A NOTE ON LINEAR RECURRENT MAHLER NUMBERS

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Abstract

A real number $\beta > 1$ is said to satisfy Property (F) if every non-negative number of $\mathbf{Z}[\beta^{-1}]$ has a finite β-expansion. Such numbers are Pisot numbers. Using results of Laurent we prove under technical hypothesis that if G is the linear numeration system canonically associated with a number β satisfying (F), if H is a linear recurrence sequence and $(n_j)_j$ is an unbounded sequence of natural integers such that $\limsup n_j = \infty$, then the real number $0{\bullet}(H_{n_0})_G(H_{n_1})_G\cdots$ does not belong to $\mathbf{Q}(\beta)$. This gives a new family of irrational numbers of Mahler's type.

1 Introduction

Let $h \geqslant 2$ be a natural integer, and let $\mathcal{A} = \{0, 1, \ldots, h-1\}$ be the *canonical alphabet* associated with the base h. Every positive integer $N \in \mathbf{N}$ has a standard h-ary expansion denoted by

$$(N)_h = \varepsilon_{\ell(N)}(N)\varepsilon_{\ell(N)-1}(N)\cdots\varepsilon_0(N)$$

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which expresses that $N = \sum_{k=0}^{\ell(N)} \varepsilon_k(N)h^k$, $\varepsilon_k(N) \in \mathcal{A}$ and $\varepsilon_{\ell(N)}(N) \neq 0$. If $N = 0$, we set $\ell(N) = \varepsilon_0(N) = 0$. If $u = (u_n)_n \in \mathbf{N}^{\mathbf{N}}$ then $(u)_h$ is defined as the infinite concatenation of the words $(u_n)_h$, and

$$0.(u)_h := 0.(u_0)_h(u_1)_h \cdots := \sum_{n=0}^{\infty} \sum_{k=0}^{\ell(u_n)} \frac{\varepsilon_k(u_n)}{h^{\ell(u_0)+\cdots+\ell(u_{n-1})+n+1+\ell(u_n)-k}} \tag{1.1}$$

is the real number whose base h expansion is the infinite concatenation of the words $(u_n)_h$. It belongs to the interval $[0, 1]$. For example, if $u_n = 2^n$ and $h = 10$, then $0.(u)_{10} = 0.1248163264128256512 1024\cdots$.

In a note published in 1981, Mahler [24] proved irrationality of real numbers $0.(u)_{10}$ for $u_n = g^n$, $g \in \mathbf{N}$, $g \geqslant 2$. In fact, his proof — based on the structure of the ring of 10-adic integers $\mathbf{Z}_{10}$ — can be easily generalized to $0.(u)_h$ for the same sequence u and any base $h \geqslant 2$. Later on, several authors have proposed various extensions of his result, using different types of arguments.

The first one, introduced by Niederreiter [25] is based on modulo 1 density. It is proved that if u is such that $\{\log_h(u_n) \pmod 1\,;\, n \geqslant 0\}$ has 0 as accumulation point then $0.(u)_h$ is irrational. Mahler's result follows as a particular case. Moreover, the argument remains valid for $0.(g^{n_1})_h(g^{n_2})_h \cdots$ for any $(n_j)_j$ of positive density, by use of uniform distribution of $\left(n \log_h g\right)_n$ modulo 1. Independently, a similar approach has been followed by Shan, see [31].

The second one applies theorems on diophantine equations. In this direction, let us mention Shan and Wang [32], Becker [7], Becker-Sander [8], Shorey-Tijdeman [34] and Bugeaud [12]. Some of these authors obtain irrationality measures of Mahler's numbers using as a by-product quantitative versions of diophantine results. The idea behind is to prove that if the number were rational, then one could find because of periodicity infinitely many solutions of some specific diophantine equation, the number of solutions of which is known to be bounded. In this way, the introductory paper [32] dealt with the Thue equation $ax^r - by^r = c$. To reach the case of recurrence sequences u, Becker used Schlickewei's more general result on S-unit equations of the form $\alpha_1 y_1 + \cdots + \alpha_n y_n = 1$. It should also be mentioned that Bundschuh [13] proposed a further approach based on Erdős-Strauss criterion and Tschakaloff's transcendental function.

In this note, we follow the diophantine approach and are interested in some non-standard representation of natural numbers. A real number $\beta > 1$ is said to satisfy Property (F) is every non-negative number of $\mathbf{Z}[\beta^{-1}]$ has a finite β-expansion (see Section 2 for definitions). It is known (see [16]) that such a β is a Pisot number. To $\beta > 1$ are associated both a way to represent real numbers of $[0, 1]$ in base β, and a way to expand natural integers in a canonical linear numeration system. The present paper is devoted to the proof of the following result.

Theorem 1. *Let β satisfying Property (F), and let $G = (G_n)_n$ be the canonical linear numeration system associated with β. Let $(H_n)_n$ be a non-degenerate recurrence sequence* *with values in* $\mathbf{Z}$ *and* $(n_j)_j$ *be an unbounded sequence of natural integers. Assume further that* $(H_n)_n$ *is not related with any sequence with minimal polynomial multiple of the minimal polynomial of* β *and dividing the minimal polynomial of* $(G_n)_n$*. Then the real number in base* β*,* $\urcorner = 0{\bullet}(|H_{n_1}|)_G(|H_{n_2}|)_G\cdots$ *does not belong to the field* $\mathbf{Q}(\beta)$*.*

A list of families of Pisot numbers satisfying Property (F) is given in Section 2.

2 Preliminaries

If $W = w_1 \cdots w_k$ and $V = v_1 \cdots v_\ell$ are finite words defined on some alphabet $\mathcal{A}$, then their concatenation is defined as $WV = w_1 \cdots w_k v_1 \cdots v_\ell$. This can be obviously extended to a finite and then to an infinite sequence of finite words $(W_n)_n$ and we can in particular define W^ω as the (infinite) word $WWW\cdots$. As usual, $|W|$ denotes the length of W, that is if $W = w_1 \cdots w_k$ then $|W| = k$. Although we will also use the same symbol for the absolute value, what is meant will always be clear from the context. In the whole paper, the alphabet is an initial interval of $\mathbf{N}$; the natural order relation on $\mathbf{N}$ extends to the set of (non-empty) words by the so-called *lexicographical order*, that will be still denoted by $\leqslant$. The reader may consult [4, Chap. 1] for a meaningful account on "stringology".

