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RECENT PROGRESS IN RAMSEY THEORY ON THE INTEGERS

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Abstract

We give a brief survey of some recent developments in Ramsey theory on the set of integers and mention several unsolved problems, giving a partial answer to one.

–For Ron Graham on his 70th birthday

1. Introduction

The purpose of this note is to provide a brief survey of some recent progress in the area of Ramsey theory on the set of integers, and to present several open problems and conjectures. Most of the results in this paper are solutions (or partial solutions) to open problems mentioned in [22].

We will use the following notation and terminology. For an arithmetic progression $A = \{a + id : 0 \leq i \leq k-1\}$, we say that A is a *k-term a.p. with gap d*. We denote the family of all arithmetic progressions by AP. For t a positive integer, we denote the set $\{1, 2, ..., t\}$ by $[1, t]$. An *r-coloring* of a set S is a function $\chi : S \to \{0, 1, ..., r-1\}$ (obviously, an r-coloring of S may be thought of as a partition of S into r subsets). A coloring χ is *monochromatic* on S if it is constant on S. For positive integers r and t, a specific r-coloring of $[1, t]$ may be denoted by an r-ary string. For example, the 3-coloring of $[1, 5]$ defined by $\chi(1) = 0$, $\chi(2) = \chi(3) = \chi(4) = 2$, and $\chi(5) = 1$ may be denoted by 02221. We shall use exponential notation to mean repetition so that, for example, $0^4 1^2 0$ represents the coloring 0000110.

In one way or another, all of the topics covered in this article are variations of the classical theorem known as van der Waerden's theorem [29], which states:

Theorem 1. *For any positive integers k and r, there exists a least positive integer $n = w(k;r)$ such that every r-coloring of $[1,n]$ admits a monochromatic k-term a.p.*

For a family $\mathcal{C}$ of sets, and r a positive integer, we say that $\mathcal{C}$ is *r-regular* if there exists an n such that every r-coloring of $[1,n]$ yields a monochromatic member of $\mathcal{C}$. If a family is r-regular for all $r \in \mathbb{Z}^+$, the family is said to be *regular*. By van der Waerden's theorem, for each k the family of k-term a.p.'s is regular. We may also say, more simply, that AP is regular. One of the most celebrated results regarding the van der Waerden numbers is the upper bound of Timothy Gowers [16], which for the case of $r = 2$ states:

$$w(k;2) \leq 2^{2^{2^{2^{2^{k+9}}}}}$$

for $k \geq 2$. In sharp contract, the best known lower bound is $w(p+1;2) \geq p2^p$ for p prime, and is due to Berlekamp [5]. Finally, we mention that an equivalent (and well-known) form of Theorem 1 is the following statement.

For all positive integers r, every r-coloring of $\mathbb{Z}^+$ yields arbitrarily long monochromatic a.p.'s.

2. Ascending Waves

By van der Waerden's theorem, any superset of AP is regular. Brown, Erdős, and Freedman [8] considered one such set, namely the family of ascending waves.

Definition. A *k-term ascending wave* is an increasing sequence $a_1, a_2, ..., a_k \in \mathbb{Z}^+$, such that $a_{i+1} - a_i \geq a_i - a_{i-1}$ for $i = 2, 3, ..., k-1$.

They showed that $k^2 - k + 1 \leq AW(k;2) \leq \frac{1}{3}(k^3 - 4k + 9)$, where $AW(k;2)$ denotes the function that differs from the function $w(k;2)$ only in that the family AP is replaced by the family of ascending waves. Alon and Spencer [1] improved the lower bound, showing that

$$AW(k;2) \geq \frac{k^3}{10^{21}}(1 + o(1)).$$

Recently, LeSaulnier and Robertson [25] improved the upper bound, and also extended upper and lower bounds to r colors. They showed

1. $AW(k;2) \leq \frac{k^3}{6}(1 + o(1))$.
2. $AW(k;r) \leq \frac{k^{2r-1}}{(2r-1)!}(1 + o(1))$.
3. For any $\epsilon > 0$, $AW(k;r) \geq k^{2r-1-\epsilon}(1 + o(1))$.

It would be desirable to find (or get closer to) a constant c such that $AW(k;2) = ck^3(1+o(1))$ (from the above discussion we know that $\frac{1}{10^{21}} \leq c \leq \frac{1}{6}$), and likewise in the general case of r colors.

