^p y) = c(y)$ holds for nonzero rational number y and integers m, n, and p if and only if m is a multiple of 3.

Lemma 29. If $c(x) \neq c(2x)$ for all nonzero rational numbers x, then $c(y) = c(-y)$ for all nonzero rational numbers y.

Finally, to finish the proof of Proposition 7 that $c_{2,3}$ and $c_{3,3}$ are the only 3-colorings of the nonzero rational numbers without a monochromatic solution to $E(\frac{3}{2}, 3)$, it suffices to prove the following lemma.

Lemma 30. If no nonzero rational number x satisfies $c(x) = c(2x)$, then $c(nx) = c(x)$ if and only if $v_2(n)$ is a multiple of 3.

Proof of Lemma 21. By induction on n, it suffices to prove that $c(4x) = c(x)$ if $c(x) = c(2x)$. Assume for contradiction that $c(x) = c(2x) \neq c(4x)$. Then we know that $c(3x)$ must further be distinct from $c(x)$ and $c(4x)$ because of the forbidden ratio from $4x$ and the solution $(3x, x, 2x)$. It follows that $c(\frac{10}{3}x) = c(x)$ because of forbidden ratios from $3x$ and $4x$. Similarly, $c(\frac{5}{2}x) = c(4x)$ because of forbidden ratios from $3x$ and $\frac{10}{3}x$. It follows that $c(\frac{13}{3}x) = c(3x)$ because of the solutions $(x, \frac{13}{3}x, \frac{10}{3}x)$ and $(\frac{5}{2}x, \frac{13}{3}x, 4x)$. Similarly, $c(\frac{9}{2}x) = c(4x)$ because of the solutions $(\frac{9}{2}x, 2x, \frac{10}{3}x)$ and $(3x, \frac{9}{2}x, \frac{13}{3}x)$. We have $c(6x) = c(3x)$ because of a forbidden ratio from $\frac{9}{2}x$ and the solution $(6x, x, \frac{10}{3}x)$. Finally, the number $\frac{3}{4}x$ cannot be colored with any colors, because of a forbidden ratio from x as well as the solutions $(\frac{9}{2}x, \frac{3}{4}x, \frac{5}{2}x)$ and $(\frac{3}{4}x, 6x, \frac{13}{3}x)$. □

Proof of Lemma 22. By the previous lemma, we have $c(2^n x) = c(x)$ for all nonnegative integers n. Since $c(\frac{5}{8}y) = c(y)$ for every nonzero rational number y, then $c(2^n 5^p x) = c(x)$ for all nonnegative integers n and p. To finish the proof, it suffices, by induction, to prove that neither $3x$ nor $9x$ is the same color as x, and $27x$ is the same color as x. Since $3x$ and $4x$ have ratio $\frac{3}{4}$, then $c(3x) \neq c(4x) = c(x)$. Since $9x$ and $16x$ have ratio $(\frac{6}{5})^{-2}(\frac{10}{9})^{-2}$, then $c(9x) \neq c(16x) = c(x)$. Since $27x$ and $64x$ have ratio $\frac{27}{64}$, then $c(27x) = c(64x) = c(x)$. □

Proof of Lemma 23. Assume for contradiction that $c(3x) \neq c(6x)$. By Lemma 22, $c(x) = c(4x) = c(8x)$. Since $4x = \frac{4}{3}(3x)$ and $8x = \frac{4}{3}(6x)$, then x, $3x$, and $6x$ are all different colors. By Lemma 22, $c(\frac{3}{2}x) \neq c(3x)$, since otherwise $c(3x) = c(6x)$. Since $\frac{3}{4}(2x) = \frac{3}{2}x$, then $\frac{3}{2}x$ and x are different colors. Hence, $c(\frac{3}{2}x) = c(6x)$. Since $\frac{64}{27}(\frac{3}{2}x) = \frac{32}{9}x$, then $\frac{32}{9}x$ is the same color as $6x$. Since $(6x, \frac{4}{3}x, \frac{32}{9}x)$ is a solution to $E(\frac{3}{2}, 3)$, then $c(\frac{4}{3}x)$ and $c(6x)$ are different colors. Since $\frac{4}{3}x$ and x have ratio $\frac{4}{3}$, then $\frac{4}{3}x$ and x are different colors. Hence, $\frac{4}{3}x$ is the same color as $3x$. Since $\frac{3}{4}x$, x, and $\frac{4}{3}x$ are all different colors, then $\frac{3}{4}x$ is the same color as $6x$. But then $\frac{3}{4}x$ and $\frac{3}{2}x$ are the same color, which implies, by Lemma 22, that $3x$ and $6x$ are the same color, a contradiction. □

Proof of Lemma 24. Clearly, from Lemma 22 and Lemma 23, we have the result for *nonnegative* integers.

Recall that, for every nonzero rational number y, we have $c(y) = c(\frac{8}{5}y)$ and we also have y, $\frac{4}{3}y$, and $\frac{16}{9}y$ are all different colors. Therefore, it suffices, by the remark above and induction to prove that $c(\frac{x}{2}) = c(x)$. Since $c(3x) = c(6x)$ and $(6x, \frac{x}{2}, 3x)$ is a solution to $E(\frac{3}{2}, 3)$, then $\frac{x}{2}$ is a different color from $3x$. We have $\frac{80}{3}x = \frac{40}{27}(18x)$, so $18x$ and $\frac{80}{3}x$ are the same color as $9x$. Since $(\frac{x}{2}, \frac{80}{3}x, 18x)$ is a solution to $E(\frac{3}{2}, 3)$, then $\frac{x}{2}$ and $9x$ are different colors. Hence, $\frac{x}{2}$ is the same color as x, which completes the proof. □

Proof of Lemma 25. Suppose c is a 3-coloring of the nonzero rational numbers without a monochromatic solution to $E(\frac{3}{2}, 3)$ and x is a nonzero rational number such that $c(x) = c(2x)$. By Lemma 24, for integers m_1, n_1, p_1, m_2, n_2, and p_2 we have $c(2^{n_1}3^{m_1}5^{p_1}x) = c(2^{n_2}3^{m_2}5^{p_2}x)$ if and only if $m_2 - m_1$ is a multiple of 3. By induction, it suffices to prove for every prime $p > 3$, that $c(x) = c(px)$. For $p = 5$, Lemma 24 implies that $c(x) = c(px)$.