2.1 Beta-expansions

For a survey on non-standard representation of numbers we refer to [23, Chap. 7]. According to Rényi [28], real numbers $x \in [0,1]$ can be represented in a real base $\beta > 1$ as an infinite convergent series $x = \sum_{n\geqslant 1} \xi_n \beta^{-n}$. Given x, the digits ξ_n are obtained recursively by the formula $\xi_n = \lfloor \beta T_\beta^{n-1}(x) \rfloor$, where T_β is the so-called β-transformation on the closed interval $[0,1]$ defined by $T_\beta(x) = \beta x - \lfloor \beta x \rfloor$. The infinite word $\xi_1\xi_2\cdots$ is said to be the β-*expansion* of x, and is denoted by $d_\beta(x)$. Note that the digits ξ_n are elements of the alphabet $\mathcal{A} = \{0, 1, \ldots, \lfloor \beta \rfloor\}$. Notice that this representation can be extended to arbitrary $x \in \mathbf{R}_+$ by rescaling; one obtains a series representation $x = \sum_{n\geqslant n_0} \xi_n \beta^{-n}$, $n_0 \in \mathbf{Z}$. It is also possible to expand an element of $\mathbf{R}_+$ thanks to a greedy algorithm [28]. Greedy expansions and β-expansions coincide for any x in $[0,1[$, however they differ for $x = 1$. For instance, if β is the golden mean, $d_\beta(1) = 110^\omega$, but the greedy expansion of 1 is just 1.

A β-*representation* of a real number x is any sequence of rational integers $(d_n)_{n\geqslant 1}$ such that $x = \sum_{n\geqslant 1} d_n \beta^{-n}$. A representation ending in infinitely many zeros is said to be *finite*, and the trailing zeros are omitted.

Denote by $\mathrm{Fin}(\beta)$ the set of non-negative real numbers having a finite β-expansion.

The number β is said to satisfy the *finiteness property* or Property (F) if

$$\text{(F)} \qquad \mathrm{Fin}(\beta) = \mathbf{Z}[\beta^{-1}]_+.$$

Recall that a Pisot number is an algebraic integer whose Galois conjugates lie inside the open unit disc. It is known (see [16]) that if β satisfies (F) then it is a Pisot number and the β-expansion of 1 is finite. Set $d_\beta(1) = a_0 a_1 \cdots a_{d-1}$, with $a_{d-1} \geqslant 1$, and let $P := X^d - a_0 X^{d-1} - \cdots - a_{d-1}$ be the *beta-polynomial* associated with $d_\beta(1)$.

The characterization of numbers satisfying (F) is an open problem. Concerning the difficulty of this problem some accounts can be found in [3]. Up to now, the following sufficient conditions are known.

1. The coefficients satisfy the decreasing condition ([16])

$$a_0 \geqslant a_1 \geqslant \cdots \geqslant a_{d-1} \geqslant 1.$$

   It is known since Brauer [11] that in this case P is the minimal polynomial of the Pisot number β.

2. The coefficients satisfy the condition ([20])

$$a_0 > a_1 + \cdots + a_{d-1}.$$

   Then P is the minimal polynomial of the Pisot number β.

3. The number β is a cubic Pisot unit, of minimal polynomial

$$M := X^3 - aX^2 - bX - 1$$

   with one of the three possibilities

   (a) $0 \leqslant b \leqslant a$, then $d_\beta(1) = ab1$

   (b) $a \geqslant 2$, $b = -1$, then $d_\beta(1) = (a-1)(a-1)01$

   (c) $a \geqslant 0$, $b = a+1$, then $d_\beta(1) = (a+1)00a1$. Note that the smallest Pisot number corresponds to $a = 0$ and $b = 1$.

   These conditions characterize the cubic Pisot units satisfying (F), see [2].

In the sequel β satisfies Property (F) and $P = X^d - a_0 X^{d-1} - \cdots - a_{d-1}$ is its beta-polynomial. Note that $\lfloor \beta \rfloor = a_0$. Therefore the alphabet of the expansions coincides with $\mathcal{A} = \{0, 1, \ldots, a_0\}$. Real numbers with finite β-expansions have two representations, namely $\xi_1 \cdots \xi_n$ itself and $\xi_1 \cdots \xi_{n-1}(\xi_n - 1)(a_0 a_1 \cdots a_{d-2}(a_{d-1} - 1))^\omega$. The second one is called *improper*, and plays a role in the proof of our result too. According to [26] a sequence of non-negative integers $(\xi_n)_n$ is the β-expansion of some number $x \in [0, 1[$ if and only if

$$\forall n \geqslant 1 \ : \ \xi_n \xi_{n+1} \cdots < (a_0 a_1 \cdots a_{d-2}(a_{d-1} - 1))^\omega. \tag{2.1}$$

It follows that the β-expansion of a number $x \in [0,1[$ is the greatest in the lexicographical order of all the β-representations of x.

Infinite words $\xi_1\xi_2\cdots$ that satisfy

$$\forall n \geqslant 1 \ : \ \xi_n\xi_{n+1}\cdots \leqslant (a_0a_1\cdots a_{d-2}(a_{d-1}-1))^\omega \tag{2.2}$$

are said to be *β-admissible*. Equivalently,

$$\forall n \in \mathbf{N}, \quad \xi_{n+1}\xi_{n+2}\cdots\xi_{n+d} \leqslant a_0a_1\cdots(a_{d-1}-1). \tag{2.3}$$

In other terms, finite words $\xi_1\xi_2\cdots\xi_k$ are β-admissible if and only if $\xi_1\xi_2\cdots\xi_k0^\omega$ is. Numbers with a β-representation satisfying (2.2) but not (2.1) actually have a finite β-expansion. Remark that $d_\beta(1) = a_0a_1\cdots a_{d-1}$ is **not** β-admissible, and that the only β-admissible representation of 1 is $(a_0a_1\cdots a_{d-2}(a_{d-1}-1))^\omega$.