3. Large Sets and Accessible Sets

While all supersets of AP are regular, for *subsets* of AP, regularity is not guaranteed. For example, the coloring 010101.... shows that the subset of AP consisting of all a.p.'s having odd gaps is not regular (it is not even 2-regular). This prompts the following notions.

If $D \subseteq \mathbb{Z}^+$, denote by AP_D the family of a.p.'s with gaps in D. For $D \subseteq \mathbb{Z}^+$ and $r \geq 1$ fixed, we say that D is *r-large* if every r-coloring of $\mathbb{Z}^+$ yields arbitrarily long monochromatic members of AP_D. We say $D \subseteq \mathbb{Z}^+$ is *large* if it is r-large for all $r \geq 1$.

Some deep results of Furstenberg [12] and Bergelson and Leibman [4], employing dynamical systems methods, provide sufficient conditions for a set to be large. Other results, obtained by combinatorial means, are given in [9]. However, the general question of *which* sets of positive integers are large is still rather wide-open.

Related to the concept of largeness, but a property that is more easily satisfied, is that of accessibility. For $D \subseteq \mathbb{Z}^+$, we call a sequence of positive integers $\{x_1, x_2, ..., x_k\}$ a *k-term D-diffsequence* if $x_i - x_{i-1} \in D$ for $2 \leq i \leq k$. Given $r \in \mathbb{Z}^+$ and D a set of positive integers, we say that D is *r-accessible* if whenever $\mathbb{Z}^+$ is r-colored, there are arbitrarily long monochromatic D-diffsequences. Further, D is called *accessible* if D is r-accessible for all positive integers r.

It is obvious that any large set is accessible. Jungic [18] recently showed that the converse is false.

It is not difficult to show that for each positive integer r there are sets that are r-accessible but that fail to be $(r+1)$-accessible (see [22]). As one example, $\{2^i : i \geq 0\}$ is 2-accessible but not 3-accessible. On the other hand, a still unsolved conjecture [9] states that whenever a set is 2-large, then it is, in fact, a large set.

It was recently shown [23] that the set of primes, P, is not 3-accessible, and that all odd translations of P are 2-accessible. It is not known whether there exist any even translations of P (including P itself) that are 2-accessible. Another open problem is determining, for an odd number c, the maximum value of r such that $P + c$ is r-accessible.

4. Generalizations of Arithmetic Progressions

An arithmetic progression $\{x + id : 0 \leq i \leq (k-1)\}$ is determined by the parameters x, d, and k. By introducing additional parameters, the van der Waerden numbers become one special case of a more general function. The following generalization of 3-term a.p.'s was considered in [21].

For $1 \leq a \leq b$, an *(a, b)-triple* is a set $\{x, ax + d, bx + 2d\}$, where $x, d \in \mathbb{Z}^+$.

Denote by $n = n(a, b; r)$ the least positive integer (if it exists) such that every r-coloring of $[1, n]$ admits a monochromatic (a, b)-triple. For convenience, we will say that (a, b) is *regular* if $n(a, b; r) < \infty$ for all $r \geq 1$. By Theorem 1, $(1, 1)$ is regular. If (a, b) is not regular, the *degree of regularity* of (a, b), denoted $\mathrm{dor}(a, b)$, is the largest integer r such that (a, b) is r-regular. In [21], the following results were obtained.

1. $\mathrm{dor}(a, b) = 1$ iff $b = 2a$.
2. $\mathrm{dor}(a, 2a - 1) = 2$ for all $a \geq 2$
3. Let $1 \leq a < b$. If $b \geq (2^{3/2} - 1)a + 2 - 2^{3/2}$, then $\mathrm{dor}(a, b) \leq \lceil 2 \log_2 c \rceil$, where $c = \lceil b/a \rceil$.

The authors of [21] conjectured that:

(i) $(1, 1)$ is the *only* regular pair, and

(ii) for all pairs (a, b), if (a, b) is not regular, then $\mathrm{dor}(a, b) \leq 2$.

Conjecture (i) has been verified, independently, in [14] and [15]. In fact, Fox and Radoicic show that $\mathrm{dor}(a, b) \leq 6$ for all $(a, b) \neq (1, 1)$. Their proof makes use of a result, due to Fox and Kleitman [13], which states that if a linear homogeneous equation in three variables is 24-regular, then it is regular. In [15], it is shown that $2 \leq \mathrm{dor}(a, 2a - 2) \leq 4$ for all $a \geq 2$; that $\mathrm{dor}(a, 2a + j) \leq 4$ for $1 \leq j \leq 5$; and that $\mathrm{dor}(a, 2a + 1) \leq 3$ for all $a \geq 1$. A counterexample to (ii) is given in [15]: $dor(2, 2) = 3$, and furthermore $n(2, 2; 3) = 88$.