The proof is by induction on the size of p. Suppose p is prime with $p > 5$. We write $p = 6a + b$, where a and b are nonnegative integers and $b \in \{1, 5\}$. The induction hypothesis is that $c(q) = c(p'q)$ for every nonzero rational number q satisfying $c(q) = c(2q)$ and prime p' satisfying $3 < p' < p$. The induction hypothesis implies that $c(q) = c(p'q)$ for every nonzero rational number q satisfying $c(q) = c(2q)$ and positive odd integer p' which is not a multiple of 3 and satisfies $p' < p$. Then we have for all p that $c(px) \neq c(9x) = c(\frac{9}{4}x)$ because of the solution $(\frac{9}{4}(p-6)x, 9x, px)$. It suffices to show that in the two cases below, $c(px) \neq c(3x)$ since $c(x)$, $c(3x)$, and $c(9x)$ are distinct.

Case 1: $p = 6a + 1$. For $a = 1$, we have $p = 7$, and $(3x, 7x, 6x)$ is a solution to $E(\frac{3}{2}, 3)$, so $7x$ and $3x$ are different colors. For $a > 1$, we have $0 < 3a + 5 < 6a + 1$, so by the induction hypothesis, we have $c((3a + 5)q) = c(q)$ for every rational number q satisfying $c(q) = c(2q)$, and in particular, for $q = \frac{8}{9}x$. The numbers $6x$ and $\frac{8}{9}x$ are the same color as the color of $3x$ by Lemma 24. Since $(px, 6x, (3a+5)\frac{8}{9}x)$ is a solution to $E(\frac{3}{2}, 3)$, then px and $3x$ are different colors. Hence, $c(px) = c(x)$.

Case 2: $p = 6a + 5$. For each prime $p > 5$ of the form $p = 6a + 5$ we have $3a + 7 < 6a + 5$, so by the induction hypothesis we have $c((3a + 5)q) = c(q)$ for every rational number q satisfying $c(q) = c(2q)$, and in particular, for $q = \frac{8}{9}x$. The numbers $6x$ and $\frac{8}{9}x$ are the same color as the color of $3x$ by Lemma 24. Since $(px, 6x, (3a + 7)\frac{8}{9}x)$ is a solution to $E(\frac{3}{2}, 3)$, then px and $3x$ are different colors. Hence, $c(px) = c(x)$. □

Proof of Lemma 26. By Lemma 25, it suffices to prove that $c(-y) = c(y)$ for all nonzero rational numbers y. By Lemma 25, we have $c(\frac{85}{6}y) = c(9y)$. Since $(-y, \frac{85}{6}y, 9y)$ is a solution to $E(\frac{3}{2}, 3)$, then $-y$ and $9y$ are different colors. By Lemma 25, we have $c(\frac{14}{9}y) = c(3y)$. Since

$(-y, 3y, \frac{14}{9}y)$ is a solution to $E(\frac{3}{2}, 3)$, then $-y$ and $3y$ are different colors. Therefore, we have $c(-y) = c(y)$, completing the proof. □

Proof of Lemma 27. It suffices to prove that $c(y) \neq c(4y)$ for all nonzero rational numbers y. So suppose for contradiction that there is a nonzero rational number y such that $c(y) = c(4y)$. Let red be the color of y, blue be the color of $2y$, and green be the remaining color. Since $3y = \frac{4}{3}(4y)$, then $3y$ is green or blue.

Case 1: $3y$ is green. Since $3y$, $4y$, and $\frac{16}{3}y$ are all different colors, then $\frac{16}{3}$y is blue. Since $\frac{9}{4}y = \frac{27}{64}(\frac{16}{3}y)$, then $\frac{9}{4}y$ is blue. Since $\frac{9}{2}y = 2(\frac{9}{4}y)$, then $\frac{9}{2}y$ is not blue. Since $(9y, 2y, \frac{16}{3}y)$ is a solution to $E(\frac{3}{2}, 3)$, then $9y$ is not blue. Since $9y = 2(\frac{9}{2}y)$, then $\frac{9}{2}y$ and $9y$ are different colors. Therefore, either $\frac{9}{2}y$ is red and $9y$ is green or $\frac{9}{2}y$ is green and $9y$ is red.

Subcase 1a: $\frac{9}{2}y$ is red and $9y$ is green. Since $6y = \frac{4}{3}(\frac{9}{2}y)$, then $6y$ is not red. Since $6y = 2(3y)$, then $6y$ is not green. Hence, $6y$ is blue. Since $12y = 2(6y)$, then $12y$ is not blue. Since $12y = \frac{4}{3}(9y)$, then $12y$ is red. Since $\frac{15}{2}y = \frac{5}{8}(12y)$, then $\frac{15}{2}y$ is red. Then $(\frac{15}{2}y, y, 4y)$ is a monochromatic solution to $E(\frac{3}{2}, 3)$, a contradiction.

Subcase 1b: $\frac{9}{2}y$ is green and $9y$ is red. Since $\frac{10}{3}y = \frac{5}{8}(\frac{16}{3}y)$, then $\frac{10}{3}y$ is blue. Since $3y = 2(\frac{3}{2}y)$ and $2y = \frac{4}{3}(\frac{3}{2}y)$, then $\frac{3}{2}y$ is red. Since $\frac{5}{3}y = \frac{10}{9}(\frac{3}{2}y)$ and $\frac{5}{3}y = \frac{5}{6}(2y)$, then $\frac{5}{3}y$ is green. Finally, $\frac{28}{9}y$ can not be colored with any colors because of the solutions $(y, 4y, \frac{28}{9}y)$, $(2y, \frac{10}{3}y, \frac{28}{9}y)$, and $(\frac{9}{2}y, \frac{5}{3}y, \frac{28}{9}y)$.