From now on, if $W = w_1w_2\cdots$ is a word on the alphabet $\{0,\ldots,a_0\}$, then we denote $0.W = 0.w_1w_2\cdots := \sum_{k=1}^{\infty} w_k\beta^{-k}$. It should be noted that the series above always converges, although in general W is not the β-expansion of its numerical value.

The set of β-expansions is endowed with the product topology and the lexicographical order. The application $\mathcal{T}_\beta$ which associates to any real number x in $[0,1[$ its β-expansion $(\lfloor \beta T_\beta^n(x)\rfloor)_{n\geqslant 0}$ is increasing, and not continuous due to the phenomenon of improper representations, but its inverse $\mathcal{T}_\beta^{-1}$ is continuous. We have the useful (although straightforward) property:

Lemma 1. *Let $u = u_1u_2\cdots$ and $v = v_1v_2\cdots$ be two different β-admissible words with $0.u \geqslant 0.v$. Let k be the greatest positive integer such that $u_j = v_j$ for all $j < k$. Then, for any non-negative integer m, if $0.u - 0.v < \beta^{-k-md}$, then the three following assertions are satisfied:*

- $u_k - v_k = 1$,
- $u_{k+1}\cdots u_{k+md} = 0^{md}$, *and*
- $v_{k+1}\cdots v_{k+md} = (a_0a_1\cdots(a_{d-1}-1))^m$.

Conversely, those assumptions together entail $0.u - 0.v < 2\beta^{-k-md}$.

Recall the following result (Proposition 2 in [16]).

Lemma 2. *Let β be a Pisot number. Denote $\mathrm{Fin}_N(\beta)$ the set of elements having β-expansion of the form $\sum_{1\leqslant n\leqslant N}\xi_n\beta^{-n}$. There exists a positive integer L depending only on β having the following property. Let x and y in $\mathrm{Fin}_N(\beta)$, $x > y$. If $x+y$ (resp. $x-y$) belongs to $\mathrm{Fin}(\beta)$, then $x+y$ (resp. $x-y$) belongs to $\mathrm{Fin}_{N+L}(\beta)$.*

From this result one derives:

Lemma 3. *Suppose that β satisfies (F). If x is of the form $\sum_{1\leqslant n\leqslant N} x_n\beta^{-n}$ and y is of the form $\sum_{1\leqslant n\leqslant N} y_n\beta^{-n}$, with x_n and $y_n \in \{0,1,\ldots,a_0\}$, and $x > y$, then there exists Λ depending only on β such that $x \pm y \in \mathrm{Fin}_{N+\Lambda}(\beta)$.*

Proof. The sum $x = \sum_{1 \leqslant n \leqslant N} x_n \beta^{-n}$ can be splitted into d terms

$$x = \sum_{\substack{1 \leqslant n \leqslant N \\ n \equiv 1 \pmod d}} x_n \beta^{-n} + \cdots + \sum_{\substack{1 \leqslant n \leqslant N \\ n \equiv d \pmod d}} x_n \beta^{-n}.$$

For each i, $1 \leqslant i \leqslant d$, denote $x^{(i)} = \sum_{\substack{1 \leqslant n \leqslant N \\ n \equiv i \pmod d}} x_n \beta^{-n}$. Then the word $x_1^{(i)} \cdots x_N^{(i)}$ defined by $x_n^{(i)} = x_n$ if $n \equiv i \pmod d$, and $x_n^{(i)} = 0$ otherwise, is the β-expansion of $x^{(i)}$, and, by Lemma 2, x belongs to $\mathrm{Fin}_{N+L(d-1)}(\beta)$. The result follows with $\Lambda = Ld$. □

2.2 Linear numeration systems

To a number β with $d_\beta(1) = a_0 a_1 \cdots a_{d-1}$ one can canonically associate a linear recurrent sequence of integers $G := (G_n)_n$ defined by

$$\begin{cases} G_0 = 1, \ G_k = a_0 G_{k-1} + \cdots + a_{k-1} G_0 + 1 & \text{for } k < d \\ G_{n+d} = a_0 G_{n+d-1} + \cdots + a_{d-1} G_n & \text{for } n \geqslant 0. \end{cases} \tag{2.4}$$

Any non-negative integer N can be uniquely *expanded in the basis* G in the following sense :

$$N = \sum_{k=0}^{\ell(N)} \varepsilon_k(N) G_k, \tag{2.5}$$

and

$$\forall k \leqslant \ell(N) \ : \ \varepsilon_k(N) \cdots \varepsilon_0(N) 0^\omega < (a_0 \cdots (a_{d-1} - 1))^\omega. \tag{2.6}$$

The digits $\varepsilon_k(N)$ are computed by the greedy algorithm (see [14], [10], [23]). The *G-expansion* of N is the finite word $(N)_G = \varepsilon_{\ell(N)}(N) \varepsilon_{\ell(N)-1}(N) \cdots \varepsilon_0(N)$ on the canonical alphabet $\mathcal{A} = \{0, 1, \ldots, a_0\}$. Finite words $\varepsilon_m \varepsilon_{m-1} \cdots \varepsilon_0$ that satisfy (2.6) are said to be *G-admissible*. We call *canonical linear numeration system associated with* β such a sequence G and the related way to represent integers. From [10] it is known that a finite word is G-admissible if and only if it is β-admissible, as expressed by (2.6).

The most popular example is the so-called *Fibonacci numeration system* associated with the golden mean $\beta = \frac{1+\sqrt{5}}{2}$ with $G_1 = 2$ and $G_{n+2} = G_{n+1} + G_n$. The corresponding G-admissible strings, called the *Zeckendorff expansions*, are the finite sequences of 0's and 1's where the pattern 11 does not occur.

Note that, even if a finite word $W = w_{k-1} \cdots w_0$ is not G-admissible, its *numerical value* in basis G can be defined as follows: $\pi(W) = \sum_{i=0}^{k-1} w_i G_i$.