Several interesting questions concerning (a, b)-triples remain unanswered. We would like to know which pairs besides $(2, 2)$ have degree of regularity greater than 2? In particular, are there infinitely many such pairs? Are there any *not* of the form $(a, 2a - 2)$?

Bialostocki, Lefmann, and Meerdink [6] considered the following generalization of 3-term arithmetic progressions. For $b \geq 0$, let $g(b)$ be the least positive integer (if it exists) such that for every 2-coloring of $[1, g(b)]$ there is a monochromatic set of the form $x, x + d, x + 2d + b$. They showed that for b even, $2b + 10 \leq g(b) \leq \frac{13}{2}b + 1$. The upper bound was improved by the present paper's second author, to $\lceil \frac{9}{4}b \rceil + 9$, who also conjectured that for $b \geq 10$ even, $g(b) = 2b + 10$ [20]. In 2004, Grynkiewicz [17] proved the conjecture.

One apparently unexplored problem related to the function $g(b)$ is that of the more general function $g_k(b)$, the van der Waerden-type number associated with sequences of the form $a, a + d, a + 2d, ..., a + (k - 2)d, a + (k - 1)d + b$. In particular, we wonder what can be said about the function $g_4(b) = g_4(b; 2)$. In [6] it is shown that $g(b; 8)$ does not exist. More generally, we wonder what can be said about the existence of $g_k(b; r)$. In particular, can the upper bound of $g(b; 3) \leq \frac{55}{6} + 1$ for b a multiple of 6, which is proved in [6], be improved?

Another way to generalize the definition of an arithmetic progression is via the following definition.

Definition. If $1 \leq s_1 < s_2 < \cdots < s_{k-1}$, then a *homothetic copy of* $(1, 1 + s_1, 1 + s_2, ..., 1 + s_{k-1})$ is a k-tuple $(a, a + ds_1, a + ds_2, \ldots, a + ds_{k-1})$ where $d \in \mathbb{Z}^+$.

Hence the family of k-term a.p.'s is precisely the family of homothetic copies of $(1, 2, \ldots, k)$. Denote by $h(s_1, s_2, \ldots, s_{k-1})$ the least positive integer n such that every 2-coloring of $[1, n]$ admits a monochromatic homothetic copy of $(s_1, s_2, \ldots, s_{k-1})$. Thus, $w(k) = h(1, 2, \ldots, k-1)$. Brown, Landman, and Mishna [10] showed that $h(s_1, s_2) \leq 4s_2 + 1$. They also showed that this is an equality for most cases, and gave tight bounds for the remaining cases. Exact values for these remaining cases have very recently been found by Kim and Rho [19]. How h behaves when $k > 2$ is a wide open problem, as is the corresponding question (even when $k = 2$) if more than 2 colors are used.

5. Mixed van der Waerden Numbers

Denote by $w(k_0, k_2, \ldots, k_{r-1}; r)$ the least positive integer n such that for every r-coloring $\chi : [1, n] \longrightarrow \{0, 1, ..., r-1\}$ there is, for some i, $0 \leq i \leq r-1$, a k_i-term arithmetic progression of color i. For example, if $k_i = k$ for all i, we have the classical van der Waerden numbers $w(k; r)$, or what might be called the "diagonal" van der Waerden numbers. For example, $w(3, 3; 2) = w(3; 2) = 9$.

All of the non-trivial exact values of $w(k_0, ..., k_{r-1}; r)$ that are known to date, are presented (chronologically) in [11, 7, 28, 3, 2, 24]. The known values cover all cases in which $s = \sum_{i=0}^{r-1} k_i \leq 10$, and those in which $s = 11$ except for $w(6, 5; 2)$. Also covered are the values of $w(k, 3; 2)$ for $9 \leq k \leq 13$, as well as the values of $w(6, 4, 2; 3)$, $w(7, 3, 2; 3)$, $w(4, 4, 2, 2; 4)$, $w(5, 3, 2, 2; 4)$, $w(4, 3, 3, 2; 4)$, $w(3, 3, 3, 3; 4)$, $w(6, 3, 2, 2)$, $w(7, 3, 2, 2)$, $w(3, 3, 2, 2, 2)$, and $w(3, 3, 3, 2, 2)$.