Case 2: $3y$ is blue. Since $3y$, $4y$, $\frac{16}{3}y$ are all different colors, then $\frac{16}{3}y$ is green. Since $\frac{16}{3}y = 2(\frac{8}{3}y)$, then $\frac{8}{3}y$ is not green. Since $\frac{8}{3}y = \frac{4}{3}(2y)$, then $\frac{8}{3}y$ is not blue. Hence, $\frac{8}{3}y$ is red. Since $\frac{3}{2}y$, $2y$, $\frac{8}{3}y$ are all different colors, then $\frac{3}{2}y$ is green. Since $\frac{9}{4} = \frac{27}{64}(\frac{16}{3}y)$, then $\frac{9}{4}y$ is green. Since $\frac{9}{2}y = 2(\frac{9}{4}y)$, then $\frac{9}{2}y$ is not green. Since $(\frac{9}{2}y, y, \frac{8}{3}y)$ is a solution to $E(\frac{3}{2}, 3)$, then $\frac{9}{2}y$ is not red. Therefore, $\frac{9}{2}y$ is blue. Since $8y = (\frac{6}{5})^2(\frac{10}{9})^2(\frac{9}{2}y)$, then $8y$ is not blue. Since $8y = 2(4y)$, then $8y$ is not red. Hence, $8y$ is green. Since $5y = \frac{5}{8}(8y)$, then $5y$ is green. Since $\frac{32}{9}y = \frac{64}{27}(\frac{3}{2}y)$, then $\frac{32}{9}$ is green. Since $(\frac{y}{2}, 5y, \frac{32}{9}y)$ is a solution to $E(\frac{3}{2}, 3)$, then $\frac{y}{2}$ is not green. Since $y = 2(\frac{y}{2})$, then $\frac{y}{2}$ is not red. Hence, $\frac{y}{2}$ is blue.

We found a contradiction in Case 1, so if y and $4y$ are the same color, then $\frac{y}{2}$, $2y$, $3y$ are the same color. Therefore, $\frac{y}{4}$, y, $\frac{3}{2}y$ must be the same color. But $\frac{3}{2}y$ is green and y is red, a contradiction. □

Proof of Lemma 28. It suffices, by induction and Lemma 27, to prove that $c(y) = c(3y) = c(5y)$ for every y. By Lemma 27, $c(y) = c(8y)$ for every y. Since $8y = \frac{8}{5}(5y)$, then $5y$ is the same color as $8y$, so $c(y) = c(5y)$ for all y. So suppose for contradiction that there is a nonzero rational number x such that $c(x) \neq c(3x)$. So, by Lemma 27, x, $2x$, and $4x$ are all different colors. Let red be the color of x, blue be the color of $2x$, and green be the color of $4x$. Since $3x$ is a different color from x and $4x$, then $3x$ is blue. Since $24x = 8(3x)$, then $24x$ is blue. Since $50x = 25(2x)$, then $50x$ is blue. Since $\frac{64}{9}x = \frac{64}{27}(3x)$, then $\frac{64}{9}x$ is blue. Since $(3x, 24x, \frac{52}{3}x)$, $(3x, 50x, \frac{104}{3}x)$, and $(3x, \frac{26}{3}x, \frac{64}{9}x)$ are solutions to $E(\frac{3}{2}, 3)$, then none of the numbers $\frac{26}{3}x$, $\frac{52}{3}x$, $\frac{104}{3}x$ is blue. But $\frac{26}{3}x$, $\frac{52}{3}x$, $\frac{104}{3}x$ are all different colors, so one of them has to be blue, a contradiction. □

Proof of Lemma 29. By Lemma 28, the numbers $2y$ and $\frac{2}{9}y$ are the same color and the numbers $4y$, $\frac{4}{3}y$, $\frac{4}{9}y$ are the same color. Since $(2y, -y, \frac{2}{9}y)$ and $(-y, \frac{4}{3}y, \frac{4}{9}y)$ are solutions to $E(\frac{3}{2}, 3)$ and y, $2y$, $4y$ are all different colors, then $c(y) = c(-y)$. □

Proof of Lemma 30. By Lemma 28 and Lemma 29, for integers l, m, n, and p and nonzero rational number y, we have $c((-1)^l 2^m 3^n 5^p y) = c(y)$ if and only if m is a multiple of 3. By induction, it suffices to prove for every odd $p \geq 3$, that $c(x) = c(px)$. For $p = 3$ or $p = 5$, Lemma 28 implies that $c(x) = c(px)$.

The proof is by induction on the size of p. Suppose p is an odd number with $p > 5$. The induction hypothesis is that $c(q) = c(p'q)$ for every nonzero rational number q and odd number p' that is less than p. We know that $c(px) \neq c(2x) = c(\frac{9}{4}x) = c(6x)$ because of the solution $(\frac{9}{4}(p-4)x, 6x, px)$. Similarly, $c(px) \neq c(4x) = c(\frac{3}{2}x) = c(\frac{9}{2}x)$ because of the solution $(\frac{9}{2}x, \frac{3}{2}(p-2)x, px)$. Since $c(x)$, $c(2x)$, and $c(4x)$ are distinct, $c(px) = c(x)$. □

3. Minimal Colorings for $x_1 + x_2 + x_3 = 4x_4$

In this section we prove that the minimal colorings for $x_1 + x_2 + x_3 = 4x_4$ are those of the form c_{Π_5}. It is a straightforward check that each of the colorings of the form c_{Π_5} is minimal for $x_1 + x_2 + x_3 = 4x_4$. We suppose for the rest of this subsection that c is a 4-coloring of the nonzero rational numbers without a monochromatic solution to $x_1 + x_2 + x_3 = 4x_4$.

Lemma 31. If x is a nonzero rational number and $r \in \{\frac{4}{3}, \frac{3}{2}, 2\}$, then $c(x) \neq c(rx)$.

Lemma 32. For every nonzero rational number x, we have $c(x) \neq c(3x)$.

Lemma 33. For every nonzero rational number x, we have $c(x) \neq c(4x)$.

Lemma 34. For every nonzero rational number x and integers m and n, we have $c(x) = c(2^m 3^n x)$ if and only if $w_5(2^m 3^n) \equiv 1 \pmod 5$.

Lemma 35. For every nonzero rational number x, we have $c(-x) = c(4x)$.

To finish the proof of Proposition 10, it suffices by Lemma 35 to prove the following lemma.

Lemma 36. If x is a nonzero rational number and n is a positive integer satisfying $v_5(n) = 0$ and $n \equiv d \pmod 5$ with $d \in \{1, 2, 3, 4\}$, then $c(nx) = c(dx)$.