From now on we fix some notations. For β a Pisot number satisfying (F) with $d_\beta(1) = a_0 a_1 \cdots a_{d-1}$, $P = X^d - a_0 X^{d-1} - \cdots - a_{d-1}$ is the beta-polynomial associated with β, and $M = \prod_{j \leqslant s}(X - \beta_j)$ is the minimal polynomial of $\beta = \beta_1$, of degree s. The minimal polynomial of the linear recurrence sequence $(G_n)_n$ defined by (2.4) is denoted $Q = \prod_{i \leqslant q}(X - \alpha_i)$.

Lemma 4. *With the notations above, it holds $M \mid Q$ and $Q \mid P$.*

Proof. By definition of $d_\beta(1)$, β is a root of P, hence M divides P. Furthermore, by non-negativity of the coefficients a_j, $|1 - z^{-d}P(z)| < 1$ for any complex number $z \neq \beta$ with $|z| \geqslant \beta$. Thus β is the dominating root of P. Computing $\beta P'(\beta) - dP(\beta) > 0$ ensures that this root is simple. Let $g(z) = \sum_{n\geqslant 0} G_n z^n$ be the generating function of G. The recurrence relation defining G yields

$$g(z) = \frac{1 - z^d}{(1-z)z^d P(z^{-1})},$$

from which one derives the existence of a non-zero constant B such that $G_n \sim B\beta^n$ (see [18], [19]). This proves that Q satisfies $M \mid Q$. Since P is the characteristic polynomial of the recurrence relation (2.4), we have $Q \mid P$. □

Example 1. There are two families of cubic Pisot units β satisfying (F) for which the minimal polynomial does not coincide with the beta-polynomial, Cases 3 (b) and (c) in Section 2 above.
1) For $M = X^3 - aX^2 + X - 1$ and $a \geqslant 2$, one has $P = X^4 - (a-1)X^3 - (a-1)X^2 - 1 = (X+1)M$. Computing $G_0 = 1$, $G_1 = a$, $G_2 = (a-1)G_1 + a = a^2$ yield

$$G_3 = (a-1)G_2 + (a-1)G_1 + 1 = a^3 - a + 1 = aG_2 - G_1 + G_0,$$

hence the minimal polynomial of $(G_n)_n$ is indeed M.
2) The second family is given by $M = X^3 - aX^2 - (a+1)X - 1$ and $a \geqslant 0$. Then $P = X^5 - (a+1)X^4 - aX - 1 = (X - e^{i\pi/3})(X - e^{-i\pi/3})M$ and one verifies that $G_3 \neq aG_2 + (a+1)G_1 + G_0$, hence the minimal polynomial of $(G_n)_n$ is P.

2.3 About Diophantine equations

Finally, we recall facts from the theory of recurrence sequences and introduce vocabulary from [29]. For a general reference on recurrence sequences, we refer to [33]. Let $(H_n)_n$ be a non constant linear recurrence sequence with coefficients and values in $\mathbf{Z}$; let $Q := \prod_{1\leqslant i\leqslant q} (X - \alpha_i)^{\sigma_i}$ be its minimal polynomial and let $\mathbf{K} := \mathbf{Q}(\alpha_1, \ldots, \alpha_q)$ be the splitting field of the polynomial Q over the rationals. Then Q has degree $s = \sum_{1\leqslant i\leqslant q} \sigma_i$ and there exists a unique q-tuple $(f_1, \ldots, f_q) \in \mathbf{K}[X]^q$ with $\deg(f_i) < \sigma_i$ for $i = 1, \ldots, q$ such that for all non-negative integer n, one has

$$H_n = \sum_{i=1}^{q} f_i(n)\alpha_i^n. \tag{2.7}$$

Then for all i we have $f_i \neq 0$: indeed, suppose $f_i = 0$ for some i; let Q_i be the minimal polynomial of α_i, and let α_j such that $Q_i(\alpha_j) = 0$. Let $\tau \in \mathrm{Gal}\big(\mathbf{K}/\mathbf{Q}\big)$ such that

$\tau(\alpha_i) = \alpha_j$. Applying τ to H_n and using uniqueness of representation (2.7), we get $f_j = 0$; thus $(H_n)_n$ is a solution of the recurrence relation of minimal polynomial Q/Q_i.

The sequence $(H_n)_n$ is said to be *non-degenerate* if its minimal polynomial Q satisfies the following condition: if $i \neq j$ then α_i/α_j is not a root of unity and if $\deg Q = 1$ then its unique root is not a root of unity (*i.e.* here neither 1 nor -1).

Given $(H_n)_n$ with minimal polynomial Q as above, let us define $\widetilde{Q}$ to be Q itself if one of the roots of Q is a root of unity and $(X-1)Q$ otherwise. After Laurent (see [21] and [22]), for linear recurrence sequences $(H_n^{(i)})_n$, $i = 1$ or 2 with $\widetilde{Q^{(i)}} = \prod_{1 \leqslant j \leqslant q^{(i)}} (X - \alpha_j^{(i)})^{\sigma_j^{(i)}}$, $(H_n^{(1)})_n$ and $(H_n^{(2)})_n$ are said to be *related* if $q^{(1)} = q^{(2)}$ and if there exist non-zero integers s and t such that after a suitable reordering of the roots of $Q^{(i)}$ one has for any j: $\left(\alpha_j^{(1)}\right)^s = \left(\alpha_j^{(2)}\right)^t$ (for $q^{(i)} = 1$, one retrieves the notion of multiplicative independence of $\alpha^{(1)}$ and $\alpha^{(2)}$, which plays a key role in the previous works on Mahler numbers). Then we have:

Lemma 5. (Part of Proposition 1, [22]) *For non-degenerate linear recurrence sequences $(H_n^{(1)})_n$ and $(H_n^{(2)})_n$, the equation $H_\ell^{(1)} = H_m^{(2)}$ has only finitely many solutions, unless $\left(H_n^{(1)}\right)_n$ and $\left(H_n^{(2)}\right)_n$ are related.*

This will be our principal tool to prove Theorem 1.