In [24], the function $w(k, 2, 2, ..., 2; r)$ is considered. For convenience, we will denote it by $w_2(k; r)$. The authors proved the following theorem which, for $k > r$, gives upper bounds and (provided k/r is sufficiently large) exact values of this function.

Before stating the theorem, we adopt the following notation. Let $p_1 < p_2 < \cdots$ be the sequence of primes. Let $\pi(r)$ denote the number of primes not exceeding r, and $\#r = p_1 p_2 \cdots p_{\pi(r)}$. Further, let $j_{k,r} = \min\{j \geq 0 : \gcd(k - j, \#r) = 1\}$, and $\ell_{k,r} = \min\{\ell \geq 0 : \gcd(k - \ell, \#r) = r\}$.

Theorem 2. *Let $k > r \geq 2$. Let $j = j_{k,r}$, $\ell = \ell_{k,r}$, and $m = \min\{j, \ell\}$. Then*

(1) $w_2(k; r) = rk$ if $j = 0$.

(2) $w_2(k; r) = rk - r + 1$ if either (i) $j = 1$; or (ii) r is prime and $\ell = 0$.

(3) If r is composite and $j \geq 2$ then $w_2(k; r) \geq rk - j(r-2)$, with equality provided either

(i) $j = 2$ and $k \geq 2r - 3$;
or (ii) $j \geq 3$ and $k \geq (\pi(r))^3(r-2)$.

(4) If r is prime, $j \geq 2$, and $\ell \geq 1$, then $w_2(k;r) \geq rk - m(r-2)$, with equality provided either (i) $\ell = 1$, (ii) $m = 2$ and $k \geq 2r-3$, or (iii) $m \geq 3$ and $k \geq (\pi(r))^3(r-2)$.

There are many open questions we may ask about mixed van der Waerden numbers. Here are some natural ones:

1. What is the value of $w(4,4,4;3)$ (this is the classical van der Waerden number $w(4;3)$)?
2. What is the value of $w(8,4;2)$?
3. Can the restrictions on the magnitude of k in comparison to r in Theorem 2 be weakened, especially to guarantee equality in Cases (3)(ii) and (4)(iii)?
4. What is the value of $w_2(k;r)$ when $k < r$?
5. Can we find a "nice" upper bound on $w(3,3,2,2,2,...,2;r)$? More generally, what can we say about the magnitude of the mixed van der Waerden number $w'(k,\ell) = w(2,2,\dots,2,3,3,\dots,3;k+\ell)$, where there are k 2's and ℓ 3's?

6. Estimates of $w_2(k;r)$

The next two theorems give a partial answer to one of the open problems mentioned in Section 5, namely, the determination of the magnitude of $w_2(k;r)$ when $k < r$.

Theorem 3. *Let $k < r < \frac{3}{2}(k-1)$. Then $w_2(k;r) \leq r(k-1)$.*

Proof. Let f be an arbitrary r-coloring of $[1, r(k-1)]$. For a contradiction, assume that f avoids k-term monochromatic a.p.'s of color 0, and 2-term monochromatic a.p.'s of colors $1, 2, ..., r-1$. We may assume that f is the string

$$0^{b_1}x_1 0^{b_2}x_2 \dots 0^{b_{r-1}}x_{r-1}0^{b_r},$$

where $0 \leq b_i \leq k-1$ for all $1 \leq i \leq r$ and where $f(x_j) = j$ for $j = 1, ..., r-1$.

Let $d_i = x_i - x_{i-1}$ for $i = 2, 3, ..., r-1$. Let $X = \{x_1, ..., x_{r-1}\}$ and let $D = \{d_2, d_3, ..., d_{r-1}\}$. Clearly, we have

$$x_i - x_{i-1} \leq k \text{ for all } 2 \leq i \leq r-1 \tag{1}$$

and

$$x_{r-1} - x_1 \geq r(k-1) - 2(k-1) - 1. \tag{2}$$

Now we consider two cases.

Case 1. At most $r-k$ members of D equal k. We consider two subcases.

Subcase 1.1. At least $r-k+1$ elements of D are equal to $k-1$.
In this case the number of distinct residue classes modulo $k-1$ that are covered by X is at most $k-2$. Hence for some j, $1 \leq j \leq k-1$,

$$\{j+i(k-1) : 0 \leq i \leq k-1\} \subseteq [1, k(k-1)]$$

is an arithmetic progression of color 0, a contradiction.