Proof of Lemma 31. Since $(\frac{4}{3}x, \frac{4}{3}x, \frac{4}{3}x, x)$, $(\frac{3}{2}x, \frac{3}{2}x, x, x)$, and $(2x, x, x, x)$ are solutions to $x_1 + x_2 + x_3 = 4x_4$, then $c(x) \neq c(rx)$ for $r \in \{\frac{4}{3}, \frac{3}{2}, 2\}$. □

Proof of Lemma 32. We suppose for contradiction that there is a nonzero rational number x

such that $c(x) = c(3x)$. Without loss of generality, we may take $x = 1$. The previous lemmas imply that the ratios 2, $\frac{3}{2}$, and $\frac{4}{3}$ are *forbidden ratios*, that is, $c(x) \neq c(rx)$ for $r \in \{2, \frac{3}{2}, \frac{4}{3}\}$. Using these forbidden ratios, we see that $c(1) = c(3)$, $c(2)$, and $c(4)$ must be different colors.

We present a computer-generated proof of the rest of this lemma in Table 1. Without loss of generality, assume that our set of colors is $\{0, 1, 2, 3\}$ and that $c(1) = c(3) = 0$, $c(2) = 1$, and $c(4) = 2$; we must derive a contradiction.

We describe briefly how to read Table 1. In the left-most column, the assumptions that we make are listed. They are structured in a tree-like fashion, reflecting the trial-and-error argument. The next column lists the current claim. If this claim is "$c(x)$!?", then this means that $c(x)$ cannot be colored with any color and a contradiction has been obtained (thus we must backtrack); if the claim is $c(x) = a$, then this will be the next assumption. If the claim is $c(x) \in \{a, b\}$, then we must consider the cases $c(x) = a$ and $c(x) = b$ separately. Finally, the last four columns describe why $c(x) \neq 0$, *etc.*, if this is needed to support the claim. An equation of the form $y \cdot r = x$ means that the forbidden ratio r forbids x and y from being the same color; a 4-tuple (x_1, x_2, x_3, x_4) means that if we colored x the color in question, then this would be a monochromatic solution to $x_1 + x_2 + x_3 = 4x_4$.

For example, the first line of the proof can be read as follows: "$c(6)$ is either 1 or 3 because of forbidden ratios from 3 and 4; we consider these possibilities separately." □

Proof of Lemma 33. For the sake of contradiction, assume that $c(x) = c(4x)$ for some x. As in the previous lemma, we will use forbidden ratios; here, they are 2, $\frac{3}{2}$, $\frac{4}{3}$, and 3. To begin, note that $c(x) = c(4x)$, $c(2x)$, $c(3x)$, and $c(6x)$ must be different colors by these forbidden ratios. Now, $c(\frac{3}{2}x) = c(6x)$ by forbidden ratios from x, $2x$, and $3x$. It follows that $c(\frac{9}{4}x) = c(2x)$ by forbidden ratios from $3x$ and $\frac{3}{2}x$ as well as the solution $(4x, 4x, x, \frac{9}{4}x)$. Then $c(\frac{9}{2}x) = c(x)$ by forbidden ratios from $\frac{9}{4}x$, $3x$, and $\frac{3}{2}x$. Next, $c(9x) = c(2x)$ by forbidden ratios from $\frac{9}{2}x$, $3x$, and $6x$. Further, $c(12x) = c(3x)$ by forbidden ratios from $4x$, $9x$, and $6x$. Also, $c(8x) = c(2x)$ by forbidden ratios from $4x$, $12x$, and $6x$. Nearing the end, $c(\frac{1}{2}x) = c(3x)$ by forbidden ratios from x and $\frac{3}{2}x$ as well as the solution $(8x, \frac{1}{2}x, \frac{1}{2}x, \frac{9}{4}x)$. Finally, there are no possibilities left for $c(\frac{3}{4}x)$ by forbidden ratios from x, $\frac{9}{4}x$, $\frac{1}{2}x$, and $\frac{3}{2}x$. □

Proof of Lemma 34. By induction, it suffices to prove that $c(x) = c(6x) = c(16x)$ for every nonzero rational number x. We have x, $2x$, $3x$, and $4x$ are all different colors by the previous three lemmas. Also, $2x$, $3x$, $4x$, and $6x$ are all different colors by Lemma 31 and Lemma 32. Hence, $c(x) = c(6x)$. By the previous three lemmas, the numbers $2x$, $4x$, $6x$, and $8x$ are all different colors. Hence, $c(8x) = c(3x)$. By the previous three lemmas, $3x$, $4x$, $6x$, and $12x$ are all different colors, so $c(12x) = c(2x)$. By the previous three lemmas, $4x$, $8x$, $12x$, and $16x$ are all different colors, so $c(16x) = c(x)$, completing the proof. □

Proof of Lemma 35. Since $c(x) = c(6x)$ and $(-x, -x, 6x, x)$ is a solution to $x_1+x_2+x_3 = 4x_4$, then $-x$ and x are different colors. Since $c(2x) = c(\frac{3}{4}x)$ and $(-x, 2x, 2x, \frac{3}{4}x)$ is a solution to $x_1 + x_2 + x_3 = 4x_4$, then $-x$ and $2x$ are different colors.

By Lemma 34, $5x$, $10x$, $15x$, and $20x$ are all different colors, so $c(3x) \in \{5x, 10x, 15x, 20x\}$. By Lemma 34, we have $c(3x) = c(8x) = c(18x) = c(\frac{27}{4}x)$. Since $(5x, -x, 8x, 3x)$,