3 Non-periodicity

All the irrationality proofs recalled in Section 2 are based on the characterization of rational numbers as the real numbers whose base h expansion is eventually periodic. Thereby the arithmetic property of $0{\scriptstyle\bullet}(u)_h$ is viewed as a combinatorial property of the infinite word $(u)_h$. A similar characterization holds for the non-integral base we are interested in and is due independently to Bertrand [9] and Schmidt [30]: for a Pisot number β, the set of the real numbers $x \in [0,1[$ whose β-expansion is eventually periodic is exactly $[0,1[\cap \mathbf{Q}(\beta)$. The following lemma is an important technical step towards an application of this result.

Lemma 6. *Assume that β satisfies Property (F). Let $(G_n)_n$ be the linear numeration system associated with β. Let $(H_n)_n$ be a non-degenerate recurrence sequence with values in $\mathbf{Z}$ which is not related with any sequence with minimal polynomial multiple of M and dividing P (notation of Section 2.2). Let W_1, W_2 and W_3 be finite G-admissible words on the alphabet $\{0, 1, \ldots, a_0\}$, with W_2 not empty. Then there exist finitely many couples of non-negative integers (n, m) such that the expansion of $|H_n|$ in basis G is $W_1 W_2^m W_3$.*

Proof. It is not unlikely that $(G_n)_n$ is degenerated (*cf.* Example 1). We define an equivalence relation on the (not necessarily different) roots α_i of $Q = \prod_{i \leqslant q} (X - \alpha_i)$ by

$\alpha_i \sim \alpha_j$ if and only if α_i/α_j is a root of unity. Note that if ℓ is any common multiple of the order of those roots of unity, then the minimal polynomials of the subsequences $(G_{\ell n+u})_n$, for $u = 0, \ldots, \ell - 1$, divide $\prod_{C \in \mathcal{C}} (X - \alpha_C^\ell)^{\mathrm{Card}(C)}$, where the product is taken on the quotient set $\mathcal{C}$ of the equivalence relation (the powers α_C^ℓ are independent of the choice of the root in the equivalence class C). Moreover, by classical facts on symmetric polynomials, $R_\ell = \prod_{j \leqslant s}(X - \beta_j^\ell)$ is a monic element of $\mathbf{Z}[X]$. Since all but one of its roots have modulus smaller than 1, it is irreducible on $\mathbf{Z}[X]$. Hence it is the minimal polynomial of the Pisot number β_1^ℓ and a divisor of the minimal polynomial of the subsequences $(G_{\ell n+u})_n$, for all $u = 0, \ldots, \ell - 1$ (using Lemma 4). Those subsequences are thus non-trivial and non-degenerate.

Let us note $W_2 = \varepsilon_{h-1} \cdots \varepsilon_1 \varepsilon_0$ and $W_1 = \zeta_{t-1} \cdots \zeta_1 \zeta_0$, and suppose that $|W_3| = r$. Let $a = \pi(W_3)$ and $V_m = \pi(W_1 W_2^m W_3)$. Then

$$V_m = a + \sum_{i=0}^{h-1} \varepsilon_i \sum_{k=0}^{m-1} G_{r+kh+i} + \sum_{i=0}^{t-1} \zeta_i G_{r+mh+i}.$$

Putting $G_n = \sum_{j=1}^q f_j \alpha_j^n$ in this formula and using geometric summations yields

$$V_m = C + \sum_{j=1}^{d} \rho_j(m) \alpha_j^{hm}, \tag{3.1}$$

with constant $C \in \mathbf{K}$ and $\rho_j \in \mathbf{K}[\mathbf{X}]$, where $\mathbf{K}$ is the splitting field of Q over the rationals. We shall apply Lemma 5 to prove that the equation $H_n = \pm V_m$ has finitely many solutions.

Let c be the l.c.m. of the order of the roots of unity arising from the minimal polynomial Q of $(G_n)_n$. For any $u = 0, \ldots, c-1$, Relation (3.1) and the discussion above show that the minimal polynomial of the sequence $(V_{cm+u})_m$ is a multiple of $R_{c\ell}$ and is non-degenerate. Assume that the sequence $(H_n)_n$ is not related with any sequence with minimal polynomial multiple of M and dividing P. Then neither are $(\pm V_{cm+u})_m$ and $(H_n)_n$, and Lemma 5 together with the pigeonhole principle give the finiteness result. $\square$

From now on, we consider the real number $\daleth = 0.(|H_{n_j}|)_G$ in accordance with the hypothesis and notations of Theorem 1 and look first at the infinite word $\Xi = (|H_{n_1}|)_G (|H_{n_2}|)_G \cdots = \xi_1 \xi_2 \cdots$ on the alphabet $\{0, 1, \ldots, a_0\}$. If Ξ were eventually periodic with period V, say, then one would have $(|H_{n_j}|)_G = W_1 V^m W_3$ with W_1, W_3 and m depending on j, W_1 and W_3 being a suffix and a prefix of V respectively. By the pigeonhole principle, this would contradict Lemma 6. Hence Ξ is not eventually periodic. In the case of h-ary expansions, this is sufficient to get irrationality.

In our case, there is a further problem coming from the fact that Bertrand and Schmidt's characterization deals with β-expansions, although Ξ has no reason to be a β-expansion. Indeed, the finite words $(|H_n|)_G$ are β-admissible, but their concatenation may introduce forbidden patterns (take for instance the Zeckendorf expansion; strings 101 and 100 are admissible, but 101100 is not). Thus one has to normalize the word Ξ to get the β-expansion of $\daleth$, and prove that this normalized expansion is not eventually periodic.

4 Normalization

Since the finite words $(|H_n|)_G$ are β-admissible, we can hope that the normalization does not perturb to much the word $(|H_{n_1}|)_G(|H_{n_2}|)_G\cdots$. In particular, if it only affects the most significant digits of the $(|H_{n_j}|)_G$'s, one should be able to apply Lemma 6 again. Of peculiar interest is a subclass of linear recurrent systems of numeration for which it is possible to follow the normalization process explicitly. These systems are called "confluent". Confluent linear numeration systems are those for which there is no propagation of the carry to the right, see Example 2 below. Since this special case also enlights the subsequent difficulties of the general case, we treat it before.