Subcase 1.2. Fewer than $r-k+1$ elements of D are equal to $k-1$.
We claim that at least $r-k+2$ elements of D equal $k-2$. We verify this claim by contradiction. If it were false, then

$$\begin{aligned} s &= \textstyle\sum_{i=2}^{r-1} d_i \leq k(r-k)+(k-1)(r-k)+ \\ &\quad (k-2)(r-k+1)+(k-3)(r-2-3r+3k-1) \\ &= kr-8k+3r+7. \end{aligned}$$

Now, by (2), $s \geq r(k-1)-2(k-1)-1 = kr-r-2k+1$. But this is a contradiction because $kr-r-2k+1 > kr+3r-8k+7$, since by hypothesis $6k > 4r+6$. Therefore, the claim is true, and hence X covers at most $r-1-(r-k+3)+1 = k-3$ different congruence classes modulo $k-2$. Therefore for some j, $1 \leq j \leq k-2$, no member of

$$A = \{j+i(k-2) : 0 \leq i \leq k-1) \subseteq [1, k-2+(k-1)(k-2)] = [1, k(k-2)]$$

intersects X. Thus A is a k-term arithmetic progression with color 0, a contradiction.

Case 2. At least $r-k+1$ members of D equal k.
This implies that at most $k-2$ different congruence classes modulo k are covered by X. Hence, for some j, $1 \leq j \leq k-1$, the arithmetic progression $\{j+ik : 0 \leq i \leq k-1\} \subseteq [1, k^2-1]$, has color 0, a contradiction. □

Note that the upper bound of Theorem 3 is less than the bounds of Theorem 2 which deals with $k > r$.

Although Theorem 3 may seem to apply to only a very restricted set of pairs (k, r), the next theorem shows that the upper bound of Theorem 3 would hold for *all* k and r such that $k < r$ provided only that it holds whenever $\frac{3}{2}(k-1) \leq r \leq 2k+1$.

Theorem 4. *Let $k \geq 2$ and assume that $w_2(k;r) \leq (k-1)r$ for $\frac{3}{2}(k-1) \leq r \leq 2k+1$. Then $w_2(k;r) \leq (k-1)r$ for all $r > k$.*

Proof. The proof is by induction on r, with k fixed. By the hypothesis and Theorem 3, the statement is true for all r such that $k < r \leq 2k+1$.

Now assume $r \geq 2k+1$ is such that $w_2(k;r) \leq (k-1)r$. To prove $w_2(k;r+1) \leq (k-1)(r+1)$, let f be any $(r+1)$-coloring of $[1,(k-1)(r+1)]$. Since $r-k \geq k+1$, by the inductive hypothesis $w_2(k;r-k) \leq (k-1)(r-k)$. Hence f must have at least $r-k$ non-zero colors in $[1,(k-1)(r-k)]$ to avoid a k-term a.p. of color 0 or a 2-term a.p. of a non-zero color. This forces f to have at most k non-zero colors in $[(k-1)(r-k)+1,(k-1)(r+1)]$. By assumption $w_2(k;k+1) \leq (k-1)(k+1)$. Therefore, f has a k-term a.p. of color 0 or a monochromatic 2-term, a.p. of a non-zero color in $[(k-1)(r-k)+1,(k-1)(r+1)]$. This completes the proof. □

Table 1 shows the mixed van der Waerden numbers $w_2(k;r)$, obtained by computer, for $k = 3,4,5$ and $k < r \leq 13$. By Theorem 4 we obtain:

Corollary 5. *Let $k = 3,4,5$. Then $w_2(k;r) \leq (k-1)r$ for all $r > k$.*

r / k	4	5	6	7	8	9	10	11	12	13
3	8	10	12	15	16	17	18	19	21	22
4		14	16	18	20	22	24	26	29	31
5			20	22	24	27	30	33	35	38

Table 1. Values of $w_2(k;r)$

7. A Conjecture

We conclude with the following conjecture (which has obvious extensions to $r > 2$ colors). The conjecture, and its extensions to $r > 2$ colors, are supported by all known values of $w(k_0,...,k_{r-1};r)$ (see [24] for a table of known values).

Conjecture. Let $k \geq \ell > 2$. Then

$$w(k,\ell;2) \geq w(k+1,\ell-1) \geq w(k+2,\ell-2) \geq \cdots .$$

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