Assumptions	Claim	Why not 0	Why not 1	Why not 2	Why not 3
$c(1)=c(3)=0, c(2)=1, c(4)=2$	$c(6)\in\{1,3\}$	$3\cdot 2=6$		$4\cdot\frac{3}{2}=6$	
$c(6)=1$	$c(8)=3$	$(3,1,8,3)$	$6\cdot\frac{4}{3}=8$	$4\cdot 2=8$	
$c(8)=3$	$c(12)\in\{0,2\}$		$6\cdot 2=12$		$8\cdot\frac{3}{2}=12$
$c(12)=0$	$c(16)=2$	$12\cdot\frac{4}{3}=16$	$(6,2,16,6)$		$8\cdot 2=16$
$c(16)=2$	$c(9)=3$	$12\cdot\frac{3}{4}=9$	$6\cdot\frac{3}{2}=9$	$(16,16,4,9)$	
$c(9)=3$	$c(\frac{9}{2})=2$	$3\cdot\frac{3}{2}=\frac{9}{2}$	$6\cdot\frac{3}{4}=\frac{9}{2}$		$9\cdot\frac{1}{2}=\frac{9}{2}$
$c(\frac{9}{2})=2$	$c(10)\in\{1,3\}$	$(1,1,10,3)$		$(4,4,10,\frac{9}{2})$	
$c(10)=1$	$c(24)=3$	$12\cdot 2=24$	$(10,6,24,10)$	$16\cdot\frac{3}{2}=24$	
$c(24)=3$	$c(32)=0$		$(6,2,32,10)$	$16\cdot 2=32$	$24\cdot\frac{4}{3}=32$
$c(32)=0$	$c(11)=2$	$(32,1,11,11)$	$(2,11,11,6)$		$(24,9,11,11)$
$c(11)=2$	$c(5)=3$	$(12,3,5,5)$	$10\cdot\frac{1}{2}=5$	$(11,\frac{9}{2},\frac{9}{2},5)$	
$c(5)=3$	$c(\frac{11}{2})$!?	$(1,\frac{11}{2},\frac{11}{2},3)$	$(10,10,2,\frac{11}{2})$	$11\cdot\frac{1}{2}=\frac{11}{2}$	$(5,9,8,\frac{11}{2})$
$c(10)=3$	$c(24)=1$	$12\cdot 2=24$		$16\cdot\frac{3}{2}=24$	$(8,8,24,10)$
$c(24)=1$	$c(7)=0$		$(24,2,2,7)$	$(\frac{9}{2},\frac{9}{2},7,4)$	$(10,10,8,7)$
$c(7)=0$	$c(18)=2$	$12\cdot\frac{3}{2}=18$	$24\cdot\frac{3}{4}=18$		$9\cdot 2=18$
$c(18)=2$	$c(\frac{27}{2})$!?	$(1,\frac{27}{2},\frac{27}{2},7)$	$(24,24,6,\frac{27}{2})$	$18\cdot\frac{3}{4}=\frac{27}{2}$	$9\cdot\frac{3}{2}=\frac{27}{2}$
$c(12)=2$	$c(16)=0$		$(6,2,16,6)$	$12\cdot\frac{4}{3}=16$	$8\cdot 2=16$
$c(16)=0$	$c(5)\in\{1,3\}$	$(16,3,1,5)$		$(12,4,4,5)$	
$c(5)=1$	$c(7)\in\{0,3\}$		$(6,7,7,5)$	$(12,12,4,7)$	
$c(7)=0$	$c(9)=3$	$(16,3,9,7)$	$6\cdot\frac{3}{2}=9$	$12\cdot\frac{3}{4}=9$	
$c(9)=3$	$c(14)=2$	$7\cdot 2=14$	$(5,5,14,6)$		$(9,9,14,8)$
$c(14)=2$	$c(20)$!?	$(7,1,20,7)$	$(2,2,20,6)$	$(14,14,20,12)$	$(8,8,20,9)$
$c(7)=3$	$c(10)=2$	$(1,1,10,3)$	$5\cdot 2=10$		$(8,10,10,7)$
$c(10)=2$	$c(14)=0$		$(5,5,14,6)$	$(12,14,14,10)$	$7\cdot 2=14$
$c(14)=0$	$c(20)=3$	$(16,20,20,14)$	$(2,2,20,6)$	$10\cdot 2=20$	
$c(20)=3$	$c(9)=0$		$6\cdot\frac{3}{2}=9$	$12\cdot\frac{3}{4}=9$	$(20,7,9,9)$
$c(9)=0$	$c(13)=2$	$(9,14,13,9)$	$(5,6,13,6)$		$(7,8,13,7)$
$c(13)=2$	$c(17)$!?	$(16,3,17,9)$	$(5,2,17,6)$	$(13,10,17,10)$	$(7,8,17,8)$
$c(5)=3$	$c(\frac{9}{2})=2$	$3\cdot\frac{3}{2}=\frac{9}{2}$	$6\cdot\frac{3}{4}=\frac{9}{2}$		$(5,5,8,\frac{9}{2})$
$c(\frac{9}{2})=2$	$c(10)=1$	$(1,1,10,3)$		$(4,4,10,\frac{9}{2})$	$5\cdot 2=10$
$c(10)=1$	$c(7)=0$		$(10,7,7,6)$	$(\frac{9}{2},\frac{9}{2},7,4)$	$(5,8,7,5)$
$c(7)=0$	$c(\frac{15}{2})$!?	$(7,7,16,\frac{15}{2})$	$10\cdot\frac{3}{4}=\frac{15}{2}$	$(\frac{9}{2},4,\frac{15}{2},4)$	$5\cdot\frac{3}{2}=\frac{15}{2}$
$c(6)=3$	$c(8)=1$	$(3,1,8,3)$		$4\cdot 2=8$	$6\cdot\frac{4}{3}=8$
$c(8)=1$	$c(\frac{9}{2})=2$	$3\cdot\frac{3}{2}=\frac{9}{2}$	$(8,8,2,\frac{9}{2})$		$6\cdot\frac{3}{4}=\frac{9}{2}$
$c(\frac{9}{2})=2$	$c(9)\in\{0,1\}$			$\frac{9}{2}\cdot 2=9$	$6\cdot\frac{3}{2}=9$
$c(9)=0$	$c(12)=2$	$9\cdot\frac{4}{3}=12$	$8\cdot\frac{3}{2}=12$		$6\cdot 2=12$
$c(12)=2$	$c(\frac{3}{2})=3$	$3\cdot\frac{1}{2}=\frac{3}{2}$	$2\cdot\frac{3}{4}=\frac{3}{2}$	$(12,\frac{9}{2},\frac{3}{2},\frac{9}{2})$	
$c(\frac{3}{2})=3$	$c(\frac{9}{4})=1$	$3\cdot\frac{3}{4}=\frac{9}{4}$		$\frac{9}{2}\cdot\frac{1}{2}=\frac{9}{4}$	$\frac{3}{2}\cdot\frac{3}{2}=\frac{9}{4}$
$c(\frac{9}{4})=1$	$c(\frac{27}{8})=0$		$\frac{9}{4}\cdot\frac{3}{2}=\frac{27}{8}$	$\frac{9}{2}\cdot\frac{3}{4}=\frac{27}{8}$	$(\frac{3}{2},6,6,\frac{27}{8})$
$c(\frac{27}{8})=0$	$c(\frac{9}{8})=2$	$(\frac{27}{8},9,\frac{9}{8},\frac{27}{8})$	$\frac{9}{4}\cdot\frac{1}{2}=\frac{9}{8}$		$\frac{3}{2}\cdot\frac{3}{4}=\frac{9}{8}$
$c(\frac{9}{8})=2$	$c(\frac{3}{4})=1$	$1\cdot\frac{3}{4}=\frac{3}{4}$		$\frac{9}{8}\cdot\frac{2}{3}=\frac{3}{4}$	$\frac{3}{2}\cdot\frac{1}{2}=\frac{3}{4}$
$c(\frac{3}{4})=1$	$c(\frac{15}{2})=3$	$(3,3,\frac{15}{2},\frac{27}{8})$	$(\frac{3}{4},\frac{3}{4},\frac{15}{2},\frac{9}{4})$	$(\frac{9}{2},4,\frac{15}{2},4)$	
$c(\frac{15}{2})=3$	$c(\frac{21}{4})$!?	$(\frac{27}{8},\frac{27}{8},\frac{21}{4},3)$	$(\frac{3}{4},2,\frac{21}{4},2)$	$(12,\frac{9}{2},\frac{9}{2},\frac{21}{4})$	$(\frac{15}{2},\frac{15}{2},6,\frac{21}{4})$
$c(9)=1$	$c(10)\in\{1,3\}$	$(1,1,10,3)$		$(4,4,10,\frac{9}{2})$	
$c(10)=1$	$c(7)\in\{0,3\}$		$(10,10,8,7)$	$(\frac{9}{2},\frac{9}{2},7,4)$	
$c(7)=0$	$c(\frac{8}{3})=3$	$(7,1,\frac{8}{3},\frac{8}{3})$	$2\cdot\frac{4}{3}=\frac{8}{3}$	$4\cdot\frac{2}{3}=\frac{8}{3}$	
$c(\frac{8}{3})=3$	$c(\frac{4}{3})=2$	$1\cdot\frac{4}{3}=\frac{4}{3}$	$2\cdot\frac{2}{3}=\frac{4}{3}$		$\frac{8}{3}\cdot\frac{1}{2}=\frac{4}{3}$
$c(\frac{4}{3})=2$	$c(\frac{16}{3})=0$		$8\cdot\frac{2}{3}=\frac{16}{3}$	$4\cdot\frac{4}{3}=\frac{16}{3}$	$\frac{8}{3}\cdot 2=\frac{16}{3}$
$c(\frac{16}{3})=0$	$c(\frac{32}{3})$!?	$\frac{16}{3}\cdot 2=\frac{32}{3}$	$8\cdot\frac{4}{3}=\frac{32}{3}$	$(\frac{4}{3},4,\frac{32}{3},4)$	$(\frac{8}{3},\frac{32}{3},\frac{32}{3},6)$
$c(7)=3$	$c(5)\in\{0,2\}$		$10\cdot\frac{1}{2}=5$		$(7,7,6,5)$
$c(5)=0$	$c(12)=2$	$(5,3,12,5)$	$9\cdot\frac{4}{3}=12$		$6\cdot 2=12$
$c(12)=2$	$c(16)$!?	$(3,1,16,5)$	$8\cdot 2=16$	$12\cdot\frac{4}{3}=16$	$(6,6,16,7)$
$c(5)=2$	$c(12)=0$		$9\cdot\frac{4}{3}=12$	$(4,4,12,5)$	$6\cdot 2=12$
$c(12)=0$	$c(\frac{7}{2})$!?	$(12,1,1,\frac{7}{2})$	$(10,2,2,\frac{7}{2})$	$(5,5,4,\frac{7}{2})$	$7\cdot\frac{1}{2}=\frac{7}{2}$
$c(10)=3$	$c(\frac{16}{3})=0$		$8\cdot\frac{2}{3}=\frac{16}{3}$	$4\cdot\frac{4}{3}=\frac{16}{3}$	$(10,6,\frac{16}{3},\frac{16}{3})$
$c(\frac{16}{3})=0$	$c(\frac{8}{3})=3$	$\frac{16}{3}\cdot\frac{1}{2}=\frac{8}{3}$	$2\cdot\frac{4}{3}=\frac{8}{3}$	$4\cdot\frac{2}{3}=\frac{8}{3}$	
$c(\frac{8}{3})=3$	$c(\frac{4}{3})=2$	$1\cdot\frac{4}{3}=\frac{4}{3}$	$2\cdot\frac{2}{3}=\frac{4}{3}$		$\frac{8}{3}\cdot\frac{1}{2}=\frac{4}{3}$
$c(\frac{4}{3})=2$	$c(\frac{32}{3})$!?	$\frac{16}{3}\cdot 2=\frac{32}{3}$	$8\cdot\frac{4}{3}=\frac{32}{3}$	$(\frac{4}{3},4,\frac{32}{3},4)$	$(\frac{8}{3},\frac{32}{3},\frac{32}{3},6)$