4.1 Proof of Theorem 1 for confluent systems

After [15], if G is given by (2.4), the system of numeration defined by G is said to be *confluent* if $a_0 = a_1 = \cdots = a_{d-2} \geqslant a_{d-1}$. For simplicity, we note $a = a_0$ and $b = a_{d-1}$. We import from [16] the following convenient notation: if $w = w_1w_2\cdots$ and $v = v_1v_2\cdots v_s$ are words of length $+\infty$ and s respectively, if n is a positive integer, then $w \underset{n}{\oplus} v$ is the infinite word defined by

$$w \underset{n}{\oplus} v = w_1\cdots w_{n-1}(w_n + v_1)\cdots(w_{n+s-1} + v_s)w_{n+s}w_{n+s+1}\cdots$$

Proof. Since the word Ξ is written on the alphabet $\{0,\ldots,a\}$, a factor $\xi_{n+1}\xi_{n+1}\cdots\xi_{n+d}$ of Ξ is not β-admissible if and only if it has the form $a^{d-1}c$, with $b \leqslant c \leqslant a$. The algorithm of normalization proceeds as follows:

Procedure 1. If all the factors $\xi_{n+1}\xi_{n+2}\cdots\xi_{n+d}$ satisfy Condition (2.3), then $\xi_1\xi_2\cdots$ is β-admissible. If not, let m be the smallest integer n such $\xi_{n+1}\xi_{n+2}\cdots\xi_{n+d} = a^{d-1}c$, with $b \leqslant c \leqslant a$. Then transform Ξ into $\Xi' = \xi'_1\xi'_2\cdots$ with $\Xi' = \Xi \underset{m}{\oplus} 1\,\overbrace{(-a)\cdots(-a)}^{d-1}(-b)$, *i.e.*

$$\xi'_i = \begin{cases} \xi_i & \text{if } i \leqslant m-1 \text{ or } i \geqslant m+d+1; \\ \xi_m + 1 & \text{if } i = m; \\ 0 & \text{if } m+1 \leqslant i \leqslant m+d-1; \\ \xi_{m+d} - b & \text{if } i = m+d. \end{cases} \tag{4.1}$$

The relation $M(\beta)=0$ ensures that $0.\xi_1\xi_2\cdots=0.\xi'_1\xi'_2\cdots$.

Procedure 2. After one iteration of Procedure 1, there are two possibilities. Either $\xi'_1\cdots\xi'_{m+2d-1}$ is β-admissible, or $\xi'_{m-d+1}\cdots\xi'_m=a^{d-1}b$. The latter happens if and only if $\xi_{m-d+1}\cdots\xi_m=a^{d-1}(b-1)$. A further application of Procedure 1 gives then $\xi''_1\xi''_2\cdots$ with $\xi''_i=0$ at least for $m-d+1\leqslant i\leqslant m+d-1$, and $\xi''_1\cdots\xi''_{m+2d-1}$ is not admissible if and only if $\xi''_{m-2d+1}\cdots\xi''_{m-d}=a^{d-1}(b-1)$, and so on: Procedure 1 is applied k times, say, giving a word $\Xi^{(k)}=\xi_1^{(k)}\xi_2^{(k)}\cdots$, until $\xi_1^{(k)}\cdots\xi_{m+2d-1}^{(k)}$ is β-admissible. This happens if and only if $\xi_{m-kd+1}\cdots\xi_{m+d}=\big(a^{d-1}(b-1)\big)^{k-1}a^{d-1}\xi_{m+d}$ and $\xi_{m+d}\geqslant b$; then, $\xi^{(k)}_{m-(k-1)d+1}\cdots\xi^{(k)}_{m+d}=0^{kd-1}(\xi_{m+d}-b)$. We call "Procedure 2" these successive applications of Procedure 1.

The algorithm consists of successive applications of Procedure 2. Call m_s the index of the s-th iteration of Procedure 2. After one iteration of Procedure 2 we have that $\xi_j^{(k+\ell)}=\xi_j^{(k)}$ for any integer j with $j\leqslant m_1+d-1$ and any non-negative integer ℓ. Hence the indices where Procedure 2 is applied satisfy $m_{s+1}\geqslant m_s+d$. Therefore the algorithm converges, say to $Z=\zeta_1\zeta_2\cdots$. For each s, $0.0^{m_s+d-1}\xi^{(k)}_{m_s+d}\xi^{(k)}_{m_s+d+1}=0.0^{m_s+d-1}\zeta_{m_s+d}\zeta_{m_s+d+1}$, which ensures that there exist infinitely many j with $\zeta_j\neq 0$.

Suppose now that Z is eventually periodic with period V of length p. We have seen that $V\neq 0^p$. Thus Procedure 2 consists of boundedly many iterations of Procedure 1. Furthermore, the β-admissibility of $(|H_{n_j}|)_G$ for all $j\geqslant 1$ shows that the first iteration of Procedure 1 in any application of Procedure 2 concerns a string $\xi_{m_s+1}\xi_{m_s+2}\cdots\xi_{m_s+d}$ which sits astride (at least) $(|H_{n_j}|)_G$ and $(|H_{n_{j+1}}|)_G$. Hence, from Ξ to Z, only boundedly many digits of each $(|H_{n_j}|)_G$ have been modified: at most the prefix of length $d-1$ and a suffix of bounded length. Thus Lemma 6 can be applied and Z is not eventually periodic. Contradiction. $\square$

4.2 Proof of Theorem 1 in the general case

Roughly speaking, the procedure of normalization in confluent systems gives rise to a carry propagation to the left, the length of which is explicit, and does not affect anything at the right of the modified string. That comes from a convenient fact: the substraction $0.0^n\xi_{n+1}\xi_{n+2}\cdots\xi_{n+d}-0.0^na^{d-1}b$ can be handled digitwise. But this does not hold in general, as the following example shows.