Table 1: The computer-generated proof of Lemma 32.

$(10x, -x, 3x, 3x)$, $(15x, -x, 18x, 8x)$, and $(20x, -x, 8x, \frac{27}{4}x)$ are solutions to $x_1+x_2+x_3 = 4x_4$, then $-x$ and $3x$ are different colors. Hence, $c(-x) = c(4x)$. □

Proof of Lemma 36. By Lemma 34, for integers m and n and nonzero rational number x, we have $c(2^m3^nx) = c(x)$ if and only if $w_5(2^m3^n) \equiv 1 \pmod 5$. By Lemma 35, we have $c(-x) = c(4x)$ for all nonzero rational numbers x. By induction, it suffices to prove for every positive integer p that is not a multiple of 5, we have, $c(px) = c(dx)$ for $d \in \{1, 2, 3, 4\}$ if and only if $p \equiv d \pmod 5$. By Lemma 34, we have already established this for $p = 2^m3^n$ and n and m are integers.

The proof is by induction on the size of p. Suppose $p > 5$ is an integer that is not a multiple of 2, 3, or 5. We write $p = 10a + b$, where a and b are nonnegative integers and $b \in \{1, 3, 7, 9\}$. The induction hypothesis is that, for $d \in \{1, 2, 3, 4\}$, we have $c(p'q) = c(dq)$ for every nonzero rational number q and positive integer p' such that $p' \equiv d \pmod 5$, $p' < p$, and p' is not a multiple of 2, 3, or 5.

By Lemma 34, the four colors $c(5x), c(10x), c(15x), c(20x)$ are distinct for every nonzero rational number x. Hence, $c(x), c(2x), c(3x), c(4x) \in \{c(5x), c(10x), c(15x), c(20x)\}$ for every nonzero rational number x.

Case 1: $p = 10a + 1$ with $a \geq 1$.

We have $c(3x) = c(8x) = c(18x)$. If $a = 1$, then $p = 11$ and $c(px) \neq c(3x)$ since $(3x, 18x, 11x, 8x)$ is a solution to $x_1 + x_2 + x_3 = 4x_4$. If $a > 1$, then $0 < 5a + 6 < p$, and by the induction hypothesis, we have $c(3x) = c((\frac{5a+6}{2})x)$. Since $(px, 3x, 8x, (\frac{5a+6}{2})x)$ is a solution to $x_1 + x_2 + x_3 = 4x_4$, then px and $3x$ are different colors.

We have $c(14x) = c(4x)$. Since $0 < 5a + 4 < p$, then by the induction hypothesis, we have $c(4x) = c((5a + 4)x)$. Since $(px, px, 14x, (5a + 4)x)$ is a solution to $x_1 + x_2 + x_3 = 4x_4$, then px and $4x$ are different colors.

By the induction hypothesis, Lemma 34, and Lemma 35, we have $c(-3x) = c(2x) = c(7x) = c(12x)$. Since $0 < 5a + 9 < p$ for $a > 1$ and $\frac{5a+9}{2} = 7 < 11$ for $a = 1$, then by the induction hypothesis, we have $c((\frac{5a+9}{2})x) = c(2x)$. Since $(5x, 12x, px, (\frac{5a+9}{2})x)$, $(10x, 7x, px, (\frac{5a+9}{2})x)$, $(15x, 2x, px, (\frac{5a+9}{2})x)$, $(20x, -3x, px, (\frac{5a+9}{2})x)$ are solutions to $x_1+x_2+x_3 = 4x_4$ and $c(2x) \in \{c(5x), c(10x), c(15x), c(20x)\}$, then px and $2x$ are different colors. Hence, $c(px) = c(x)$.

Case 2: $p = 10a + 3$ with $a \geq 1$.

Since $0 < 5a + 2 < p$, then by the induction hypothesis, we have $(5a + 2)x$ and $2x$ are the same color. Since $(px, px, 2x, (5a + 2)x)$ is a solution to $x_1 + x_2 + x_3 = 4x_4$, then px and $2x$ are different colors.

Since $p = 10a+3 \geq 13$, then by the induction hypothesis, we have $c(4x) = c(9x) = c(14x)$. For $a = 1$, we have $p = 13$, and $(13x, 14x, 9x, 9x)$ is a solution to $x_1 + x_2 + x_3 = 4x_4$, so $13x$ and $4x$ are different colors. For $a > 1$, we have $0 < 5a + 8 < p$, so by the induction hypothesis, we have $(\frac{5a+8}{2})x$ and $4x$ are the same color. Since $(px, 9x, 4x, (\frac{5a+8}{2})x)$ is a solution to $x_1 + x_2 + x_3 = 4x_4$, then px and $4x$ are different colors.

Since $p = 10a + 3 \geq 13$ and $0 < 5a + 7 < p$, then by the induction hypothesis, we have $c(x) = c(6x) = c(\frac{7}{2}x) = c(\frac{72}{7}x) = c((\frac{5a+7}{2})x)$, $c(\frac{5}{7}x) = c(15x)$, and $c(\frac{15}{2}x) = c(20x)$. Since the tuples $(5x, px, 6x, (\frac{5a+7}{2})x)$, $(10x, px, x, (\frac{5a+7}{2})x)$, $(\frac{5}{7}x, px, \frac{72}{7}x, (\frac{5a+7}{2})x)$, and $(\frac{15}{2}x, px, \frac{7}{2}x, (\frac{5a+7}{2})x)$ are solutions to $x_1 + x_2 + x_3 = 4x_4$, and $c(x) \in \{c(5x), c(10x), c(15x), c(20x)\}$, then px and x are different colors. Hence, $c(px) = c(3x)$.

Case 3: $p = 10a + 7$ with $a \geq 0$.

We have $c(-4x) = c(x)$. Since $0 < 5a + 2 < 10a + 7$, then by the induction hypothesis, we have $c((\frac{5a+2}{2})x) = c(x)$. Since $(px, -4x, x, (\frac{5a+2}{2})x)$ is a solution to $x_1 + x_2 + x_3 = 4x_4$, then px and x are different colors.

We have $c(-2x) = c(3x)$. Since $0 < 5a + 3 < 10a + 7$, then by the induction hypothesis, we have $c(5a+3)x) = c(3x)$. Since $(px, px, -2x, (5a+3)x)$ is a solution to $x_1 + x_2 + x_3 = 4x_4$, then px and $3x$ are different colors.

Note that $c(-16x) = c(-6x) = c(-\frac{9}{4}x) = c(-\frac{8}{3}x) = c(4x)$, $c(\frac{5}{3}x) = c(10x)$, and $c(\frac{5}{4}x) = c(20x)$. Since $0 < 5a + 3 < 10a + 7$, then by the induction hypothesis, we have $(\frac{5a+3}{2})x$ and $4x$ are the same color. Since $(5x, px, -6x, (\frac{5a+3}{2})x)$, $(\frac{5}{3}x, px, -\frac{8}{3}x, (\frac{5a+3}{2})x)$, $(15x, px, -16x, (\frac{5a+3}{2})x)$, and $(\frac{5}{4}x, px, -\frac{9}{4}x, (\frac{5a+3}{2})x)$ are solutions to $x_1 + x_2 + x_3 = 4x_4$ and $c(4x) \in \{c(5x), c(10x), c(15x), c(20x)\}$, then px and $4x$ are different colors. Hence, $c(px) = c(2x)$.

Case 4: $p = 10a + 9$ with $a \geq 1$.

We have $c(x) = c(6x)$. Since $0 < 5a + 6 < p$, then by the induction hypothesis, we have $(5a + 6)x$ and x are the same color. Since $(px, px, 6x, (5a + 6)x)$ is a solution to $x_1 + x_2 + x_3 = 4x_4$, then px and x are different colors.

We have $c(-3x) = c(2x)$. Since $0 < 5a + 4 < p$, then by the induction hypothesis, we have $(\frac{5a+4}{2})x$ and $2x$ are the same color. Since $(px, 2x, -3x, (\frac{5a+4}{2})x)$ is a solution to $x_1 + x_2 + x_3 = 4x_4$, then px and $2x$ are different colors.