Example 2. Take $d=3$ and $(a_0,a_1,a_2)=(3,2,1)$, that is, $G_{n+3}=3G_{n+2}+2G_{n+1}+G_n$. Then

$$0.0330=0.0321+0.0010-0.0001=0.1000+0.000321-0.0001=0.100221.$$

Hence, for instance, the normalization of $0.0330(120)^\omega$ gives $0.1(003)^\omega$; thus a single non β-admissible string 330 gives rise to infinitely many modifications to the right.

Proof. Let $\daleth = 0.(|H_{n_1}|)_G(|H_{n_2}|)_G \cdots = 0.\Xi$. Let $(t_j)_j$ be an increasing sequence of positive integers such that $|H_{n_{t_j}}|$ goes to infinity. Denote by N_k the length of $(|H_{n_1}|)_G \cdots (|H_{n_{t_k-1}}|)_G$. We will get the β-expansion of $\daleth$ by executing

$$\begin{array}{cllll}
 & 0. & (|H_{n_1}|)_G \cdots (|H_{n_{t_1-1}}|)_G & & \\
+ & 0. & 0^{N_1} & (|H_{n_{t_1}}|)_G \cdots (|H_{n_{t_2-1}}|)_G & \\
+ & 0. & 0^{N_1} & 0^{N_2-N_1} & (|H_{n_{t_2}}|)_G \cdots (|H_{n_{t_3-1}}|)_G \\
+ & 0. & \cdots & \cdots & \cdots \\
\vdots & & \vdots & \vdots & \vdots \quad \cdots
\end{array}$$

The additions are performed one after each other following the order of the rows above, and the result is normalized. Henceforth, after k steps, one obtains a finite word A_k which is the β-expansion of $0.(|H_{n_1}|)_G \cdots (|H_{n_{t_{k+1}-1}}|)_G$. For short, we set $p_k = n_{t_k}$. From Lemma 3, an immediate induction on k shows that for every k the length of A_k is at most $N_k + \Lambda$. By completing with zeros on the right, we assume from now on that $|A_k| = N_k + \Lambda$. The $(k-1)$-th addition looks like that

$$\begin{array}{cll}
 & 0. & \overbrace{A_{k-1}}^{N_{k-1}+\Lambda} \\
+ & 0. & 0^{N_{k-1}} \quad \overbrace{(|H_{p_{k-1}}|)_G \cdots (|H_{p_k-1}|)_G}^{N_k-N_{k-1}} \\
= & 0. & \underbrace{\qquad\qquad A_k \qquad\qquad}_{N_k+\Lambda}
\end{array} \tag{4.2}$$

We claim that if $\daleth \in \mathbf{Q}(\beta)$, then there exists a positive number Γ dependent only on β and $\daleth$ such that the prefix of A_{k-1} of length $N_{k-1} - \Gamma$ is a prefix of A_k too.

Proof of the claim: if $\daleth$ indeed is in $\mathbf{Z}[\beta^{-1}]_+$, then its β-expansion is finite, but we choose its improper representation, ending by $(a_0 \cdots a_{d-2}(a_{d-1}-1))^\omega$. Now, let ℓ be the length of the longest factor of Ξ of the form 0^ℓ. By hypothesis ℓ is finite. We have that $0 < \daleth - 0.A_k \ll \beta^{-N_k}$. By Lemma 1, there exists a constant Γ such that for all k, the chosen representation of $\daleth$ coincides with A_k at least for the first $N_k - \Gamma$ digits. Moreover, write $A_k = a_1^{(k)} \cdots a_{N_k}^{(k)}$. Since the real sequence $(0.A_k)_k$ is increasing, then for any j, the sequence of words $a_1^{(k)} \cdots a_j^{(k)}$ is increasing too with respect to the lexicographic order. And the claim follows.

In particular, the sequence of words $(A_k 0^\omega)_k$ converges to an admissible word A, which is the β-expansion of $\daleth$. By hypothesis on $\daleth$ there are finite words W and V such that $A = WV^\omega$. More precisely, one can write $A_k = WV^{m_k}R_k$ with $\Lambda \leqslant |R_k| \leqslant \Lambda + \Gamma + |V| - 1$. On the other hand, let B_k be the admissible word such that

$$\daleth - 0.A_k = 0.0^{N_k}B_k = 0.0^{N_k}(|H_{p_k}|)_G(|H_{p_k+1}|)_G \cdots (|H_{p_{k+1}}|)_G \cdots$$

Then there exists a constant Π such that for every k, the words B_k and $(|H_{p_k}|)_G$ have a common prefix of length $|(|H_{p_k}|)_G| - \Pi$. Otherwise, as above, Lemma 1 would ensure the existence of arbitrary large strings of the form 0^ℓ in $(|H_{p_k}|)_G$, which would yield arbitrary large strings of the same type in A, and this cannot happen.

Since $\urcorner - 0.A_k = 0.0^{N_k}B_k$, we have $0.WV^\omega - 0.WV^{m_k}R_k = 0.0^{N_k}B_k$, which can be renormalized as

$$0.V^\omega - 0.R_k = 0.0^{|R_k|}B_k. \tag{4.3}$$

Since $0.V^\omega - 0.R_k \in \mathbf{Q}(\beta)$, B_k is eventually periodic. But there are only finitely many distinct R_k, hence the lengths of the preperiod and of the period of B_k are both bounded. The existence of the integer Π in the paragraph above and the pigeonhole principle yield a contradiction with Lemma 6. □

5 The related case. Examples.

When $(H_n)_n$ and $(G_n)_n$ are related, there should be hard to get such a systematic result. Actually, we failed in extending Lemma 6 even to simply related sequences (we refer to the papers of Laurent for a definition). Whatever it would look like, a result taking care of related sequences would be probably much more complicated.

Example 3. Indeed, let $(G_n)_n$ be an arbitrary recurrence sequence satisfying the conditions of Section 2.2. Let us take $H_n = G_{n+2d} + G_d$. Then any concatenation of words of the type $(H_n)_G$ is admissible. Hence $\urcorner \notin \mathbf{Q}(\beta)$, although $(H_n)_n$ and $(G_n)_n$ are related. Using the proof of Section 4.2, it extends to sequences of the type $H_n = G_n + C$ where C is an arbitrary non-negative integer and more generally to any type of sequence $(H_n)_n$ whose G-expansion contains sufficiently large strings of zeros. On the other hand, let $p \geqslant 2$ and $H_n := \sum_{k=1}^n G_{kp-1}$. Then if $(H_n)_n$ (resp. $(G_n)_n$) has minimal polynomial Q_1 (resp. $Q = \prod_{1\leqslant j\leqslant d}(X-\alpha_j)$), then $\widetilde{Q_1} = \prod_{1\leqslant j\leqslant d}(X-\alpha_j^p)$ or $\widetilde{Q_1} = (X-1)\prod_{1\leqslant j\leqslant d}(X-\alpha_j^p)$ and $\urcorner = 0.\big(H_{n_j}\big)_G \in \mathbf{Q}(\beta)$ for any choice of $(n_j)_j$.

Example 4. (Continuation of Example 3.) An extreme case of relatedness is that of equality between the polynomials $\widetilde{Q}$ defined in Section 2.3 and the previous example presents some cases of that type. Equivalently,

$$G_n = \sum_{j=1}^q f_j(n)\alpha_j^n \quad \text{and} \quad H_n = \rho_0(n) + \sum_{j=1}^q \rho_j(n)\beta_j^n,$$

with $(f_1,\ldots,f_q,\rho_0,\ldots,\rho_q) \in \mathbf{K}[X]^{2q+1}$, where $\mathbf{K} = \mathbf{Q}(\alpha_1,\ldots,\alpha_q)$. Since $\alpha_1 = \beta$ is a simple dominating root of Q and $G_n \sim C\beta^n$, the polynomial f_1 is indeed a non-zero real number, hence ρ_1 too (see the proof of Lemma 6). In the trivial case $H_n = G_n + C$ of Example 3, one has $f_1 = \rho_1$ (and this latter equality conversely implies that $H_n - G_n$ is constant in the case Q and the beta-polynomial associated with β are equal). More generally, if $\mu_1/\lambda_1 \in \mathbf{Z}[\beta^{-1}]$, its β-expansion, which is finite, has the form $\mu_1/\lambda_1 = \sum_{k=-u}^v \varepsilon_k\beta^k$. Following an idea of [17], taking conjugates and summing up shows that the G-expansion of H_n has a bounded number of non-zero digits, hence $\urcorner$ does not belong to $\mathbf{Q}(\beta)$.

Example 5. A less immediate example is the following: let $(G_n)_n = (F_n)_n$ be the usual Fibonacci sequence with convention $F_0 = 1$ and $F_1 = 2$; take $H_n := \frac{F_{4n-2}}{3}$. We claim that $H_n \in \mathbf{N}$ for every $n \in \mathbf{N}$ and

$$(H_{2n})_G (H_0)_G^4 = (1010^5)^n, \tag{5.1}$$

which gives unbounded subsequences $(H_{n_j})_j$ for which the corresponding real number $\daleth$ belongs to $\mathbf{Q}\left(\frac{1+\sqrt{5}}{2}\right)$.

To prove (5.1) we start with the identity

$$\frac{F_{k+8} - F_k}{3} = F_{k+5} + F_{k+3},$$

which is true for all $k \in \mathbf{Z}$. It is well known (or easily checked) that $(F_n \pmod 3)_n$ has period 8 and that $3 \mid F_k$ if and only if $k \equiv 2 \pmod 4$. Thus

$$H_{2n+2} = \frac{F_{8n+6}}{3} = \frac{F_{8n-2}}{3} + F_{8n+3} + F_{8n+1} = H_{2n} + F_{8n+3} + F_{8n+1}.$$

Then inequalities $3F_k < F_{k+3} < 3F_{k+1}$ applied to H_{2n} yield $(H_{2n+2})_G = 1010^5 (H_{2n})_G$. We have $H_2 = 7 = F_3 + F_1$, hence $(H_{2n})_G = (1010^5)^n 1010$. Concatenating this word with $(H_0)_G^4 = 0^4$ gives (5.1).

The minimal polynomial of the sequence $(H_n)_n$ above is $X^2 - 7X + 1$, as $(H'_n)_n$, where $H'_n = F_{4n}$. Nevertheless, none of the words $(H_{n_j})_G$ is eventually periodic provided that $(n_j)_j$ is not. That example shows that the minimal polynomial of $(H_n)_n$ is not sufficient to state whether $\daleth$ belongs to $\mathbf{Q}(\beta)$ or not; for (some) related sequences everything depends on the initial values of $(H_n)_n$ too.

Remark 1. Niederreiter's approach (see [25]) can be straightforward adapted for sequence $(n_j)_j$ of positive density as soon as the minimal polynomial of $(H_n)_n$ has a dominating root α_1 such that $\log_\beta(\alpha_1) \notin \mathbf{Q}$. Note that this is a particular case of Theorem 1. As in [25], one obtains that the word $|H|_G$ contains arbitrary large strings of zeros. In fact, as proved in Proposition 1 of [6], for any positive integer k, any admissible word of length at most k occurs in $(|H_n|)_G$ if n is large enough. See also Remark 2 of the same paper.

Remark 2. It is a natural question whether a property stronger than irrationality (respectively the property of belonging to $\mathbf{Q}(\beta)$) could be accessible, even for the original Mahler numbers. In [5], Bailey, Borwein, Crandall and Pomerance proved the following: If y is a real algebraic of degree $D > 1$, then there exists a positive number C (depending only on y) such that for sufficiently large N the number $\#(|y|, N)$ of 1's in the binary expansion of $|y|$ through the N-th bit position satisfies $\#(|y|, N) > CN^{1/D}$. We note that in the case of the simplest Mahler number $0.(2^n)_2 = 0.1101001000100001\cdots$, an exponent $1/2 + \varepsilon$ in their result (instead of $1/2$) would be necessary to imply that $0.(2^n)_2$ is not quadratic. Thus transcendance of Mahler numbers seems to be far of reach at the moment. For recent beautiful results on transcendance in relation with the complexity of digital expansions, we refer to the paper of Adamczewski, Bugeaud and Luca [1].

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