Since $p = 10a + 9 \geq 19$, then by the induction hypothesis, we have $c(3x) = c(-2x) = c(-7x) = c(-12x) = c(-17x)$. Since $0 < 5a + 6 < p$, then $(\frac{5a+6}{2})x$ and $3x$ are the same color. Since $(5x, px, -2x, (\frac{5a+6}{2})x)$, $(10x, px, -7x, (\frac{5a+6}{2})x)$, $(15x, px, -12x, (\frac{5a+6}{2})x)$, and $(20x, px, -17x, (\frac{5a+6}{2})x)$ are solutions to $x_1 + x_2 + x_3 = 4x_4$, then px and $3x$ are different colors. Hence, px and $4x$ are the same color. $\square$

4. Minimal colorings of the Nonzero Reals

Let $A = \{a_1, \ldots, a_n\}$ be a nonempty finite multiset of nonzero real numbers. For example, $\{2, 3\}$ and $\{2, 2, 3\}$ are considered to be different multisets. We define a coloring c to be *strongly free* of monochromatic solutions to $a_1x_1 + \cdots + a_nx_n = 0$ if for every nonempty subset $A' \subseteq A$, the coloring c has no monochromatic solutions to the equation $E(A')$ defined by

$$E(A')\colon \sum_{a_i \in A'} a_ix_i = 0.$$

Proposition 37. Suppose $c\colon \mathbb{Q}\setminus\{0\} \to \{1,\ldots,r\}$ is an r-coloring of the nonzero rational numbers with $r > 1$, and c is *strongly free* of monochromatic solutions to $a_1x_1+\cdots+a_nx_n = 0$. Then in ZFC, there exists $2^{2^{\aleph_0}}$ r-colorings of the nonzero real numbers that are strongly free of monochromatic solutions to $a_1x_1 + \cdots + a_nx_n = 0$.

Proof. In ZFC, every vector space has a basis. Viewing $\mathbb{R}$ as a $\mathbb{Q}$-vector space, there is a well-ordered basis $B = \{b_j\}_{j<2^\omega}$ for $\mathbb{R}$ as a $\mathbb{Q}$-vector space. So every real number x has a unique representation as $x = \sum_{j<2^\omega} q_jb_j$, where each q_j is rational and $q_j \neq 0$ for only finitely many j. For a nonzero real number x, let $j(x)$ be the least ordinal j such that q_j is nonzero. For a finite set S of nonzero real numbers, let $j(S) = \min_{x\in S} j(x)$. Let $\Pi = (\pi_j)_{j<2^\omega}$ be any sequence of permutations π_j of the set $\{1,\ldots,r\}$ with π_0 being the identity permutation. Define the coloring C_Π by $C_\Pi(x) = \pi_{j(x)}(c(q_{j(x)}))$.

Suppose $x_1,\ldots,x_n$ are nonzero real numbers and $A' \subseteq A$ is a nonempty subset such that $\sum_{a_i\in A'} a_ix_i = 0$. Let $q_{i,j}$ be the coefficient of b_j in the representation of x_i. Since $\sum_{a_i\in A'} a_ix_i = 0$, then $\sum_{a_i\in A'} a_iq_{i,j(A')} = 0$. Letting $A'' \subset A'$ be those $a_i \in A'$ such that $q_{i.j(A')} \neq 0$, we see that A'' is a nonempty subset of A and $\sum_{a_i\in A''} a_iq_{i,j(A')} = 0$. Since $C_\Pi(x_i) = \pi_{j(A')}(c(q_{i,j(A')}))$ for $a_i \in A''$ and c is strongly free of monochromatic solutions to $a_1x_1 + \cdots + a_nx_n = 0$, then the set $\{x_i\}_{a_i\in A''}$ is not monochromatic and C_Π is strongly free of monochromatic solutions to $a_1x_1 + \cdots + a_nx_n = 0$. Since there are $2^{2^{\aleph_0}}$ nonisomorphic r-colorings of the form C_Π and there are a total of $2^{2^{\aleph_0}}$ nonisomorphic r-colorings of the real numbers, then there are exactly $2^{2^{\aleph_0}}$ colorings of the nonzero real numbers that are strongly free of monochromatic solutions to $a_1x_1 + \cdots + a_nx_n = 0$. □

For p a prime number and $n \geq 3$, each of the colorings $c_{p,n}$ is strongly free of monochromatic solutions to $E(p,n)$, so there are $2^{2^{\aleph_0}}$ different n-colorings of the nonzero real numbers without a monochromatic solution to $E(p,n)$. In general, define the n-coloring $c_{p,v,n}\colon \mathbb{Q}\setminus\{0\} \to \{0,\ldots,n-1\}$ by $c_{p,v,n}(x) \equiv \left\lfloor \frac{v_p(x)}{v} \right\rfloor \pmod{n}$. Notice that if q is a nonzero rational number and $v_p(q) \neq 0$, then $c_{p,v,n}$ with $v = v_p(q)$ is strongly free of monochromatic solutions to $E(q,n)$.

We now turn our attention to the ZF+LM system of axioms. We prove Lemma 13 using the following multiplicative version of a theorem of Steinhaus [Ste20]. For a set A of real numbers, define $A/A = \{a/a' : a, a' \in A, a' \neq 0\}$.

Theorem 38 (Steinhaus's Theorem)**.** If A is a set of real numbers with positive Lebesgue measure, then A/A contains an entire interval $(1-\epsilon, 1+\epsilon)$ for some $\epsilon > 0$.

Proof of Lemma 13:. Suppose for contradiction that all color classes of c are Lebesgue measurable. Since c partitions $\mathbb{R}$ into countably many color classes, and Lebesgue measure is countably additive, then at least one color class C has positive Lebesgue measure. Since $x, da^m b^n x$ are different colors for all integers m and n, then C/C and $\{da^m b^n : m, n \in \mathbb{Z}\}$ are disjoint sets. Since $\log_a b$ is irrational, then the set $\{da^m b^n : m, n \in \mathbb{Z}\}$ contains numbers arbitrarily close to 1. But by Steinhaus's theorem, C/C contains an entire interval around 1, so C/C and $\{da^m b^n : m, b \in \mathbb{Z}\}$ are not disjoint, a contradiction. □

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